GEOMETRY: HOMEWORK 5

DUE OCTOBER 17

(1) The complex polynomial $w = u + iv \mapsto w^2$ can be extended to ∞ as a smooth map from $\mathbb{S}^2$ to itself. In terms of the standard coordinate of $\mathbb{R}^3$, the map takes the form:

$$\begin{aligned} F: \quad \mathbb{S}^2 \quad &\to \quad \mathbb{S}^2 \\ (x,y,z) \quad &\mapsto \quad \frac{1}{1+z^2}(x^2-y^2, 2xy, 2z) \ . \end{aligned}$$

Choose the outer unit normal as the orientation of $\mathbb{S}^2$. Calculate the degree of F by evaluating the integral of $\mathscr{J}_F$.

Remark:
$$\begin{aligned} (x^2-y^2)^2 + (2xy)^2 + (2z)^2 &= (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 + 4z^2 \\ &= (x^4 + 2x^2y^2 + y^4) + 4z^2 \\ &= (x^2+y^2)^2 + 4z^2 \\ &= (1-z^2)^2 + 4z^2 \qquad (\text{if } x^2+y^2+z^2 = 1) \\ &= (1+z^2)^2 \ . \end{aligned}$$

(2) (a) Endow the orientation $(0,0,1)$ for the xy-plane. Fix $\delta > \varepsilon > 0$, and choose a smooth function $h(r)$ for $r \geq 0$ which satisfies
- $h'(r) \leq 0$ for any $r > 0$;
- $h(r) = 1$ for $r \in [0, \varepsilon]$;
- $h(r) = -1$ for $r \geq \delta$.

Namely, $h(r)$ decreases from 1 to -1 as r increases. Let $r = \sqrt{x^2+y^2}$, and consider the map

$$\begin{aligned} F: \quad \mathbb{R}^2 \quad &\mapsto \quad \mathbb{S}^2 \\ (x,y) \quad &\mapsto \quad \left(\sqrt{\frac{1-h^2(r)}{r^2}}\, x, \sqrt{\frac{1-h^2(r)}{r^2}}\, y, h(r) \right) \ . \end{aligned}$$

Note that $F(\{r \leq \varepsilon\}) = (0,0,1)$, and thus it is smooth at the origin. Evaluate $\frac{1}{4\pi} \int\int_{\mathbb{R}^2} \mathscr{J}_F \, \mathrm{d}x \, \mathrm{d}y$.

(b) Given any $n \in \mathbb{N}$ and any compact regular surface S (with an orientation N), explain a strategy to construct a map from S to $\mathbb{S}^2$ with degree n.

Remark: In [MR, §8.2], the notion of degree is defined for maps between compact regular surfaces. That is to say, the target space need not to be $\mathbb{S}^2$. The above property is not true in general. For instance, one can prove (with the help of some topology knowledge) that any smooth map from $\mathbb{S}^2$ to a surface with genus≥ 1 must have degree 0.

(3) Consider $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$ with the first fundamental form

$$I_{\mathbb{H}} = \frac{1}{y^2}(\mathrm{d}x \cdot \mathrm{d}x + \mathrm{d}y \cdot \mathrm{d}y) \ .$$

It means that for a tangent vector (a, b) at (x, y), its length is $y^{-1}\sqrt{a^2 + b^2}$ but not $\sqrt{a^2 + b^2}$. The goal of this exercise is to demonstrate that the concept of distance of $I_{\mathbb{H}}$ is different from that of the standard one, $\mathrm{d}x \cdot \mathrm{d}x + \mathrm{d}y \cdot \mathrm{d}y$.

For a smooth curve $\gamma(t) = (x(t), y(t))$, the tangent vector is $\gamma'(t) = (x'(t), y'(t))$. Its arc-length is defined to be

$$\mathrm{L}[\gamma] = \int \sqrt{I_{\mathbb{H}}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t = \int \sqrt{\frac{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}{\big(y(t)\big)^2}}\, \mathrm{d}t \ .$$

Consider an arc of the unit circle, $\sigma(t) = (\cos t, \sin t)$ for $t \in [\alpha, \beta]$, where $0 < \alpha < \beta < \pi$. Its arc-length is

$$\mathrm{L}[\sigma] = \int_\alpha^\beta \frac{1}{\sin t} \mathrm{d}t = \int_\alpha^\beta \csc t\, \mathrm{d}t = -\log(\csc t + \cot t)\big|_{t=\alpha}^\beta \ .$$

Show that any smooth curve $\gamma(t)$ in $\mathbb{H}$ which connects $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$ must have arc-length no less than $\mathrm{L}[\sigma]$.

Hint: You may use the polar coordinate to describe the curve,

$$\gamma(t) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t))) : [0, 1] \to \mathbb{H} \ .$$

The conditions are $\theta(0) = \alpha$, $\theta(1) = \beta$, and $r(0) = 1 = r(1)$.