GEOMETRY: HOMEWORK 4

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1. Some linear algebra

1.1. **Change of basis.** Let V be an n-dimensional vector space over $\mathbb{R}$. Let $\{\mathbf{v}_i\}_{i=1}^n$ and $\{\mathbf{w}_i\}_{i=1}^n$ be two bases for V. Then,

$$\mathbf{v}_i = \sum_{j=1}^n L_i^j\,\mathbf{w}_j \tag{1.1}$$

for some $[L_i^j] \in \mathrm{GL}(n;\mathbb{R}) = \{\text{invertible } n \times n \text{ matrices}\}$. It follows that

$$\sum_{i=1}^n a^i\,\mathbf{v}_i = \sum_{j=1}^n \left(\sum_{i=1}^n L_i^j\,a^i\right)\mathbf{w}_j\ .$$

In other words, $\sum_{i=1}^n a^i\,\mathbf{v}_i = \sum_{j=1}^n b^j\,\mathbf{w}_j$ if and only if

$$\sum_{i=1}^n L_i^j\,a^i = b^j\ . \tag{1.2}$$

In the concrete case, $\mathbb{R}^n$, (1.1) and (1.2) read

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n\end{bmatrix}\begin{bmatrix} L_1^1 & L_2^1 & \cdots & L_n^1 \\ L_1^2 & L_2^2 & \cdots & L_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ L_1^n & L_2^n & \cdots & L_n^n\end{bmatrix}\ ,$$

$$\begin{bmatrix} L_1^1 & L_2^1 & \cdots & L_n^1 \\ L_1^2 & L_2^2 & \cdots & L_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ L_1^n & L_2^n & \cdots & L_n^n\end{bmatrix}\begin{bmatrix}a^1\\ a^2\\ \vdots\\ a^n\end{bmatrix} = \begin{bmatrix}b^1\\ b^2\\ \vdots\\ b^n\end{bmatrix}\ .$$

1.2. **Change of basis in dual space.** Its dual vector space, $V^* = \mathrm{Hom}(V;\mathbb{R})$, has two bases[1], $\{\mathbf{v}_i^*\}_{i=1}^n$ and $\{\mathbf{w}_i^*\}_{i=1}^n$. From

$$\mathbf{w}_i^*(\mathbf{v}_j) = \mathbf{w}_i^*(\sum_{k=1}^n L_j^k\,\mathbf{w}_k) = L_j^i\ ,$$

[1] Be careful about the dual basis. For instance, consider $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ for $\mathbb{R}^2$. Use the standard inner product to identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$. Their dual bases are $\{(1,0),(0,1)\}$ and $\{(1,-1),(0,1)\}$, respectively. From this example, the notation $(1,0)^*$ only makes sense with a choice of basis.

one finds that

$$\mathbf{w}_i^* = \sum_{j=1}^n L_j^i \, \mathbf{v}_j^* \ .$$

Sometimes the index is raised to make the notation more consistent. Write $\mathbf{v}_i^*$ by $\hat{\mathbf{v}}^i$, and $\mathbf{w}_i^*$ by $\hat{\mathbf{w}}^i$. The equation is

$$\hat{\mathbf{w}}^i = \sum_{j=1}^n L_j^i \, \hat{\mathbf{v}}^j \ . \tag{1.3}$$

Similarly, $\sum_{i=1}^n \hat{a}_i \, \hat{\mathbf{v}}^i = \sum_{j=1}^n \hat{b}_j \, \hat{\mathbf{w}}^j$ is equivalent to

$$\sum_{j=1}^n L_i^j \, \hat{b}_j = \hat{a}_i \ . \tag{1.4}$$

1.3. **Bilinear form on V.** The set of bilinear forms on V is a vector space of dimension n^2. It has the following basis[2]: $\{\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j\}_{i,j=1}^n$, which are defined on the basis by

$$(\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j)(\mathbf{v}_k, \mathbf{v}_\ell) = \begin{cases} 1 & \text{when } i = k \text{ and } j = \ell; \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $(\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j)(\mathbf{v}_k, \mathbf{v}_\ell) = \delta_k^i \delta_\ell^j$. Another basis, $\{\mathbf{w}_k\}_{k=1}^n$, of V leads to another basis, $\{\hat{\mathbf{w}}^k \otimes \hat{\mathbf{w}}^\ell\}_{k,\ell}$, of bilinear forms on V. One can check that

$$\hat{\mathbf{w}}^i \otimes \hat{\mathbf{w}}^j = \sum_{k,\ell=1}^n L_k^i \, L_\ell^j \, (\hat{\mathbf{v}}^k \otimes \hat{\mathbf{v}}^\ell) \ . \tag{1.5}$$

The set of bilinear forms on V has a natural decomposition into symmetric and anti-symmetric part. The symmetric part has dimension $\frac{1}{2}n(n+1) = n + \frac{1}{2}n(n-1)$. The basis is constituted of $\{\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^i\}_{i=1}^n$ with $\{\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j + \hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j\}_{i \neq j}$. The anti-symmetric part has dimension $\frac{1}{2}n(n-1)$, and has basis $\{\hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j - \hat{\mathbf{v}}^i \otimes \hat{\mathbf{v}}^j\}_{i \neq j}$.

2. Tangent planes of a regular surface

Now, let S be a regular surface. Suppose that $(\Omega_1, \mathbf{X})$ and $(\Omega_2, \mathbf{Y})$ be two coordinate chart for S, and $\mathbf{X}(\Omega_1) \cap \mathbf{Y}(\Omega_2) \neq \varnothing$. Denote the coordinate for $\Omega_1 \subset \mathbb{R}^2$ by (u^1, u^2), and the coordinate for $\Omega_2 \subset \mathbb{R}^2$ by (ξ^1, ξ^2).

For any $p \in \mathbf{X}(\Omega_1) \cap \mathbf{Y}(\Omega_2)$, apply the discussion of section 1 to $V = T_pS$. The two bases are

$$\left\{ \mathbf{v}_i = \frac{\partial \mathbf{X}}{\partial u^i} \right\}_{i=1}^2 \qquad \text{and} \qquad \left\{ \mathbf{w}_j = \frac{\partial \mathbf{Y}}{\partial \xi^j} \right\}_{j=1}^2 \ . \tag{2.1}$$

[2]During the class, we use the notation $\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^j$. The standard notation is $\otimes$.

By applying the chain rule on $\mathbf{X} = \mathbf{Y} \circ (\mathbf{Y}^{-1} \circ \mathbf{X})$,

$$\frac{\partial \mathbf{X}}{\partial u^i} = \sum_j \frac{\partial \mathbf{Y}}{\partial \xi^j} \frac{\partial \xi^j}{\partial u^i}$$

Comparing it with (1.1),

$$L_i^j = \frac{\partial \xi^j}{\partial u^i} \tag{2.2}$$

in this current setting.

2.1. **Tangent vector.** The modern notation for the above bases of T_pS are

$$\frac{\partial}{\partial u^i} \quad \text{and} \quad \frac{\partial}{\partial \xi^i} \text{ , respectively .}$$

A vector of T_pS can be written as $\sum_i a^i \frac{\partial}{\partial u^i}$, which just means $\sum_i a^i \frac{\partial \mathbf{X}}{\partial u^i}$.

The first advantage for this notation is for taking derivative. Suppose that $f : \mathbb{R}^3 \supset S \to \mathbb{R}$ is a smooth function. Then, the differential of f in the direction of $\sum_i a^i \frac{\partial \mathbf{X}}{\partial u^i}$ is

$$(Df)|_p(\sum_i a^i \frac{\partial \mathbf{X}}{\partial u^i}) = (D(f \circ \mathbf{X})) \begin{bmatrix} a^1 \\ a^2 \end{bmatrix} = \sum_i a^i \left. \frac{\partial (f \circ \mathbf{X})}{\partial u^i} \right|_p .$$

The second advantage is related to the first one. The relation (1.2) says that

$$\sum_i a^i \frac{\partial}{\partial u^i} = \sum_j b^j \frac{\partial}{\partial \xi^j} \qquad \text{if and only if} \qquad \sum_i \frac{\partial \xi^j}{\partial u^i} a^i = b^j . \tag{2.3}$$

This can be seen by re-doing the chain rule computation:

$$\begin{aligned} (Df)(\sum_i a^i \frac{\partial \mathbf{X}}{\partial u^i}) &= \sum_i a^i \frac{\partial (f \circ \mathbf{X})}{\partial u^i} \\ &= \sum_i a^i \frac{\partial \big((f \circ \mathbf{Y}) \circ (\mathbf{Y}^{-1} \circ \mathbf{X})\big)}{\partial u^i} \\ &= \sum_{i,j} a^i \frac{\partial (f \circ \mathbf{Y})}{\partial \xi^j} \frac{\partial \xi^j}{\partial u^i} \\ = (Df)(\sum_j b^j \frac{\partial \mathbf{Y}}{\partial \xi^j}) &= \sum_j b^j \frac{\partial (f \circ \mathbf{Y})}{\partial \xi^i} . \end{aligned}$$

People usually abuse the notation, and write it as

$$\sum_i a^i \frac{\partial f}{\partial u^i} = \sum_{i,j} a^i \frac{\partial f}{\partial \xi^j} \frac{\partial \xi^j}{\partial u^i} = \sum_j b^j \frac{\partial f}{\partial \xi^j} .$$

By ignoring f, one finds the relation between a^i and b^j, (2.3).

2.2. **Cotangent vector.** A cotangent vector is an element of the dual space of T_pS. The dual space is denoted by T_p^*S. It has basis $\{\hat{\mathbf{v}}^i\}_{i=1}^2$ which is defined by

$$\hat{\mathbf{v}}^i\left(\frac{\partial \mathbf{X}}{\partial u^k}\right) = \begin{cases} 1 & \text{if } i = k \ ; \\ 0 & \text{otherwise} \ . \end{cases}$$

Another basis $\{\hat{\mathbf{w}}^j\}_{j=1}^2$ is defined by the same way. The modern notation for them are

$$\mathrm{d}u^i \quad \text{and} \quad \mathrm{d}\xi^j \ , \text{ respectively} \ .$$

In the modern notation, the defining relation reads

$$\mathrm{d}u^i\left(\frac{\partial}{\partial u^k}\right) = \delta^i_k \ .$$

By (1.3) and (2.2), the transition rule between different bases is

$$\mathrm{d}\xi^i = \sum_j \frac{\partial \xi^i}{\partial u^j}\,\mathrm{d}u^j \ , \tag{2.4}$$

which looks just like the chain rule.

2.3. **Symmetric bilinear forms.** The first and second fundamental forms are symmetric bilinear forms. That is to say, for each $p \in S$, they are symmetric bilinear forms on T_pS, and are smooth in the sense of smooth coefficient functions in coordinate charts. Denote

$$\langle \frac{\partial \mathbf{X}}{\partial u^i}, \frac{\partial \mathbf{X}}{\partial u^j} \rangle \text{ by } g_{ij}(u) \ , \text{ and } \langle \frac{\partial \mathbf{Y}}{\partial \xi^k}, \frac{\partial \mathbf{Y}}{\partial \xi^\ell} \rangle \text{ by } \tilde{g}_{k\ell}(\xi) \ .$$

Due to (2.4) and (1.5),

$$\begin{aligned} \sum_{k,\ell} \tilde{g}_{k\ell}\,\mathrm{d}\xi^k \otimes \mathrm{d}\xi^\ell &= \sum_{k,\ell,i,j} \tilde{g}_{k\ell}\,\frac{\partial \xi^k}{\partial u^i}\,\frac{\partial \xi^\ell}{\partial u^j}\,\mathrm{d}u^i \otimes \mathrm{d}u^j \\ &= \sum_{i,j}\left(\sum_{k,\ell} \tilde{g}_{k\ell}\,\frac{\partial \xi^k}{\partial u^i}\,\frac{\partial \xi^\ell}{\partial u^j}\right)\mathrm{d}u^i \otimes \mathrm{d}u^j = \sum_{i,j} g_{ij}\,\mathrm{d}u^i \otimes \mathrm{d}u^j \ . \end{aligned} \tag{2.5}$$

By matching the coefficients in front of the basis,

$$\sum_{k,\ell} \tilde{g}_{k\ell}\big(\xi(x)\big)\,\frac{\partial \xi^k}{\partial u^i}\,\frac{\partial \xi^\ell}{\partial u^j} = g_{ij}(x) \ . \tag{2.6}$$

3. Homework

(1) Consider the spherical coordinate for $\mathbb{S}^2$,

$$(\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)\ .$$

(a) Work out the expression of the first fundamental form in terms of this spherical coordinate.

(b) Consider the stereographic projection, $\frac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2)$. Find $\rho(u,v)$ and $\theta(u,v)$, and then apply (2.5) and (2.6) to find the expression of the first fundamental form in terms of the stereographic projection.

(2) Consider the Poincaré metric[3] on the unit disk $D = \{(u,v) \in \mathbb{R}^2 | u^2+v^2 < 1\}$:

$$\frac{4}{(1-u^2-v^2)^2}(\mathrm{d}u\cdot\mathrm{d}u + \mathrm{d}v\cdot\mathrm{d}v)\ ,$$

and consider the following metric on the upper half plane $\mathbb{H} = \{(\xi,\eta) \in \mathbb{R}^2 | \eta > 0\}$:

$$\frac{1}{\eta^2}(\mathrm{d}\xi\cdot\mathrm{d}\xi + \mathrm{d}\eta\cdot\mathrm{d}\eta)\ .$$

From Gauss's Theorema Egregium, these data determine their Gaussian curvatures, K_D and $K_{\mathbb{H}}$ (by some complicated formula we did not do in class).

(a) Consider the map

$$\begin{array}{ccc} D & \to & \mathbb{H} \\ (u,v) & \mapsto & \dfrac{1}{(1-u)^2+v^2}(-2v, 1-u^2-v^2)\ . \end{array}$$

One can check that this map is a diffeomorphism (you do not have to do this part). Check that this map defines a local isometry.

(b) For any $a > 0$, show that $(\xi,\eta) \to (a\xi, a\eta)$ defines a (local) isometry of $\mathbb{H}$. For any $b \in \mathbb{R}$, show that $(\xi,\eta) \to (\xi+b,\eta)$ defines a (local) isometry of $\mathbb{H}$.

(Hint: To avoid confusion, you can denote the coordinate of the domain by (ξ_1,η_1), and the coordinate of the codomain by (ξ_2,η_2).)

(c) Use part (a) and (b) to conclude[4] that $K_D = K_{\mathbb{H}} =$ constant.

(Remark: The same argument can be used to prove that a round sphere has constant Gaussian curvature.)

(d) Recall Beltrami's pseudo-sphere S:

$$(\alpha(t)\cos\theta, \alpha(t)\sin\theta, \beta(t))$$

[3] It just means given a first fundamental form abstractly.

[4] In fact, the statement for Theorema Egregium is only for regular surface in $\mathbb{R}^3$. Let us pretend the statement works in this abstract setting as well, and see what can be learnt from it.

where $\alpha(t) = e^t$ and $\beta(t) = \int_0^t \sqrt{1 - e^{2s}}\mathrm{d}s$. We calculate this example in class. Its first fundamental form and Gaussian curvature are

$$I = \mathrm{d}t \cdot \mathrm{d}t + e^{2t}\,\mathrm{d}\theta \cdot \mathrm{d}\theta \qquad \text{and} \qquad K \equiv -1\ ,$$

respectively. Construct a local isometry between open subsets of S and $\mathbb{H}$. What is the constant in part (c)?

(3) Let S be a regular surface.

(a) Suppose that the normal of S at some point p is $(0, 0, 1)$. Prove that on a neighborhood of p, S can be described by the graph of some function over the xy-plane.

(b) Let Γ be a 2-plane passing through p, and is *not* (parallel to) T_pS. Prove that for any open neighborhood U of p in S, the points of U cannot be on only one side of Γ.

(Hint: Let N_Γ be a normal vector of Γ. The condition says that there exists some $V \in T_pS$ such that $\langle V, N_\Gamma\rangle \neq 0$. From the definition of a regular surface, there exists a curve $\gamma(t)$ on S such that $\gamma(0) = p$ and $\gamma'(0) = V$.)

(4) Let S be a regular surface.

(a) At some $p \in S$, suppose that there exist some other point $p_0 \in \mathbb{R}^3$ and an open neighborhood U of p in S such that

$$|q - p_0| \leq |p - p_0| \quad \text{for any} \quad q \in U\ .$$

Denote $|p - p_0|$ by r. Show that $p - p_0$ is parallel to the normal of S at p, and the Gaussian curvature of S at p is no less than $1/r^2$, $K(p) \geq \frac{1}{r^2}$.

(Hint: Remember that any symmetric matrix is diagonalizable by an orthonormal basis. Part (3a) and (3b) may help you.)

(b) Suppose that S is *compact*. Prove that there exists some $p \in S$ where $K(p) > 0$.

(Hint: Consider $r = \max_{\mathbf{x}\in S}|\mathbf{x}|$. By compactness, r is finite. Consider the sphere of radius r centered at the origin.)

(5) Is the converse of (4a) true? Namely, suppose that $K(p) \geq \frac{1}{r^2}$ for some $r > 0$. Can you find some $p_0 \in \mathbb{R}^3$ such that

$$|q - p_0| \leq r$$

for any q on an open neighborhood of p in S? You only need to give a heuristic argument.

(Remark: There is a similar statement for plane curves.)