GEOMETRY: HOMEWORK 3

DUE OCTOBER 3

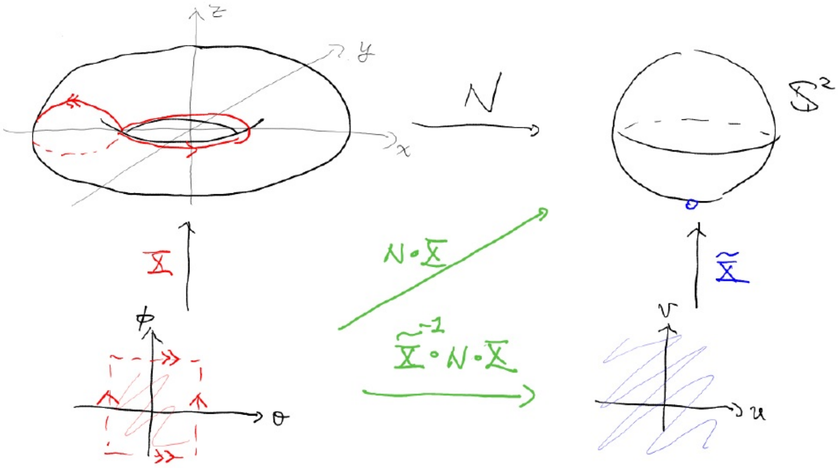
(1) Let S be the torus $\{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$. The map

$$\begin{array}{rccl} \mathbf{X}: & (-\pi,\pi)\times(-\pi,\pi) & \to & S \\ & (\theta,\phi) & \mapsto & ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta) \end{array}$$

serves as a coordinate chart for S. Also, recall the stereographic projection for $\mathbb{S}^2$:

$$\begin{array}{rccl} \tilde{\mathbf{X}}: & \mathbb{R}^2 & \to & \mathbb{S}^2 \\ & (u,v) & \mapsto & \dfrac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2)\ . \end{array}$$

One can solve for $\tilde{\mathbf{X}}^{-1} : (x, y, z) \mapsto (x, y)/(1 + z)$.



Denote by $\mathbf{N}$ the outer unit normal of S. Let $P = (3, 0, 0) \in S$, $Q = (1, 0, 0) \in \mathbb{S}^2$, $O = (0, 0) \in \theta\phi$-plane and $E = (1, 0) \in uv$-plane.

(a) Work out $(\mathbf{N} \circ \mathbf{X})(\theta, \phi)$, and calculate $D(\mathbf{N} \circ \mathbf{X})|_O$ (a 3×2 matrix).

(b) Work out $(\tilde{\mathbf{X}}^{-1} \circ \mathbf{N} \circ \mathbf{X})(\theta, \phi)$ and calculate $D(\tilde{\mathbf{X}}^{-1} \circ \mathbf{N} \circ \mathbf{X})|_O$ (2×2).

(c) Calculate $D\tilde{\mathbf{X}}|_E$ (3×2), and check that

$$D(\mathbf{N} \circ \mathbf{X})|_O = D\tilde{\mathbf{X}}|_E \cdot D(\tilde{\mathbf{X}}^{-1} \circ \mathbf{N} \circ \mathbf{X})|_O\ .$$

(d) Note that $\mathbf{N}(P) = Q$ and $T_P S = \{yz\text{-plane}\} = T_Q\mathbb{S}^2$. Write down $D\mathbf{N}|_P$ as a 2×2 matrix by using the basis $\{(0, 1, 0), (0, 0, 1)\}$.

Hint: One way to do this is to calculate $D\mathbf{X}|_O$ and use part (a).

(e) Note that

$$\frac{1}{2}\nabla((\sqrt{x^2+y^2}-2)^2+z^2) = \left(\frac{\sqrt{x^2+y^2}-2}{\sqrt{x^2+y^2}}x, \frac{\sqrt{x^2+y^2}-2}{\sqrt{x^2+y^2}}y, z\right)$$

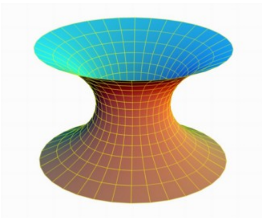
is a smooth map from $\mathbb{R}^3\backslash\{z\text{-axis}\}$ to $\mathbb{R}^3$. Denote this map by $\bar{\mathbf{N}}$. It is not hard to check that $\bar{\mathbf{N}}|_S$ is the Gauss map. Or, $\bar{\mathbf{N}}$ is an *extension* of $\mathbf{N}$ to an open set containing S. Calculate $D\bar{\mathbf{N}}|_P$ (3×3). Check that $D\bar{\mathbf{N}}|_P$ maps the yz-plane to itself, and coincides with $D\mathbf{N}|_P$.

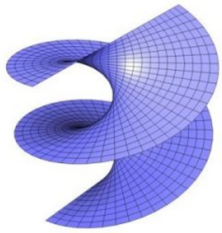
Remark: By the chain rule, and the definition of tangent space and differential map, one can verify that any smooth extension of $\mathbf{N}$ has the above property.

(2) For the following surfaces, find the first and second fundamental forms, and calculate their Gaussian and mean curvatures. You can choose $\mathbf{N}$ for your convenience.

(a) *Catenoid*: the surface S_1 given by rotating the curve $\{r = \cosh t, z = t\}$ on the rz-plane along the z-axis.

(b) *Helicoid*: the surface S_2 given by $\{(s\cos t, s\sin t, t) : s > 0, t \in \mathbb{R}\}$.





(3) Construct a local isometry between (open subsets of) the catenoid S_1 and the helicoid S_2.

Note: You can compare the fundamental forms and the curvatures at the corresponding points.

(4) Let S be a connected regular surface, with an orientation $\mathbf{N}$. Suppose that at any $p \in S$, the differential of the Gauss map is always a multiple of the identity map. Namely, there exists $\lambda : S \to \mathbb{R}$ such that $D\mathbf{N}|p(V) = \lambda(p)\, V$ for any $V \in T_pS$. Prove that S is part of a plane, or part of a sphere.

Hint: At first, prove that λ is in fact a constant. If $\lambda \equiv 0$, show that S belongs to a plane. If the constant is not zero, make a guess on the *center* of the sphere, and prove that your expression actually a (constant) point.