GEOMETRY: HOMEWORK 2

DUE SEPTEMBER 26

You do not have to submit (4) and (5).

(1) [Tubular neighborhood, from [MR, §4.5]] Let S be a *compact* surface in $\mathbb{R}^3$, and let N be the unit outward[1] normal vector field.

(a) For any $q \notin S$, let $\operatorname{dist}(q, S) = \inf_{p \in S} |p - q|$. Show the the infimum is always achieved by some $p_0 \in S$, and the line connecting p_0 and q is orthogonal to $T_{p_0}S$. (Hint: Try to consider $f(x) = |x - q|^2$ on S.)

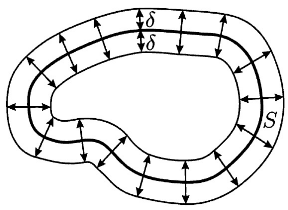
Fix $\delta > 0$. Let

$$B_\delta(S) = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, S) < \delta\}$$

and

$$N_\delta(S) = \{p + tN(p) : p \in S, |t| < \delta\} .$$

(b) Prove that $B_\delta(S) = N_\delta(S)$. (Hint: Clearly, $N_\delta(S) \subset B_\delta(S)$. To prove $N_\delta(S) \supset B_\delta(S)$, we may use part (a).)



Consider the smooth map $F(p, t) = p + tN(p)$ from $S \times (-\delta, \delta)$ to $N_\delta(S) \subset \mathbb{R}^3$.

(c) For any $p_0 \in S$, show that there exists an open neighborhood, $\mathcal{W}$, of p_0 in S and $\delta_{p_0} > 0$ such that $F : \mathcal{W} \times (-\delta_{p_0}, \delta_{p_0}) \to \mathbb{R}^3$ is diffeomorphic to its image, $N_{\delta_{p_0}}(\mathcal{W})$.

(d) Prove that there exists $\varepsilon > 0$ such that $F : S \times (-\varepsilon, \varepsilon) \to N_\varepsilon(S)$ is injective (and thus bijective). Geometrically, it means that the normal segments $N_\varepsilon(p) = \{p + tN(p) : |t| < \varepsilon\}$ do not intersect with each other for different $p \in S$.

(Hint: Suppose this is not true. It means that for any $n \in \mathbb{N}$, there exist $p_n, q_n \in S$ with $p_n \neq q_n$ but $N_{1/n}(p_n) \cap N_{1/n}(q_n) \neq \varnothing$. By compactness and passing to a subsequence, we may assume $p_n \to p$ and $q_n \to q$. Are they the same? If so, what happens if you apply part (c) to this point?)

[1]This is given by the Jordan–Brouwer separation theorem. It is the direction point toward "infinity".

(2) Show that the two-sheeted cone,

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\} ,$$

is not a regular surface. (Hint: Instead of using the definition directly, there are some basis properties we do in class. You can try to show that this cone violates one of them.)

(3) For any real numbers a, b, c, consider the regular surface Γ given by

$$z = a\,x^2 + 2b\,xy + c\,y^2 .$$

Denote $(0, 0, 0)$ by q.

(a) Find out the upward[2] unit normal N.

(b) Since N is smooth and unit-normed, it can be regarded as a smooth map from Γ to S^2. Calculate $DN|_q$.
(Hint: It is not hard to see that $T_q\Gamma$ is the xy-plane. Note that $N(q)$ is $(0, 0, 1)$. Coincidently, $T_{N(q)}S^2$ is also the xy-plane (geometrically, the origin is shifted to $(0, 0, 1)$). Hence, $DN|_q : T_q\Gamma \to T_{N(q)}S^2$ is an endomorphism of $\mathbb{R}^2$. Use the standard basis, and represent your answer as a 2×2 matrix.)

(4) Consider the following coordinate chart for S^2:

$$\begin{aligned}
X^+(u, v) &= \frac{1}{1 + u^2 + v^2}(2u, 2v, 1 - u^2 - v^2) , \\
X^-(u, v) &= \frac{1}{1 + u^2 + v^2}(2u, -2v, 1 - u^2 - v^2) , \\
Y^+(\xi, \eta) &= \frac{1}{1 + \xi^2 + \eta^2}(2\xi, -2\eta, -1 + \xi^2 + \eta^2) , \\
Y^-(\xi, \eta) &= \frac{1}{1 + \xi^2 + \eta^2}(2\xi, 2\eta, -1 + \xi^2 + \eta^2) .
\end{aligned}$$

(a) Check that the Jacobian of $(Y^+)^{-1} \circ X^+$ has positive determinant. And both $\partial_u X^+ \times \partial_v X^+$ and $\partial_\xi Y^+ \times \partial_\eta Y^+$ are in the direction of the outward normal.

(b) Check that the Jacobian of $(Y^+)^{-1} \circ X^-$ has negative determinant. And $\partial_u X^- \times \partial_v X^-$ is in the direction of inward normal.

(5) [from [MR, Example 2.17]] Fix two positive numbers $a > b$. Consider

$$S = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2\} .$$

(a) Check that S is a regular surface.

(b) Check that S is orientable, by writing down a unit normal vector field.

[2] Namely, the z-component is positive.