GEOMETRY: HOMEWORK 1

DUE SEPTEMBER 19

(1) Let $\mathrm{SO}(2) = \{A \in M_{2\times 2}(\mathbb{R}) : A^T A = \mathbf{I}\ , \det A = 1\}$.

(a) Given a smooth function $\kappa(s) : \mathbb{R} \to \mathbb{R}$, find $\Gamma(s) : \mathbb{R} \to \mathrm{SO}(2)$ such that

$$\Gamma(0) = \mathbf{I} \quad \text{and} \quad \Gamma'(s) = \Gamma(s) \begin{bmatrix} 0 & -\kappa(s) \\ \kappa(s) & 0 \end{bmatrix} .$$

(b) Could $\kappa(s)$ in part (a) always be the curvature of some plane curve? Give your reason.

(2) Show that a curve $\gamma : I \to \mathbb{R}^2$ parametrized by arc length is part of a straight line or a circle if all its tangent lines are equi-distant from a given point.

(From (1), a plane curve is part of a straight line or a circle if and only if its curvature is a constant.)

(3) (a) For any two positive numbers a, b, consider the *circular helix*:

$$\gamma(s) = \left(a\,\cos\frac{s}{\sqrt{a^2+b^2}}, a\,\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{b\,s}{\sqrt{a^2+b^2}} \right) .$$

Calculate its curvature and torsion.

(b) A curve $\gamma(s) : I \to \mathbb{R}^3$ parametrized by arc length (with $\gamma''(s) \neq 0$) is called a *general helix* if all its tangent vectors, $T(s)$, make a constant angle with a given direction. Show that γ is a general helix if and only if τ/κ is a constant.

(For the circular helix in part (a), the direction is $(0,0,1)$. For "$\Rightarrow$", you may assume the direction is $(0,0,1)$. For "$\Leftarrow$", think about what the direction shall be in terms of T, N, B.)

(4) Let $\gamma(s) : I \to \mathbb{R}^3$ be a curve parametrized by arc length with $\gamma''(s) \neq 0$. We call the line passing through $\gamma(s)$ with direction $B(s)$ the *binormal line* of γ at s. Suppose that γ lies in the unit sphere and that all its binormal lines are tangent to this sphere. Show that γ is an arc of a great circle.

(A great circle on a sphere is the intersection of the sphere with a plane passing through the center of the sphere.)

(5) Let $\gamma(s) = (x(s), y(s), z(s)) : (-1, 1) \to \mathbb{R}^3$ be a curve parametrized by arc length with $\gamma''(s) \neq 0$. Suppose that

$$\gamma(0) = (0,0,0)\ , \qquad T(0) = (1,0,0)\ , \qquad N(0) = (0,1,0)\ , \qquad B(0) = (0,0,1)\ .$$

Construct the Taylor series expansion of $\gamma(s)$ up to the s^3 terms. You will need $\kappa(0)$, $\tau(0)$ and $\kappa'(0)$.