HOMEWORK 0

GEOMETRY

(1) Suppose that $\gamma(t) = (x(t), y(t)) : (0, \ell) \subset \mathbb{R}^1 \to \mathbb{R}^2$ is a smooth curve with unit speed, i.e.

$$\gamma'(t) = (x'(t), y'(t)) \quad \text{is a unit vector for any } t\ .$$

Define its *curvature* by

$$\kappa(t) = -x''(t)\, y'(t) + y''(t)\, x'(t)\ .$$

Prove that if $\kappa(t)$ is a positive constant, $\gamma(t)$ belongs to a circle.

(2) Denote by $\mathbf{I}$ the identity matrix. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$; regard $\mathbf{v}$ as an $n \times 1$ matrix. Define the matrix

$$G_{\mathbf{v}} = \mathbf{I} + \mathbf{v}\,\mathbf{v}^T\ .$$

Find the expression for the determinant and the inverse of $G_{\mathbf{v}}$.

(3) Let $f(x, y)$ and $g(x, z)$ be two smooth functions. Consider the following sets in $\mathbb{R}^3$:

$$\Gamma_f = \{z = f(x, y)\} \quad \text{and} \quad \Gamma_g = \{y = g(x, z)\}\ .$$

Suppose that $(a, b, c) \in \Gamma_f \cap \Gamma_g$, and $\partial_x f|_{(a,b)} = 0 = \partial_x g|_{(a,c)}$. Give a sufficient condition under which $\Gamma_f \cap \Gamma_g$ is a curve near (a, b, c).

(4) Consider the subsets in $M_{3\times3}(\mathbb{R}) \cong \mathbb{R}^9$ consisting of special orthogonal matrices:

$$\mathrm{SO}(3) = \{A \in M_{3\times3}(\mathbb{R}) : A\,A^T = \mathbf{I} \quad \text{and} \quad \det A = 1\}\ .$$

Prove that it needs "3 parameters" to describe SO(3) near $\mathbf{I}$. (You shall also give a precise statement.)