tangent bundle and smooth map (some terminologies, partially for Whitney's theorem)

§I. tangent vector

tangent vector → derivative of curves: geometric picture

tangent vector → can be used to taking directional derivative: formal property, easier to work with

1° <u>defn</u> M: smooth manifold of dim n, $p_0 \in M$.

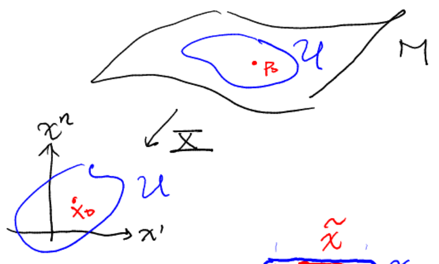
as the surface case

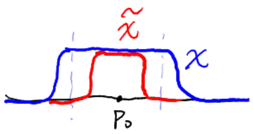
$\delta: C^\infty(M) \to \mathbb{R}$: linear$/\mathbb{R}$ map is called a derivation at p_0

if $\delta(f\cdot g) = (\delta f)\, g(p_0) + f(p_0)\cdot(\delta g)$ Leibniz rule

2° Choose a coordinate chart $\mathcal{U} \xrightarrow{\Phi} \mathbb{R}^n$

$p_0 \longmapsto x_0$





i) $\delta(1) = \delta(1\cdot 1) = 2\,\delta(1) \Rightarrow \delta(1) = 0$

$\Rightarrow \delta(\text{const}) = 0$

ii) δ is a local operator (at p_0)

Choose $\chi \in C^\infty(M) \rightarrow \chi = \begin{cases} 1 & \text{near } p_0 \\ 0 & \text{away from } p_0 \end{cases}$

and $\tilde\chi \in C^\infty(M)$. $\tilde\chi = 1$ near p_0

$\chi = 1$ on $\operatorname{supp}\tilde\chi$

$$\tilde\chi(f-\chi f) \equiv 0$$

$$\Rightarrow 0 = \delta(\tilde\chi(f-\chi f)) = (\delta\tilde\chi)\cdot\overset{0}{\cancel{(f-\chi f)(p_0)}} + \overset{1}{\tilde\chi(p_0)}\cdot\delta(f-\chi f)$$

$$\Rightarrow \delta(f) = \delta(\chi f)$$

iii) By Taylor, $f\circ\Phi^{-1}(x^1,\dots,x^n) = \underbrace{f(p_0)}_{\text{const}} + \sum_{j=1}^{n} x^j \frac{\partial f}{\partial x^j}\Big|_{p_0} + \underbrace{\sum_{k,l} a_{kl}(x)\,x^k x^l}_{\text{quadratic}}$

$$\Rightarrow \delta f = 0 + \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}\Big|_{x_0} \delta(x^j) + 0$$

Hence, a derivation at p_0 is determined on how it acts on x^j's

iv) <u>defn</u> Let $\frac{\partial}{\partial x^i}$ the derivation at p_0 which sends x^j to δ_i^j

$\in$ locally defined smooth function at p_0

<u>Upshot</u> If $\delta(x^1) = a^1, \dots, \delta(x^n) = a^n$

the derivation at p_0, $\delta = \sum_j a^j \frac{\partial}{\partial x^j}$

It sends f to $\sum_j a^j \frac{\partial f}{\partial x^j}\Big|_{x_0}$

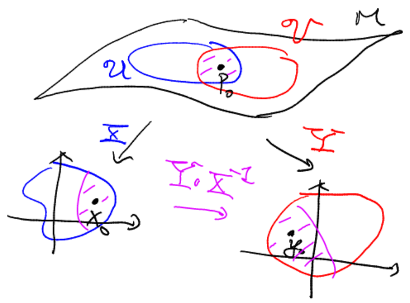
3° <u>Cor</u> / <u>defn</u> All the derivations at p_0 is an n-dim'l vecter space.

Call this space the tangent space of M at p_0, $T_{p_0}M$

A coordinate chart $(\mathcal{U}, \Phi)$ gives a basis $\{\frac{\partial}{\partial x^i}\}$ for $T_{p_0}M$

Change of basis formula between bases? ← chain rule

Say, $(\mathcal{V}, \Psi)$ is another coordinate chart
$\quad\;\; \Psi \quad\; \Psi$
$\quad\;\; p_0 \quad (y^1, \dots, y^n)$



$$\delta = a^i \frac{\partial}{\partial x^i}, \quad \delta f = \sum_i a^i \frac{\partial f(\circ \Phi^{-1})}{\partial x^i}\bigg|_{x_0}$$

$$\begin{pmatrix} f\circ\Phi^{-1} \\ = (f\circ\Psi^{-1})(\Psi\circ\Phi^{-1}) \end{pmatrix} \qquad = \sum_{i,j} a^i \frac{\partial f}{\partial y^j}\bigg|_{y_0} \frac{\partial y^j}{\partial x^i}\bigg|_{x_0}$$

$$= \sum_j \left(\sum_i a^i \frac{\partial y^j}{\partial x^i}\bigg|_{x_0} \right) \frac{\partial f}{\partial y^j}\bigg|_{y_0}$$

There fore. $\quad a^i \frac{\partial}{\partial x^i}\Big|_{x_0} = a^i \frac{\partial y^j}{\partial x^i}\Big|_{x_0} \frac{\partial}{\partial y^j}\Big|_{y_0}$

4° **defn** Let $TM = \coprod_{p\in M} \{\text{all the derivation at } p\} = \coprod_{p\in M} T_pM$

Call it the tangent bundle of M (切叢)

prop • TM is itself a smooth manifold of dim $2n$.

• $\pi: TM \to M$ is smooth ⟶ as before. in terms of coordinate charts
$\quad T_pM \longmapsto p$

rmk For each $p\in M$. $\pi^{-1}(p) = T_pM$ is a vector space.
This is a special example of vector bundle (defined later)

sketch of pf: $(\mathcal{U}, \Phi)$: coordinate chart of M

Denote $\Phi(\mathcal{U}) \subset \mathbb{R}^n$ by $\Omega_{\mathcal{U}}$

Consider $\Omega_{\mathcal{U}} \times \mathbb{R}^n \to \pi^{-1}(\mathcal{U}) \subset TM$

$(x, (a^1 \cdots a^n)) \longmapsto$ the derivation $a^j \frac{\partial}{\partial x^j}$ at $\Phi^{-1}(x)$

transition to $\Omega_{\mathcal{V}} \times \mathbb{R}^n$ is given by
$\qquad\quad (y, (b^1, \dots, b^n))$

$$\begin{cases} y = y(x) \\ b^i = \sum_j a^j \left(\frac{\partial y^i}{\partial x^j}\Big|_x \right) \end{cases}$$

※

defn a (smooth) tangent vector field of M. V is a smooth map from M to TM. such that $\pi\circ V: M \to M$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ identity

Denote the (∞-dim vector) space of smooth vector fields on M by $\Gamma(TM)$

rmk • Equivalently $\Gamma(TM)$ is the space of derivations on M

$$V \in \Gamma(TM) \iff \begin{cases} V: \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \quad \mathbb{R}\text{-linear} \\ \text{and} \quad V(f\cdot g) = V(f)\cdot g + f\cdot V(g) \end{cases}$$

§II. cotangent and differential

1° Given $f \in C^\infty(M)$, we can associate an element $df|_p$ in $(T_pM)^*$ (dual space$/\mathbb{R}$) $\forall p \in M$

by $(df|_p)(V|_p) = V(f)|_p$ (of course, $\mathbb{R}$-linear in V)

In terms of coordinate chart, $(df)(\frac{\partial}{\partial x^i}) = \frac{\partial f}{\partial x^i}(\cdot\, \Xi^{-1})$

If we denote the dual basis of $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ by $\{dx^i\}_{i=1}^n$

then $df = \frac{\partial f}{\partial x^i} dx^i$

at each $p \in \mathcal{U}$, there is a linear map on T_pM and these linear maps are smooth in p

2° <u>defn</u> $(T_pM)^*$ is usually denoted by T_p^*M.

The space $T^*M = \coprod_{p\in M} T_p^*M$ is called the cotangent bundle of M

prop $\begin{cases} T^*M \text{ is a smooth manifold of dim } 2n. \\ \pi: T^*M \to M \text{ is a smooth map.} \\ \forall p \in M,\ \pi^{-1}(p) = T_p^*M \text{ has a structure of } n\text{-dim'l vector space} \end{cases}$

pf: similar as TM: $\Omega_u \times \mathbb{R}^n \longleftrightarrow \Omega_v \times \mathbb{R}^n$

$\xi_j dx^j @ \Xi^{-1}(x) \longleftarrow (x, (\xi_1, \dots, \xi_n))$ $\qquad (y, (\eta_1, \dots, \eta_n))$

$$\left(\eta_j dy^j = \eta_j \frac{\partial y^j}{\partial x^i} dx^i = \xi_i dx^i\right) \rightsquigarrow \begin{cases} y = y(x) \\ \eta_j \frac{\partial y^j}{\partial x^i} = \xi_i \end{cases} \quad ※$$

<u>defn</u> A (smooth) 1-form α is a smooth map from M to T^*M such that $\underbrace{\pi \circ \alpha}_{= \text{identity}} : M \to M$

The space of all 1-forms is denoted by $\Omega^1(M)$ or $\Gamma(T^*M)$

<u>rmk</u> $f \in C^\infty(M) \rightsquigarrow df \in \Omega^1(M)$

3° Suppose that $F: M \to N$ is a smooth map between two smooth manifolds. What is DF, the differential of F?

$M \xrightarrow{F} N \xrightarrow{f} \mathbb{R}^1$

$p_0 \longmapsto q_0$

Suppose that δ is a derivation on M, $F(p_0) = q_0$ and $f \in C^\infty(N)$

We can associate a differential on N at q_0 by $\tilde{\delta}(f) = \delta(f \circ F)$:

$$\tilde{\delta}(f \cdot g) = \delta((f\circ F)\cdot(g \circ F))$$
$$= \underbrace{\delta(f\circ F)}_{= \tilde{\delta}(f)} g(q_0) + f(q_0)\underbrace{\delta(g\circ F)}_{= \tilde{\delta}(g)}$$

<u>Upshot/notation</u> This gives a map, $DF|_p$, from T_pM to $T_{F(p)}N$

One can check that the map is linear $\forall p \in M$

Choose coordinate chart $(\mathcal{U},\Phi)$ on M^n and $(\mathcal{V},\Psi)$ on N^ℓ

$\Rightarrow \delta = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_{x_0}$ $F: y = y(x)$, f: smooth function in y

$$\tilde{\delta}(f) = \delta(f\circ F) = \sum_{i=1}^{n} a^i \frac{\partial f(y(x))}{\partial x^i}\Big|_{x_0} = \sum_{j=1}^{\ell}\Big(\sum_{i=1}^{n} a^i \frac{\partial y^j}{\partial x^i}\Big|_{x_0}\Big)\frac{\partial f}{\partial y^j}\Big|_{y_0}$$

i.e. $\tilde{\delta} = \sum_{j=1}^{\ell}\Big(\sum_{i=1}^{n} a^i \frac{\partial y^j}{\partial x^i}\Big|_{x_0}\Big)\frac{\partial}{\partial y^j}\Big|_{y_0}$

$$\begin{tikzcd}
p_0 \in M \arrow[r, "F"] \arrow[d, "\Phi"] & q_0 \in N \arrow[r, "f"] \arrow[d, "\Psi"] & \mathbb{R}^1 \\
x \in \mathbb{R}^n \arrow[r] & y = y(x) \in \mathbb{R}^\ell \arrow[ur] &
\end{tikzcd}$$

x_0 $\qquad$ y_0

4° example : Hopf fibration $\quad S^3 = \{(z,w)\in\mathbb{C}^2 \mid |z|^2+|w|^2 = 1\}$

$F: S^3 \to S^2$

$(z,w) \mapsto (2\,\mathrm{Re}(\bar{z}w),\ 2\,\mathrm{Im}(\bar{z}w),\ |z|^2-|w|^2)$

i) $|F(z,w)|^2 = 4|z|^2|w|^2 + (|z|^2-|w|^2)^2 = (|z|^2+|w|^2)^2 = 1$

ii) If $F(z,w) = (0,0,-1)$, $\Leftrightarrow \begin{cases}\bar{z}w = 0\\ |z|^2-|w|^2 = -1\end{cases} \Leftrightarrow \begin{matrix}|z| = 0\\ |w| = 1\end{matrix}$

In fact, $\forall q \in S^2$, $F^{-1}(q) \simeq S^1$

iii) $(0,\pi/2)\times(-\pi,\pi)\times(-\pi,\pi) \to S^3$ $\quad$ a coordinate chart

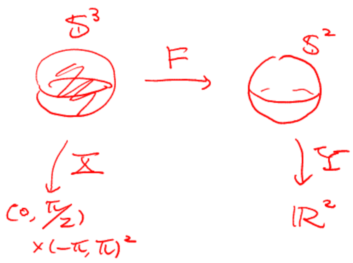
$(\rho,\ \alpha,\ \beta) \mapsto (\sin\rho\, e^{i\alpha},\ \cos\rho\, e^{i\beta})$

$\mathcal{U} = \{z,w \in \mathbb{C}\setminus\{\text{negative real axis}\}\}$

$(\arctan\frac{|z|}{|w|},\ \arg z,\ \arg w) \leftarrow (z,w)$

For S^3, use inverse stereographic projection $\quad (u^1,u^2,u^3) \longrightarrow \left(\frac{u^1}{1+u^3}, \frac{u^2}{1+u^3}\right)$

$(0,0,-1)$ $\not\equiv$



$(\sin\rho\, e^{i\alpha},\ \cos\rho\, e^{i\beta}) \xrightarrow{F} \big(\sin(2\rho)\cos(\beta-\alpha),\ \sin(2\rho)\sin(\beta-\alpha),\ -\cos(2\rho)\big)$

$\downarrow$ inverse stereographic proj.

$(\rho,\alpha,\beta) \xrightarrow{\Psi\circ F\circ\Phi^{-1}} \big(\cos\rho\cos(\beta-\alpha),\ \cos\rho\sin(\beta-\alpha)\big)$

$n=3$, $\ell=2$. $\quad DF$ is local a 2×3 matrix, $\left[\frac{\partial y}{\partial x}\right]$

the Jacobian of the above map

§III. Immersion, embedding and submanifold

1° Defn. $F: M\to N$ is called an immersion (浸入)

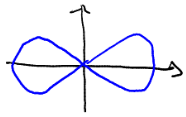
if $DF|_p : T_pM \to T_{F(p)}N$ is injective $\forall p\in M$

- An immersion is called an embedding (嵌入) if M is homeomorphic to $F(M)\subset N$ w.r.t. the subspace topology
- In the embedding case, $F(M)\subset N$ is called an (embedded) submanifold of N

• We also say $S \subset N$ is a submanifold if S is itself a manifold wrt the subspace topology, and $S \hookrightarrow N$ is an embedding

2° By IFT, if $F: M \to N$ is an immersion
then, $\forall p \in M$, $\exists\, \mathcal{U}$: nbd of p
such that $F: \mathcal{U} \to N$ is an embedding

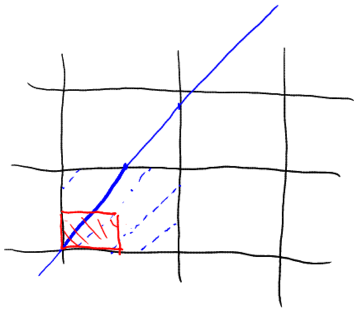
e.g. $S^1 \to \mathbb{R}^2$
$e^{it} \longmapsto (\sin t, \sin t \cos t)$



is an immersion $\boxed{\text{check}}$
But not an embedding
(∵ not injective)

$\boxed{Q}$ injective immersion $\overset{?}{\Leftrightarrow}$ embedding

Nope! e.g. irrational slope in a torus

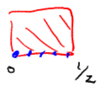


$F: \mathbb{R}^1 \longrightarrow T^2 = S^1 \times S^1$
$t \longmapsto (e^{it}, e^{\sqrt{3}it})$

immersion: direct calculation
injective: $e^{it_1} = e^{it_2}$, $e^{\sqrt{3}it_1} = e^{\sqrt{3}it_1}$

$\Rightarrow t_1 - t_2 \in 2\pi\mathbb{Z}$, $t_1 - t_2 \in \frac{2\pi}{\sqrt{3}}\mathbb{Z} \Rightarrow t_1 = t_2$

embedding ? Suppose yes. $\square$: compact in T^2
$\Rightarrow \square \cap F(\mathbb{R}^1)$ is compact in $F(\mathbb{R}^1)$
But $F^{-1}(\square)$ consists of countably many closed intervals
it cannot be compact in $\mathbb{R}^1$ $\rightarrow\leftarrow$

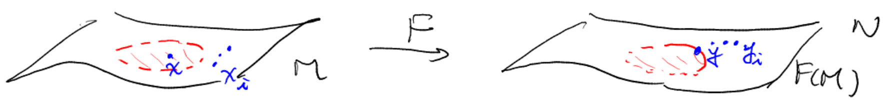
( $e^{\sqrt{3}it} = 1 \Rightarrow t \in \frac{2\pi}{\sqrt{3}}\mathbb{Z}$. also $t \in [0, \pi] + 2\pi\mathbb{Z}$)

3° For $F: M \to N$, if M is homeomorphic to $F(M) \subset N$ by F.
then F is proper, i.e. F^{-1}(compact set) is still compact

Prop if $F: M \to N$, injective immersion and proper.
then, F is an embedding

Pf: homeomorphism: remain to show $F^{-1}: F(M) \to M$ is continuous.
$\Leftrightarrow$ if $\mathcal{U} \subset M$ is open. $F(\mathcal{U})$ is open in $F(M)$

Suppose NOT, $\exists\, \mathcal{U} \subset M$: open. but $F(\mathcal{U})$ is NOT open in $F(M)$



Hence $\exists\, y \in F(\mathcal{U})$ $\rightarrow \forall$ Ball at y. N is locally Euclidean
$\exists\, y_i \in F(M)$, but $y_i \notin F(\mathcal{U})$

$\Rightarrow$ We can choose $\{y_i\} \in F(M \setminus \mathcal{U})$. $y_i \to y$ as $i \to \infty$.

(This happens on a coordinate chart at y $\cong$ open set in $\mathbb{R}^\ell$)

$\Rightarrow$ $\{y_i\} \cup \{y\}$ is compact and in $F(M)$

By injectivity, $\exists!$ $\{x_i\}$ and $x \in M$ $\to$ $F(x_i) = y_i$, $F(x) = y$

By construction, $x_i \notin \mathcal{U}$, $x \in \mathcal{U}$

By proper $\bigcup_{i=1}^{\infty}\{x_i\} \cup \{x\}$ is compact $\Rightarrow$ $\exists$ subsequence $\{x_{i_j}\} \to \tilde{x}$

$\Rightarrow$ $F(x_{i_j}) = y_{i_j} \to F(\tilde{x})$

Since $y_{i_j} \to y = F(x)$ and F is injective, $\tilde{x} = x$

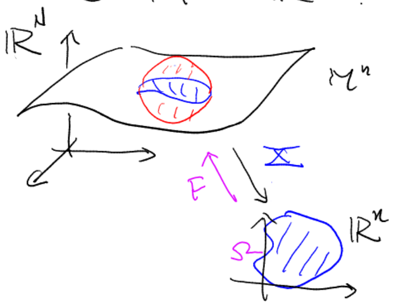
Therefore, $x_{i_j} \in \mathcal{U}$ for $j \gg 1$ $\rightarrow\leftarrow$

Cor $F: M \to N$ injective immersion and M is compact

$\Rightarrow$ F is an embedding

§IV submanifold in $\mathbb{R}^N$

1° $M^n \subset \mathbb{R}^N$, a submanifold



$(\mathcal{U}, X)$: coordinate chart Ω

$\Omega = X(\mathcal{U})$: open set in $\mathbb{R}^n$, $F = X^{-1} : \Omega \to \mathcal{U}$

Since $X \circ F(x^1, \cdots, x^n) = (x^1, \cdots, x^n)$, DF is injective

$n \leftarrow N \leftarrow n$

(as the regular surface case)

$\Rightarrow$ One can check : regular surface $=$ 2-dim'l submanifold in $\mathbb{R}^3$.

2° $TM = \left\{ (p, v) \in \mathbb{R}^N \times \mathbb{R}^N \;\middle|\; p \in M,\ v \in \text{image of } DF \text{ for a coordinate chart at } p \right\}$

as the regular surface case
the image does NOT depend on
the choice of coordinate chart

In this "special" case, $TM = T^*M$

Since for any $v, w \in T_pM$

$T_pM \times T_pM \to \mathbb{R}$

$(v, w) \mapsto \langle v, w \rangle$ is non-degenerate

$\Rightarrow$ It provides an isomorphism $T_pM \to T_p^*M$

(There is NO canonical isomorphism between TM and T^*M.
But any choice of metric, I, would provide one)