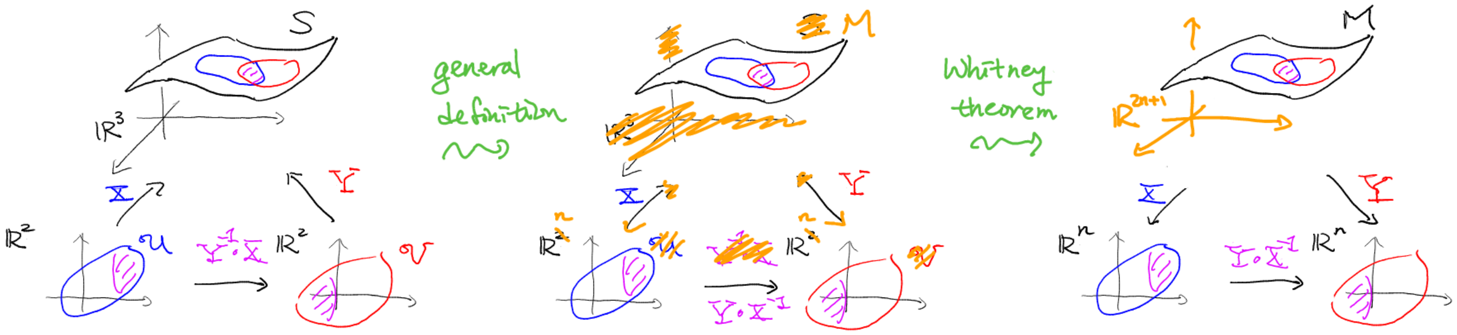
Manifold 流形 [DFN 2, §1~5 of ch.1]

some formal similarities among certain objects / sets in various situations ⤳ then refined as the modern definition



(much more work: improve to 2n)

§I. examples (before definition)

Key property locally Euclidean.

But we will first do the discussion set-theoretically, without worrying continuity and smoothness

1° $n \in \mathbb{N}$, $\{\text{all lines in } \mathbb{R}^{n+1}\} = \mathbb{P}(\mathbb{R}^{n+1}) = \mathbb{R}\mathbb{P}^n$

1-diml vector subspaces projectify n-diml

0) ∀ line, choose a unit vector spanning it : $\mathbb{R}\langle\vec{v}\rangle$ $\begin{cases}\vec{v}\in\mathbb{R}^{n+1}\\ |\vec{v}|=1\end{cases}$

$\mathbb{R}\langle\vec{v}\rangle = \mathbb{R}\langle\vec{w}\rangle \iff \vec{v}=\vec{w}$ or $\vec{v}=-\vec{w}$

Hence. $\mathbb{R}\mathbb{P}^n = S^n/\pm 1$

i) different viewpoint: any vector $(v_0, v_1, \dots, v_n)$ is non zero iff some $v_j \neq 0$

Say $v_0 \neq 0$. $\vec{v}=(v_0, v_1, \dots, v_n)$ spans the same line as $(1, \frac{v_1}{v_0}, \dots, \frac{v_n}{v_0})$

One can see that $\mathbb{R}^n \longrightarrow \mathbb{R}\mathbb{P}^n$

$(x^1, \dots, x^n) \longmapsto$ line spanned by $(1, x^1, \dots, x^n)$ is injective

$[1 : x^1 : \dots : x^n]$

ii) $\mathbb{R}\mathbb{P}^n = \mathbb{R}^{n+1}\setminus\{0\} \Big/ \vec{v}\sim s\vec{v},\ s\in\mathbb{R}\setminus\{0\}$

notation $[\vec{v}] = [v_0 : v_1 : \cdots : v_n] = [1 : \frac{v_1}{v_0} : \cdots : \frac{v_n}{v_0}]$

iii) Similarly we also have $(y^1, \dots, y^n) \longmapsto [y^1 : 1 : y^2 : \cdots : y^n]$

$\{x^1\neq 0\}$ and $\{y^1 \neq 0\}$ have the same image.

Since $[1 : x^1 : \cdots : x^n] = [\frac{1}{x^1} : 1 : \frac{x^2}{x^1} : \cdots : \frac{x^n}{x^1}]$

$\mathbb{x} = (x^1, x^2, \dots, x^n)$ and $\mathbb{y} = (\frac{1}{x^1}, \frac{x^2}{x^1}, \dots, \frac{x^n}{x^1})$ correspond to the same point

2° Grassmannian $Gr(2,4) = \{\text{all 2-planes in } \mathbb{R}^4\}$

i) P : 2-plane. choose a basis $\{u, v\}$

$u = (u^1, u^2, u^3, u^4)$, $v = (v^1, v^2, v^3, v^4)$

linear independent $\iff$ some 2×2 minor of $\begin{bmatrix} u^1 & v^1 \\ u^2 & v^2 \\ u^3 & v^3 \\ u^4 & v^4 \end{bmatrix}$ is invertible ($\det \neq 0$)

⤳ This gives a way to represent a 2-plane

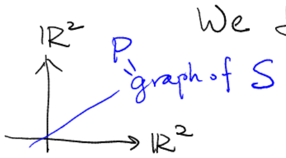
ii) change of basis : $\{au+bv, cu+dv\}$ $ad-bc\neq 0$

$\Rightarrow$ any 2x2 det of $[au+bv \;\; cu+dv]$

(ad-bc) · corresponding det of $[u \; v]$

iii) Suppose that $\begin{vmatrix} u^1 & u^2 \\ v^1 & v^2 \end{vmatrix} \neq 0$.

$$\text{2-plane of } \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} = \text{2-plane of } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix}$$



We find that $\mathbb{R}^4 \to Gr(2,4)$

$(s^1, s^2, s^3, s^4) \mapsto$ 2-plane spanned by $\begin{bmatrix} 1 \\ 0 \\ s^1 \\ s^2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ s^3 \\ s^4 \end{bmatrix}$

is injective

Moreover, $P \in$ image if and only if the orthogonal projection onto the first-two coordinate plane is surjective

iv) Similarly, we have in total $6 = \binom{4}{2}$ such maps.

The union of their image is the whole space, $Gr(2,4)$

eg $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s^1 & s^3 \\ s^2 & s^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t^1 & t^3 \\ 0 & 1 \\ t^2 & t^4 \end{bmatrix}$ $t^3 \neq 0$ $s^3 \neq 0$ (by looking at det)

$$\begin{bmatrix} 1 & 0 \\ s^1 & s^3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{s^1}{s^3} & 1/s^3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s^1 & s^3 \\ s^2 & s^4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{s^1}{s^3} & \frac{1}{s^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{s^1}{s^3} & \frac{1}{s^3} \\ 0 & 1 \\ s^2 - \frac{s^1 s^4}{s^3} & \frac{s^4}{s^3} \end{bmatrix} \Rightarrow \left| \begin{array}{l} t^1 = -\frac{s^1}{s^3} \\ t^2 = s^2 - \frac{s^1 s^4}{s^3} \\ t^3 = \frac{1}{s^3} \\ t^4 = \frac{s^4}{s^3} \end{array} \right.$$

corresponds to the same 2-plane

Upshot $Gr(2,4)$ has 4-dim'l freedom

v) There are 6 2x2-determinats :

- Not all of them are zero.
- For different choices of basis (of P), the determinats differs by a rescaling (ad-bc in ii)

$\begin{bmatrix} u^1 & v^1 \\ u^2 & v^2 \\ u^3 & v^3 \\ u^4 & v^4 \end{bmatrix}$

Namely, $Gr(2,4) \to \mathbb{P}(\mathbb{R}^6) = \mathbb{RP}^5 \ni [w^0 : w^1 : \cdots w^5]$

$P = \operatorname{span}\{u, v\} \mapsto [u^1v^2 - u^2v^1 : u^1v^3 - u^3v^2 : u^1v^4 \cdots : u^2v^3 \cdots : u^2v^4 \cdots : u^3v^4 \cdots]$

4D to 5D

vi) The map is called the Plücker map.

It is injective, and the image is ???

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ s^1 & s^3 \\ s^2 & s^4 \end{bmatrix}$

<u>key computation</u> Consider $u = (1, 0, s^1, s^2)$, $v = (0, 1, s^3, s^4)$

$$\operatorname{span}\{u,v\} \mapsto [\overset{w^0}{1} : \overset{w^1}{s^3} : \overset{w^2}{s^4} : \overset{w^3}{-s^1} : \overset{w^4}{-s^2} : \overset{w^5}{s^1s^4 - s^2s^3}]$$

$\Rightarrow$ image $\subset \{w^0w^5 + w^2w^3 - w^1w^4 = 0\}$

<u>rmk</u> homogeneous equation is well-defined on $\mathbb{RP}^*$

5D cut out by 1 equation $\rightsquigarrow$ 4D

this subset in $\mathbb{RP}^5$ is called the Klein quadric and is the same (set-theoretically) as $Gr(2,4)$

vii) • For a 2-plane in $\mathbb{R}^4$, "nearby 2-planes" looks like $\mathbb{R}^4$

• There is a natural map $Gr(2,4) \to \mathbb{RP}^5$. (no nat'l map to $\mathbb{R}^n$)

3° SU(2) and SO(3)

$$SU(2) = \{ A \in M(2\times 2; \mathbb{C}) \mid A^*A = \mathbb{I},\ \det(A) = 1 \}$$

i) $A^*A = \mathbb{I} \Leftrightarrow$ column vectors are unitary basis for $\mathbb{C}^2$

If 1st column $= \begin{bmatrix} z \\ w \end{bmatrix}$ with $|z|^2 + |w|^2 = 1$

2nd column // $\begin{bmatrix} -\bar{w} \\ \bar{z} \end{bmatrix} \Rightarrow A = \begin{bmatrix} z & -e^{i\theta}\bar{w} \\ w & e^{i\theta}\bar{z} \end{bmatrix}$

But $\det A = 1 \Rightarrow e^{i\theta} = 1 \Rightarrow A = \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix}$

Hence, $SU(2) \simeq S^3$ (again, set-theoretically)

ii) SU(2) is a group: $\begin{bmatrix} z_1 & -\bar{w}_1 \\ w_1 & \bar{z}_1 \end{bmatrix} \begin{bmatrix} z_2 & -\bar{w}_2 \\ w_2 & \bar{z}_2 \end{bmatrix} = \begin{bmatrix} z_1 z_2 - \bar{w}_1 w_2 & \vdots \\ w_1 z_2 + \bar{z}_1 w_2 & \vdots \end{bmatrix}$

$$\begin{bmatrix} z_1 & -\bar{w}_1 \\ w_1 & \bar{z}_1 \end{bmatrix}^* = \begin{bmatrix} \bar{z}_1 & \bar{w}_1 \\ -w_1 & z_1 \end{bmatrix}$$

"looks like" smooth maps.

iii) IFT point of view (sketch)

$\sigma(t)$: a "curve" in SU(2), $\sigma(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P'(0) = ?$

$$\begin{cases} \sigma^*\sigma = \mathbb{I} \rightsquigarrow \frac{d}{dt}\big|_{t=0} \text{ gives } (\sigma'(0))^* + \sigma'(0) = 0 \\ \det(\sigma(t)) = 1 \rightsquigarrow \frac{d}{dt}\big|_{t=0} \text{ gives } \operatorname{tr}(\sigma'(0)) = 0 \end{cases}$$

Write down

$\rightsquigarrow \sigma(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$

$\Rightarrow \det \sigma = ad - bc$

$\Rightarrow (\det \sigma)'(0) = a'(0) + d'(0)$

$\Rightarrow$ "tangent space" of SU(2) at $\mathbb{I}$

is traceless, skew-Hermitian matrices

$$\mathbb{R}^3 \cong P = \{ B + B^* = 0,\ \operatorname{tr}(B) = 0 \} \Leftrightarrow B = \begin{bmatrix} iz & -x+iy \\ x+iy & iz \end{bmatrix}$$

iv) Consider the Adjoint action of $SU(2) \curvearrowright P$

SU(2)
→ GL(3;ℝ)
group
homomorphism

$$SU(2) \times P \to P$$

$(A, B) \mapsto ABA^*$ clearly a linear map on $P \cong \mathbb{R}^3$ ($\forall A$)

$\mathbb{R}^3 \cong P$. the standard inner product of $\mathbb{R}^3$ is $\langle B_1, B_2 \rangle$
$= \frac{1}{2} \operatorname{Re}(\operatorname{tr}(B_2^* B_1))$

$$B_j = \begin{bmatrix} iz_j & -x_j + iy_j \\ x_j + iy_j & iz_j \end{bmatrix}$$

$$B_2^* B_1 = \begin{bmatrix} -iz_2 & x_2 - iy_2 \\ -x_2 - iy_2 & +iz_2 \end{bmatrix} \begin{bmatrix} iz_1 & -x_1 + iy_1 \\ x_1 + iy_1 & -iz_1 \end{bmatrix}$$

$$= \begin{bmatrix} z_1 z_2 + x_1 x_2 + y_1 y_2 + i(\cdots) & (\cdots) \\ (\cdots) & x_1 x_2 + y_1 y_2 + z_1 z_2 + (\cdots) \end{bmatrix}$$

$$\langle AB_1A^*, AB_2A^* \rangle = \frac{1}{2} \operatorname{Re}(\operatorname{tr}(AB_2^*A^*AB_1A^*))$$

$$= \frac{1}{2} \operatorname{Re}(\operatorname{tr}(AB_2^*B_1A^*))$$

$$= \frac{1}{2} \operatorname{Re}(\operatorname{tr}(B_2^*B_1A^*A)) = \frac{1}{2} \operatorname{Re}(\operatorname{tr}(B_2^*B_1)) = \langle B_1, B_2 \rangle$$

$SU(2) \to O(3)$

$\Rightarrow$ The action of $SU(2)$ on $\Gamma \cong \mathbb{R}^3$ preserves the inner product!

But $SU(2) \cong S^3$ is connected (Yes. we use topology here)

$SU(2) \to SO(3)$

Hence. the image belongs to $SO(3)$ $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$

v) Kernel of $SU(2) \to SO(3)$

i.e. $A B A^* = B \quad \forall B \in \Gamma$

$B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \bar{z} & \bar{w} \\ -w & z \end{bmatrix} = \begin{bmatrix} iz & i\bar{w} \\ iw & -i\bar{z} \end{bmatrix} \begin{bmatrix} \bar{z} & \bar{w} \\ -w & z \end{bmatrix}$

$$= \begin{bmatrix} i(|z|^2 - |w|^2) & 2iz\bar{w} \\ 2i\bar{z}w & -i(|z|^2 - |w|^2) \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\Rightarrow |z|^2 - |w|^2 = 1 , \bar{z}w = 0 .$$

But $|z|^2 + |w|^2 = 1 \Rightarrow |z|^2 = 1 , |w|^2 = 0 \Rightarrow A = \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \quad \bar{z} = z^{-1}$

$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{z} & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 0 & -z \\ \bar{z} & 0 \end{bmatrix} \begin{bmatrix} \bar{z} & 0 \\ 0 & z \end{bmatrix}$

$$= \begin{bmatrix} 0 & -z^2 \\ \bar{z}^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow z^2 = 1 = \bar{z}^2 \quad \Rightarrow z = \pm 1$$

Same for $B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ clearly a normal subgroup of $SU(2)$

<u>Hence</u>, only $\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is maps to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In fact. one can check $SU(2) \to SO(3)$ is surjective.

$$\Rightarrow SO(3) \cong SU(2) / \pm 1$$

$$\cong S^3 / \pm 1 = \mathbb{R}P^3$$

<u>point</u>: matrix groups provides many examples of manifolds

§II. manifold (modern definition)

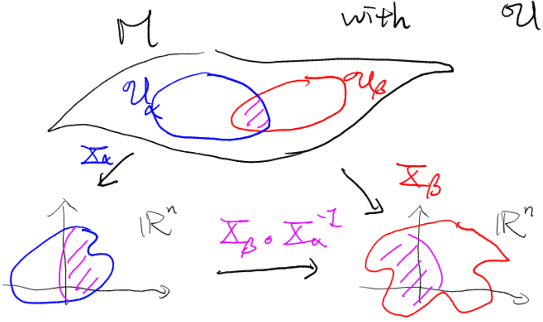
1° <u>defn</u> i) a paracompact, Hausdorff topological space M is called a n-dimensional topological manifold

$\forall p \in M$. $\exists$ open neighborhood $\mathcal{U}$ of p and

$X = (x^1, \cdots, x^n) : \mathcal{U} \subset M \to \mathbb{R}^n$ is homeomorphic onto an open subset in $\mathbb{R}^n$

such a pair is called a coordinate chart. (direction changing)

ii) a topological manifold M is called a smooth manifold if for any two coordinate charts $(\mathcal{U}_\alpha, X_\alpha)$. $(\mathcal{U}_\beta, X_\beta)$ with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$



$X_\beta \circ X_\alpha^{-1} : X_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to X_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

is a diffeomorphic (between open subsets of $\mathbb{R}^n$)

rmk manifold with boundary will be defined later; does NOT satisfy the definition above

2° digression on the dimension: Suppose M is connected.

- For smooth ones, dimension is well-defined due to the IFT and chain rule ⋯
- For topological ones, it needs Brouwer's invariance of domain

thm $F: \underset{\text{open}}{U \subset} \mathbb{R}^n \to \mathbb{R}^n$ injective and continuous then, $V = F(U)$ is open in $\mathbb{R}^n$ and $F: U \to V$ is a homeomorphism

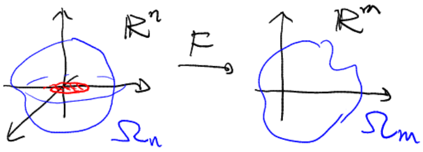
[pf: needs Brouwer fixed point, which needs singular homology]

idea $m<n$
if $B^m \overset{\text{homeo}}{\cong} B^n$
$B^m \to B^m \times \{0\} \subset B^n$
$\downarrow \wr$ homeo
B^m

something wrong here

cor For $n \neq m$, open subsets in $\mathbb{R}^n$ and $\mathbb{R}^m$ are never homeomorphic to each other

pf: Say $m<n$, $F: \Omega_n \to \Omega_m$ homeomorphism



B^m: m-dim'l ball
$\varphi \downarrow$
Ω_n

Construct an injective, continuous map, whose image is NOT open in Ω_n (or $\mathbb{R}^n$)

Then, $F \circ \varphi : B^m \to \mathbb{R}^m$ is injective and continuous, but image is not open $\rightarrow\!\leftarrow$

3° three major ways to construct/describe topology on a set M

i: base) defn a collection of subset $\mathcal{B} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ is called a base if

- $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M$
- $\forall\ U_\alpha, U_\beta$ with $U_\alpha \cap U_\beta \neq \phi$, $\exists\ \{U_\gamma\}_{\gamma \in P}$: subcollection such that $\bigcup_{\gamma \in P} U_\gamma = U_\alpha \cap U_\beta$

⇒ define open sets to be any union of (finite intersection of members of $\mathcal{B}$)

e.g. $M = \mathbb{R}^n$. $\mathcal{B}_1 = \{\text{(open) balls}\}$, $\mathcal{B}_2 = \{\text{(open) cubes}\}$ ⇝ generate the same topology

e.g. product topology. $M \times N$. $\mathcal{B}_{M\times N} = \{U_\alpha \times V_\beta\}$
open set in M
open set in N

ii: subspace topology) $M \subset N$ ⇜ topological space
open sets in M = (open set in N) $\cap M$

iii: quotient topology) $M = W/\sim$ some equivalence relation
↳ will NOT talk too much during this class

next week

4° Grassmannian $Gr(k,n) = \{$ k-planes in $\mathbb{R}^n \}$ is a smooth manifold

i) We use $\binom{n}{k}$ coordinate charts as follows.

$\mathbb{R}^n \ni (w^1, \cdots, w^n)$ Choose k of them. Denote that $\mathbb{R}^k$ by Γ

For simplicity, suppose that $\Gamma = x^1, \cdots, x^k$ -plane

let $\mathcal{U}_\Gamma = \{ P \in Gr(k,n) \mid$ orthogonal proj (P) is onto $\Gamma \}$

$$P \in \mathcal{U}_\Gamma \Leftrightarrow P = \text{span} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \\ & & * & \end{bmatrix} \text{(column vectors)} \quad \begin{bmatrix} I_k \\ \boxed{(n-k)\times k} \end{bmatrix}$$

reduced column echelon form (?)

For any $\times \in M((n-k)\times k; \mathbb{R}) \cong \mathbb{R}^{(n-k)k}$, $\xmapsto{\Phi_\Gamma^{-1}}$ span $\begin{bmatrix} I \\ \times \end{bmatrix}$ is bijective to $\mathcal{U}_\Gamma$

(by linear algebra)

base for topology $= \{ \Phi_\Gamma^{-1}(\Omega) \mid \Omega$: open in $\mathbb{R}^{(n-k)k}$.

Γ: coordinate k-plane$\}$

$\boxed{\text{check}}$ it indeed is a base,

(w.r.t. $Gr(k,n)$ is Hausdorff and compact (hence paracompact))

ii) transition is smooth. (by linear algebra and calculus)

e.g.

$$\text{span} \begin{array}{c|c} & \begin{bmatrix} \begin{array}{c|c} \boxed{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}} & \boxed{0} \\ \hline 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline \boxed{X_1} & \boxed{X_2} \\ \hline X_3 & X_4 \end{array} \end{bmatrix} \end{array} = \text{span} \begin{bmatrix} \begin{array}{c|c} \boxed{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}} & \boxed{0} \\ \hline \boxed{y_1} & \boxed{y_2} \\ \hline 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline y_3 & y_4 \end{array} \end{bmatrix}$$

$k-2$, 2, 2, $n-2-k$; $k-2$, 2

column operation

intersection: $\det(X_2) \neq 0$ $\qquad$ $\det(y_2) \neq 0$

by inverting

$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

$$\begin{bmatrix} \begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 \\ \hline 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline X_1 & X_2 \\ \hline X_3 & X_4 \end{array} \end{bmatrix} \begin{bmatrix} \begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 \\ \hline -X_2^{-1}x_1 & X_2^{-1} \end{array} \end{bmatrix} = \begin{bmatrix} \begin{array}{c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 \\ \hline -X_2^{-1}X_1 & X_2^{-1} \\ \hline 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline X_3 - X_4X_2^{-1}X_1 & X_4X_2^{-1} \end{array} \end{bmatrix}$$

Hence
$$\begin{cases} y_1 = -X_2^{-1}x_1 \\ y_2 = x_2^{-1} \\ y_3 = X_3 - X_4X_2^{-1}X_1 \\ y_4 = X_4X_2^{-1} \end{cases}$$

Therefore, $Gr(k,n)$ is a smooth manifold of dimension $(n-k)\times k$