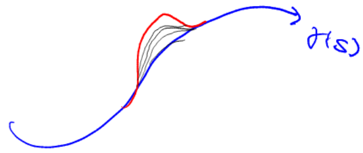
geodesic 測地線

§I. Euler – Lagrange equation

general setting in the calculus of variation

$\mathcal{L}$: function on $\mathbb{R}^n \times \mathbb{R}^n \ni (u^1, \dots, u^n, v^1, \dots, v^n)$
position velocity



For a curve in $\mathbb{R}^n$, $\gamma(s) = (u^1(s), \dots, u^n(s))$

associate $E[\gamma] = \int \mathcal{L}(\gamma(s), \gamma'(s))\, ds$

What is the "critical state" of E?

Namely. $\forall\, (a^1(s), \dots, a^n(s))$: compact support. $\left.\frac{d}{dt}\right|_{t=0} E[\gamma + t a] = 0$

$$0 = \left.\frac{d}{dt}\right|_{t=0} \int \mathcal{L}(\gamma + ta, \gamma' + t a')\, ds$$

$$= \int \frac{\partial \mathcal{L}}{\partial u^k}(\gamma(s), \gamma'(s))\, a^k(s) + \frac{\partial \mathcal{L}}{\partial v^k}(\gamma(s), \gamma'(s))\, (a^k)'(s)\, ds$$

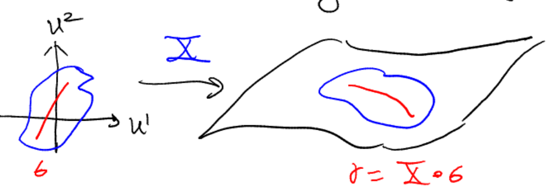
$$= \int \left(\frac{\partial \mathcal{L}}{\partial u^k} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial v^k} \right) \right) a^k\, ds$$

True for any $\{a^i\}$ $\Rightarrow$ $\dfrac{\partial \mathcal{L}}{\partial u^k} - \dfrac{d}{ds} \dfrac{\partial \mathcal{L}}{\partial v^k} = 0$ ——— (EL)

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial v^k}(\gamma(s), \gamma'(s)) \right) = \frac{\partial^2 \mathcal{L}}{\partial v^k \partial u^j} u^j_{'} + \frac{\partial^2 \mathcal{L}}{\partial v^k \partial v^j} u^j_{''}$$

<u>upshot</u> a critical state must be a solution to the Euler-Lagrange equation.

§II. geodesic



On a regular surface S. work out the Euler – Lagrange equation for the distance functional

$$I = g_{ij}(u)\, du^i \cdot du^j$$

$$\text{Arc-length} = \int \left(g_{ij}(u(s)) \frac{du^i}{ds} \frac{du^j}{ds} \right)^{\frac{1}{2}} ds$$

That is to say. $\mathcal{L}(u, v) = \left(g_{ij}(u)\, v^i v^j \right)^{\frac{1}{2}}$

For convenience. we assume the critical state is parametrized by arc-length, $g_{ij}(u(s)) \dfrac{du^i}{ds} \dfrac{du^j}{ds} = \underline{\text{Const.}} > 0$

$$\frac{\partial \mathcal{L}}{\partial u^k} = \frac{1}{2} \square^{-\frac{1}{2}} \frac{\partial g_{ij}}{\partial u^k} v^i v^j$$

$$\rightsquigarrow \frac{\partial \mathcal{L}}{\partial u^k}(\gamma(s), \gamma'(s)) = \frac{1}{2} \square^{-\frac{1}{2}} \frac{\partial g_{ij}}{\partial u^k}(u(s)) \frac{du^i}{ds} \frac{du^j}{ds}$$

$$\frac{\partial \mathcal{L}}{\partial v^k} = \frac{1}{2} \square^{-\frac{1}{2}} \left(g_{kj} v^j + g_{ik} v^i \right) = \square^{-\frac{1}{2}} g_{ik} v^i$$

$$\rightsquigarrow \frac{\partial \mathcal{L}}{\partial v^k}(\gamma(s), \gamma'(s)) = \frac{1}{2} \square^{-\frac{1}{2}} \left(g_{kj}(u(s)) \frac{du^j}{ds} + g_{ik}(u(s)) \frac{du^i}{ds} \right)$$

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial v^k} \right) = \frac{1}{2} \square^{-\frac{1}{2}} \left(\frac{\partial g_{kj}}{\partial u^i}(u(s)) \frac{du^i}{ds} \frac{du^j}{ds} + \frac{\partial g_{ik}}{\partial u^j}(u(s)) \frac{du^j}{ds} \frac{du^i}{ds} + 2 g_{kj}(u(s)) \frac{d^2 u^j}{ds^2} \right)$$

$\square$ is assumed to be constant along $\gamma(s)$.

$$\Rightarrow \quad g_{kj}\frac{d^2u^j}{ds^2} + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial u^i}\frac{du^i}{ds}\frac{du^j}{ds} + \frac{\partial g_{ik}}{\partial u^j}\frac{du^j}{ds}\frac{du^i}{ds} - \frac{\partial g_{ij}}{\partial u^k}\frac{du^i}{ds}\frac{du^j}{ds}\right) = 0$$

Only k is not summed. two equations $k=1,2$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} \frac{d^2(u^1)}{ds^2} + \cdots \\ \frac{d^2(u^2)}{ds^2} + \cdots \end{bmatrix} = 0$$

positive definite $\Rightarrow$ invertible

symmetric also symmetric

multiply the inverse of $[g_{ij}]$: $[g^{-1}]_{k\ell} = [g^{k\ell}]$ (notation

$$g^{\ell k}\Big(\underbrace{g_{kj}}_{\sum_k \to \delta^\ell_j}\underbrace{\frac{d^2u^j}{ds^2}}_{\sum_j \to \frac{d^2u^\ell}{ds^2}} + \cdots \Big) = 0$$

$$\Rightarrow \quad \frac{d^2u^\ell}{ds^2} + \frac{1}{2}g^{\ell k}\left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right)\frac{du^i}{ds}\frac{du^j}{ds} = 0$$

rmk.
$$\sum_{i,j}\frac{\partial g_{kj}}{\partial u^i}\frac{\partial u^i}{\partial s}\frac{\partial u^j}{\partial s} = \sum_{i,j}\frac{\partial g_{ik}}{\partial u^j}\frac{\partial u^j}{\partial s}\frac{\partial u^i}{\partial s}$$
For $\Gamma^\ell_{ij} = \Gamma^\ell_{ji}$
we do NOT simplify

<u>defn</u> Denote $\frac{1}{2}\sum_\ell g^{\ell k}\left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right)$ by Γ^ℓ_{ij}

They are called the Christoffel symbols of the 1st fundamental form

The geodesic equation is the following system

$$\frac{d^2u^\ell}{ds^2} + \sum_{i,j}\Gamma^\ell_{ij}\frac{du^i}{ds}\frac{du^j}{ds} = 0 \quad \text{for } \ell = 1,2. \qquad \text{——} \quad (*)$$

<u>Cor</u> i) By ODE, given initial position & velocity

$\rightsquigarrow$ locally $\exists!$ geodesic

ii) sol'n to $(*)$ must be parametrized by arc-length

Pf:
$$\frac{d}{ds}(g_{ij}\dot u^i \dot u^j) = \frac{\partial g_{ij}}{\partial u^k}\dot u^k \dot u^i \dot u^j + g_{ij}\ddot u^i \dot u^j + g_{ij}\dot u^i \ddot u^j$$
$$= \frac{\partial g_{ij}}{\partial u^k}\dot u^k \dot u^i \dot u^j - g_{ij}\Gamma^i_{k\ell}\dot u^k \dot u^\ell \dot u^j - g_{ij}\Gamma^j_{k\ell}\dot u^i \dot u^k \dot u^\ell$$

$$g_{ij}\Gamma^i_{k\ell} = \frac{1}{2}\overbrace{g_{ij}g^{im}}^{\delta^m_j}\left(\frac{\partial g_{km}}{\partial u^\ell} + \frac{\partial g_{\ell m}}{\partial u^k} - \frac{\partial g_{k\ell}}{\partial u^m}\right)$$
$$= \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial u^\ell} + \frac{\partial g_{\ell j}}{\partial u^k} - \frac{\partial g_{k\ell}}{\partial u^j}\right)$$

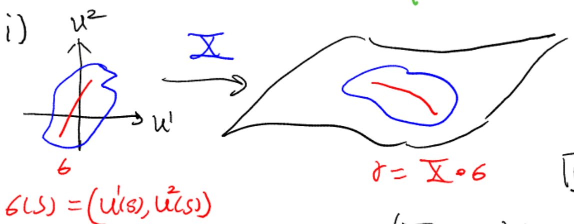
$$= \frac{\partial g_{ij}}{\partial u^k}\dot u^k \dot u^i \dot u^j - \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial u^\ell} + \frac{\partial g_{\ell j}}{\partial u^k} - \frac{\partial g_{k\ell}}{\partial u^j}\right)\dot u^k \dot u^\ell \dot u^j$$
$$- \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^\ell} + \frac{\partial g_{\ell i}}{\partial u^k} - \frac{\partial g_{k\ell}}{\partial u^i}\right)\dot u^i \dot u^k \dot u^\ell$$

$\leftarrow$ after re-naming indices
all terms are the same
$1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = 0$

$= 0$ #

§III. curvature of space curve ?

i)



<u>recall</u> $\kappa = |\gamma''(s)|$

$\kappa \equiv 0 \Rightarrow$ straight line.

$\boxed{Q}$ relation between $(*)$ and $\gamma''(s)$?

$$\gamma(s) = (\mathbf{X}\circ\sigma)(s)$$
$$\Rightarrow \quad \dot\gamma = \frac{\partial \mathbf{X}}{\partial u^1}\dot u^1 + \frac{\partial \mathbf{X}}{\partial u^2}\dot u^2 = \frac{\partial \mathbf{X}}{\partial u^j}\dot u^j$$

$$\Rightarrow \ddot{\gamma} = \frac{\partial X}{\partial u^j}\ddot{u}^j + \frac{\partial^2 X}{\partial u^i \partial u^j}\dot{u}^i \dot{u}^j \qquad \text{some similarity to } (*)$$

What are $\frac{\partial^2 X}{\partial u^i \partial u^j}$? In any case, we can express it as a linear combination of $\{\underbrace{\frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}}_{\text{tangent}}, \underbrace{N}_{\text{normal}}\}$

ii) $\langle \partial_i \partial_j X, N \rangle = h_{ij}$

$$\langle \partial_i \partial_j X, \partial_k X \rangle = \partial_i(\langle \partial_j X, \partial_k X\rangle) - \langle \partial_j X, \partial_i \partial_k X\rangle$$

but we can also do))

$$\langle \partial_j \partial_i X, \partial_k X \rangle = \partial_j \langle \partial_i X, \partial_k X\rangle) - \langle \partial_i X, \partial_j \partial_k X\rangle$$

$$\Rightarrow 2\langle \partial_i \partial_j X, \partial_k X\rangle = \partial_i g_{jk} + \partial_j g_{ik} - (\underbrace{\langle \partial_j X, \partial_i \partial_k X\rangle + \langle \partial_i X, \partial_j \partial_k X\rangle}_{\partial_k(\langle \partial_i X, \partial_j X\rangle)})$$

tangential component

$$= \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}$$

$$(\partial_i \partial_j X)^T = P^m_{ij}\partial_m X \Rightarrow 2 g_{mk} P^m_{ij} = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}$$

$$\Rightarrow \underbrace{g^{kl} g_{mk}}_{\delta^l_m} P^m_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

$$\Rightarrow P^l_{ij} = \Gamma^l_{ij}$$

Hence $(\partial_j \partial_i X)^T = \Gamma^k_{ij}\partial_k X$

iii) Sum up

$$\ddot{\gamma} = \ddot{u}^k \partial_k X + \dot{u}^i \dot{u}^j \partial_i \partial_j X$$
$$= (\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i \dot{u}^j)\partial_k X + (h_{ij}\dot{u}^i \dot{u}^j) N$$

prop γ = geodesic $\Leftrightarrow$ $\ddot{\gamma}$ has no tangential component

cor great circles on round sphere are geodesics ($\ddot{\gamma} \parallel \gamma = N$)

§IV. Gauss equation

i) We have $\partial_i \partial_j X = \Gamma^k_{ij}\partial_k X + h_{ij} N$

3-equations. $(i,j) := (1,1)$, $(1,2) = (2,1)$, $(2,2)$

What happens if we take one more partial derivative?

ii) recall. $\partial_i N = a_i^j \partial_j X = -h_{ik} g^{kj}\partial_j X$

$$\Rightarrow \partial_l \partial_i \partial_j X = (\partial_l \Gamma^k_{ij})\partial_k X + \Gamma^k_{ij}\partial_l \partial_k X + (\partial_l h_{ij}) N + h_{ij}\partial_l N$$
$$= (\partial_l \Gamma^k_{ij})\partial_k X + \Gamma^{\cancel{k}\,m}_{ij}\Gamma^{\cancel{m}\,k}_{l\cancel{k}\,m}\partial_{\cancel{m}\,k} X + \Gamma^k_{ij} h_{kl} N - h_{ij} h_{lm} g^{mk}\partial_k X + (\partial_l h_{ij}) N$$

$$\Rightarrow (\partial_l \partial_i \partial_j X)^T = (\partial_l \Gamma^k_{ij} + \Gamma^m_{ij}\Gamma^k_{lm} - h_{ij} h_{lm} g^{mk})\partial_k X$$

iii) But $\partial_2 \partial_1 \partial_1 X = \partial_1 \partial_2 \partial_1 X$

$l=2 \quad i=j=1 \qquad i=2,\ l=j=1$

$$\Rightarrow (\partial\Gamma, \Gamma)^k - h_{11}h_{2m}g^{mk} = (\partial\Gamma, \Gamma)^k - h_{12}h_{1m}g^{mk}$$ $\partial_k X$ coefficients

$$\Rightarrow (h_{11}h_{2m} - h_{12}h_{1m})\, g^{mk} = \text{some function in } \partial\Gamma \text{ and } \Gamma$$

For $k=2$

$$\begin{aligned} m=1 \quad & (h_{11}h_{21} - h_{12}h_{11})\, g^{12} \\ m=2 \quad & + (h_{11}h_{22} - h_{12}h_{12})\, g^{22} = (\partial\Gamma, \Gamma) \end{aligned}$$

the Gauss equation.

But $g^{22} > 0 \Rightarrow \det([h_{ij}])$ can be computed from $(\partial^2 g, \partial g, g)$

<u>Cor</u> Gauss's Theorema Egregium: K is determined by the 1st fundamental form