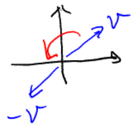
Poincaré–Hopf and Gauss–Bonnet

§I. applications of degree

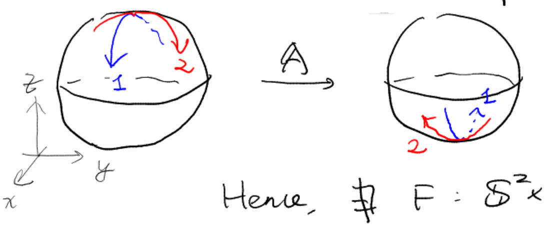
1° Brouwer's hairy ball theorem: Any "tangent" vector field on S^2 must have zero.

pf: i) In 1D. v cannot be deformed to $-v$ without passing 0.

In 2D.  it is okay by rotation.

ii) S^2, outer normal (position) $\mathbb{I}(p)=p$ has degree 1

The anti-podal map $A(p)=-p$ has degree -1



the orientation is reversed.

(or. $\delta_A = -1$)

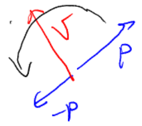
Hence, $\nexists\ F: S^2 \times [0,1] \to S^2$

such that $F(p,0)=p$. $F(p,1)=-p$.

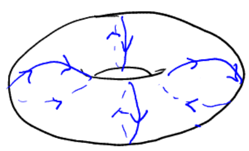
iii) Suppose $\exists\ V$: tangent vector field on S^2. with $V(p) \neq 0\ \forall p$

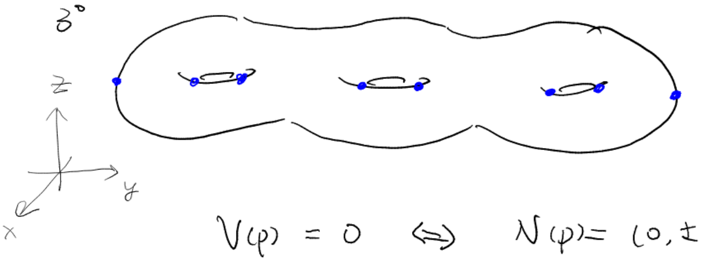
$(V(p) \perp N(p)=p \quad \forall p)$

Then, consider $F(p,t) = \cos(\pi t)\, p + \sin(\pi t)\, V(p)$ / itself



$\rightarrow\leftarrow$

2° on torus?  or  okay!

3° 

Say $(0,\pm 1,0)$ is a regular value of the Gauss map N

Consider $V(p)$ = orthogonal projection of $(0,1,0)$ onto T_pS

$V(p)=0 \iff N(p)=(0,\pm1,0)$

What does V looks like near its zeros?

Local model. (rotate it)

$z=\frac{1}{2}(x^2+y^2)$ $\quad N=\dfrac{(-x,-y,1)}{\sqrt{1+x^2+y^2}}$

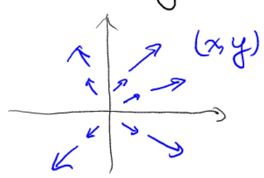
orthogonal projection of $(0,0,1)$

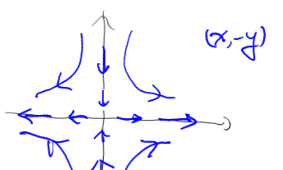
$=(x,\ y,\ x^2+y^2)/(1+x^2+y^2)$

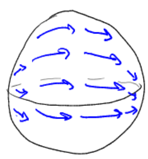
$z=\frac{1}{2}(x^2-y^2)$ $\quad N=\dfrac{(-x,y,1)}{\sqrt{1+x^2+y^2}}$

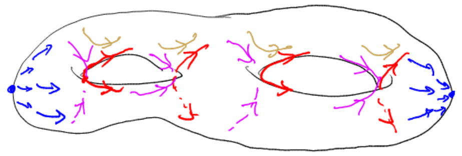
$\rightsquigarrow (x,\ -y,\ x^2+y^2)/(1+x^2+y^2)$

Since tangent is close to the xy-plane. focus on the first two components







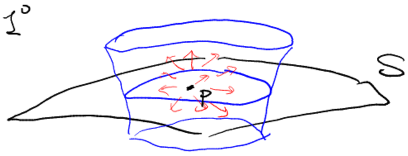


$\boxed{Q}$ any constraint on the zeros of tangent vector field?

§II. index of tangent vector field

$\boxed{\text{goal}}$ for each zeros of V, assign an integer to describe its non-triviality, and show their sum is constraint by the global shape (topology) of S

1°



$V(p_0) = 0$. $\mathcal{U}$: small open nbd of p_0 in S

$\mathcal{U} \times (-\delta, \delta) \qquad \mathbb{R}^3$

$(p, t) \longmapsto p + t N(p)$ injection

image Ω : open nbd of p_0 in $\mathbb{R}^3$

We can naturally associate a vector field $\tilde{V}$ on Ω

by $\tilde{V}(p + tN) = V(p) + t N(p)$

Since $V(p) \perp N(p)$, $V(p) = 0$ only at p_0, $|N(p)| = 1$

$\tilde{V} = 0$ only at $p + 0 \cdot N = p$

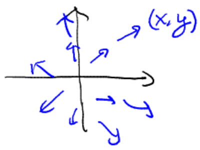
<u>defn</u> The index of V at p is defined to be

$$i(V; p_0) = \deg_{S^2(p_0; \varepsilon)} \left(\tilde{V} / |\tilde{V}| \right) \quad \text{for } \varepsilon << \delta$$

2° <u>examples</u> on the xy-plane with $N = (0, 0, 1)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $x_y + tN$: add third component $= z$

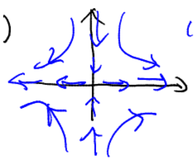
i)



$\rightsquigarrow (x, y, z) \rightsquigarrow$ index $= +1$

Similarly $(-x, -y) \rightsquigarrow (-x, -y, z)$, index $= +1$

ii)



$\rightsquigarrow (x, -y, z)$ only $(0,0,1)$ is sent to $(0,0,1)$

but orientation is reversed. $\Rightarrow$ index $= -1$

iii) In general $\vec{F} = (P(x,y), Q(x,y), z)$, we may count the sign of the pre-image of $(1,0,0)$.

$T_{(1,0,0)} S^2 = \{(0,1,0), (0,0,1)\}$ $\quad$ but $\frac{\partial F}{\partial z} = (0,0,1)$

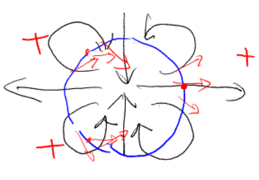
$(\sqrt{1-y^2-z^2}, y, z)$

It remains to see on the xy-plane, how (P, Q) changes near a point with value $(1,0)$

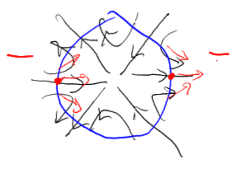
At $(x_0, y_0, 0)$, <u>oriented</u> basis $= \{(-y_0, x_0, 0), (0,0,1)\}$

$\hookrightarrow X = (x_0, y_0, 0)$: outer normal

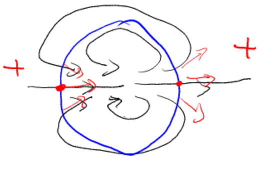
counter clockwise



index $= 3$



index $= -2$



index $= +2$

$z \mapsto z^n$

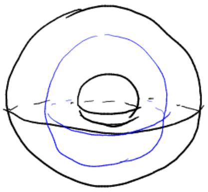
$(\mathrm{Re}(x+iy)^n, \mathrm{Im}(x+iy)^n)$

has index n at $(0,0)$

go around the blue circle counterclockwise near $\rightarrow$, if it goes from $\begin{cases} \mathrm{IV} \text{ to } \mathrm{I} \rightsquigarrow +1 \\ \mathrm{I} \text{ to } \mathrm{IV} \rightsquigarrow -1 \end{cases}$

§III. further discussion on degree

1° $\Omega = \{(x,y,z) \mid a^2 \le x^2+y^2+z^2 \le b^2\} = \coprod_{r\in[a,b]} S_r$



Suppose that there is a smooth map $F: \Omega \to S^2$

With outer normal, we can calculate $\deg(F|_{S_r})$ for each $r \in [a,b]$

Again, continuity + integer value, $\deg(F|_{S_r})$ is independent of r

2° recall divergence theorem. Suppose that V is a vector field on Ω

$$\iiint_{\Omega} \operatorname{div} V \, dx\,dy\,dz = \iint_{\partial\Omega} V \cdot \underbrace{d\vec{S}}_{\vec{N}\, dA}$$

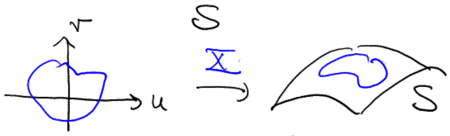
$\vec{N}$: outer w.r.t. Ω

goal There is a divergence theorem argument for 1°

3° Key associate a suitable V of $F: \Omega \to S^2 \subset \mathbb{R}^3$

$$\deg(F) = \iint_S k_F \, dA \,/\, 4\pi$$

when $\langle X_u \times X_v, N \rangle > 0$



$$k_F = \frac{\langle F_u \times F_v, F \rangle}{|X_u \times X_v|} \qquad F_u = \frac{\partial}{\partial u}(F \circ X)$$

Now, F is defined not only on a surface, but on an open subset of $\mathbb{R}^3$.

chain rule $\Rightarrow \begin{cases} \dfrac{\partial F}{\partial u} = \dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial u} \\ \dfrac{\partial F}{\partial v} = \dfrac{\partial F}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial v} \end{cases}$ components of $\dfrac{\partial X}{\partial u}$

$$\langle F_u \times F_v, F \rangle = \det\begin{bmatrix} | & | & | \\ F_u & F_v & F \\ | & | & | \end{bmatrix} = \det\begin{bmatrix} x_u F_x & x_v F_x & \\ +y_u F_y & +y_v F_y & F \\ +z_u F_z & +z_v F_z & \end{bmatrix}$$

$$= x_u y_v \det[F_x \ F_y \ F] + x_v y_u \det[F_y \ F_x \ F] + \cdots$$

$$= \frac{\partial(x,y)}{\partial(u,v)} \det[F_x \ F_y \ F] + \frac{\partial(y,z)}{\partial(u,v)} \det[F_y \ F_z \ F] + \frac{\partial(z,x)}{\partial(u,v)} \det[F_z \ F_x \ F]$$

$$d\vec{S} = \frac{X_u \times X_v}{|X_u \times X_v|} |X_u \times X_v| \, du\,dv = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv$$

It follows that $\iint_S k_F \, dA$ is the flux integral of the vector field

$$V = \left(\det[F_y \ F_z \ F], \ \det[F_z \ F_x \ F], \ \det[F_x \ F_y \ F] \right)$$

4° $\operatorname{div} V = \partial_x(|F_y \ F_z \ F|) + \partial_y(|F_z \ F_x \ F|) + \partial_z(|F_x \ F_y \ F|)$

$$= \cancel{|F_{yx} \ F_z \ F|} + \cancel{|F_y \ F_{zx} \ F|} + |F_y \ F_z \ F_x|$$
$$+ \cancel{|F_{zy} \ F_x \ F|} + \cancel{|F_z \ F_{xy} \ F|} + |F_z \ F_x \ F_y|$$
$$+ \cancel{|F_{xz} \ F_y \ F|} + \cancel{|F_x \ F_{yz} \ F|} + |F_x \ F_y \ F_z|$$
$$= 3 \det[F_x \ F_y \ F_z]$$

But $|F|^2 = 1 \Rightarrow F_x, F_y, F_z \perp F \Rightarrow \det[F_x \ F_y \ F_z] = 0 \Rightarrow \operatorname{div} V = 0$

prop $\quad \Omega \subset \mathbb{R}^3$, open. whose boundary consists of compact regular surfaces
$F : \Omega \to S^2$, smooth map. Then. $\deg_{\partial\Omega}(F) = 0$

(endow each component of $\partial\Omega$ the "outer" normal)

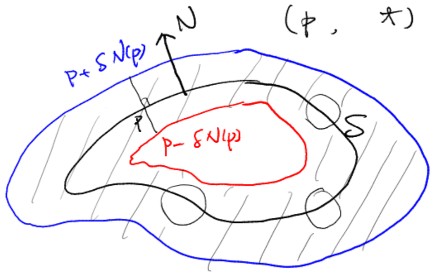
$\rightsquigarrow$ gives another proof of I°

§IV. Poincaré – Hopf theorem.

thm $\quad S$: compact regular surface with orientation N
V : tangent vector field on S. with isolated zeros.

Then, $\displaystyle\sum_{q \in V^{-1}(0)} i(V, q) = \frac{1}{2\pi} \iint_S K \, dA \quad (= 2(1-g))$

pf: Use the previous homework. Work on the image of
$S \times (-\delta, \delta) \to \mathbb{R}^3$
$(p, t) \longmapsto p + t N(p)$ $\quad$ Denote the image by Ω



i) Define $\tilde{V}$: a vector field on Ω by
$\tilde{V}(p + tN(p)) = V(p) + tN(p)$

Since $V \perp N$, $|N| = 1$, $\tilde{V}(q + tN(q)) = 0$
if and only if $V(q) = 0$, $t = 0$

ii) Apply the prop for $\tilde{V}/|\tilde{V}|$ on $\Omega \setminus \bigcup_{q \in \tilde{V}^{-1}(0)} B(q;\varepsilon)$

$\partial\left(\Omega \setminus \bigcup_{q \in \tilde{V}^{-1}(0)} B(q;\varepsilon)\right)$ consists of $S + \delta N$, $S - \delta N$, $S^2(q;\varepsilon)$

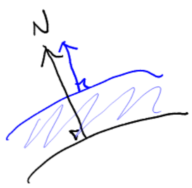
$$\deg_{S+\delta N} \tilde{V}/|\tilde{V}| + \deg_{S-\delta N} \tilde{V}/|\tilde{V}| + \sum_{q \in \tilde{V}^{-1}(0)} \deg_{S^2(q;\varepsilon)} \tilde{V}/|\tilde{V}| = 0$$

(w.r.t. the orientation away from Ω)

iii) Due to orientation, last term $= -\displaystyle\sum_{q \in \tilde{V}^{-1}(0)} i(V, q)$

iv) On $S + \delta N$. $\quad \tilde{V} = \dfrac{V + \delta N}{|V + \delta N|} = \dfrac{\delta^{-1} V + N}{|\delta^{-1} V + N|}$

$\dfrac{N + s\,\delta^{-1} V}{|N + s\,\delta^{-1} V|}$: nowhere zero, $\tilde{V}$ when $s = 1$. N when $s = 0$



$\Rightarrow \deg_{S+\delta N} \dfrac{\tilde{V}}{|\tilde{V}|} = \deg_{S+\delta N} N = \deg_S N = \dfrac{1}{4\pi} \displaystyle\iint_S K \, dA$

orientation away from
Ω is N + (some tangent correction)

v) Similarly, on $S - \delta N$. $\quad \tilde{V} = S - \delta N$

$\Rightarrow \deg_{S-\delta N} \dfrac{\tilde{V}}{|\tilde{V}|} = \deg_S(-N) = \dfrac{1}{4\pi} \displaystyle\iint_S K \, dA$

orientation away from
Ω is $-N$ + (some tangent correction)

Finally) $\frac{1}{4\pi}\iint_S K\,dA + \frac{1}{4\pi}\iint_S K\,dA - \sum_{q\in V^{-1}(0)} i(V,q) = 0$ ※

§V. Gauss–Bonnet (pt. 2)

Now, $\iint_S K\,dA = 2\pi \sum_{q\in V^{-1}(0)} i(V,q)$

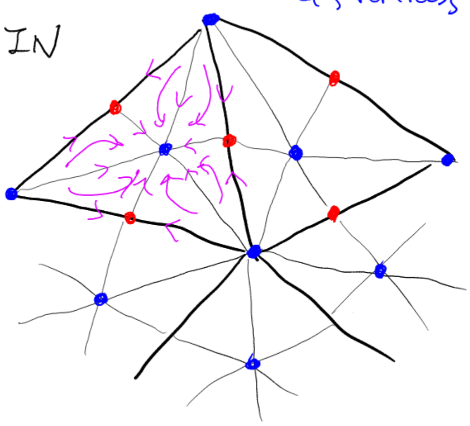
1° The total indices of V would balance each other.

2° Now, suppose that S has a triangulation (which is fine enough)

Then, by topology, $V - E + F$ = constant (independent of the triangulation)

$\#$ {vertices} {edges}, {faces}

ZOOM IN



We can draw a vector field according to the triangulation with

$$\begin{cases} \text{vertex} = \text{source} & +1 \\ \text{center of face} = \text{sink} & +1 \\ \text{center of edge} = \text{saddle point} & -1 \end{cases}$$

Therefore, $V - E + F$ = total indices of the vector field

$= \frac{1}{2\pi}\iint_S K\,dA \ (= 2-2g)$

§IV. some other application

Here is another Corollary of the Proposition in §II.

Cor $\bar{B} = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

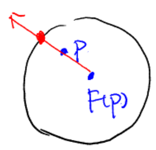
$\nexists\, F : \bar{B} \to S^2$ such that $F|_{S^2} = \mathrm{Id}$.

Pf: such an F has $\deg = 0$ on $\partial\bar{B} = S^2$ ※

thm (3D Brouwer fixed point) $F : \bar{B} \to \bar{B}$ smooth ← in fact, continuous is enough.

Then, F must have a fixed point.

pf:



Suppose not: $f(p) \neq p \quad \forall p \in \bar{B}$.

Consider $\tilde{F} : p \mapsto$ intersection of $\overrightarrow{F(p)\,p}$ with S^2

Solve $|t\,p + (1-t)F(p)|^2 = 1$

$= |t(p-q) + q|^2 \qquad q = F(p)$

$= t^2|p-q|^2 + 2t\langle p-q, q\rangle + |q|^2$

$|p-q|^2 t^2 + \langle p-q, q\rangle 2t - (1-|q|^2) = 0$

nonzero two root: $t \le 0$, $t \ge 1$ ← take this root.

If $|p| = 1$, $\tilde{F}(p) = p$ $\rightarrow\leftarrow$

Cor (a theorem of (Perron–) Frobenius in 4D)

$[A] \in M(4\times 4; \mathbb{R})$, all entries are positive. $A_{ij} > 0$, $i, j \in \{1, 2, 3, 4\}$

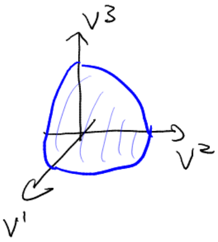
not quite
need continuous
version

Then. $[A]$ has a positive eigenvalue

pf: Consider $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$\vec{v} \longmapsto A\vec{v}$ $\qquad \vec{v} = (v^1, v^2, v^3, v^4)$

Positive entries $\Rightarrow$ It maps $I = \{\vec{v} \in \mathbb{R}^4 \mid v^j \geq 0 \ \ j \in \{1,2,3,4\}\}$ to itself



Let $\tilde{B} = \{\vec{v} \in I \mid |\vec{v}| = 1\}$

$\tilde{B}$ is homeomorphic to the 3-ball, $\bar{B}$.

$\tilde{A} : \tilde{B} \to \tilde{B}$ is continuous.

$\vec{v} \longmapsto A\vec{v} / |A\vec{v}|$

By Brouwer, $\tilde{A}$ has a fixed point.

Namely, $\exists\, \vec{v}_0 \in \tilde{B}$, $\rightarrow$ $A\vec{v}_0 / |A\vec{v}_0| = \vec{v}_0$

※