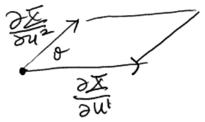
Degree of a smooth map to $\mathbb{S}^2$.

§0 Total integral of the Gaussian curvature

<u>recall</u> $dA = \left|\frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}\right| du^1 du^2 = \sqrt{\det(g)}\, du^1 du^2$

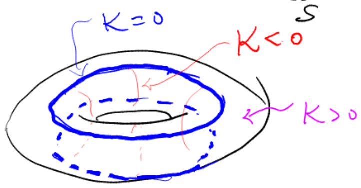


$$
\begin{aligned}
|\partial_1 X \times \partial_2 X|^2 &= (|\partial_1 X|\,|\partial_2 X| \sin\theta)^2 \\
&= |\partial_1 X|^2 |\partial_2 X|^2 (1-\cos^2\theta) \\
&= |\partial_1 X|^2 |\partial_2 X|^2 - (\langle \partial_1 X, \partial_2 X\rangle)^2 \\
&= g_{11} g_{22} - (g_{12})^2 = \det(g)
\end{aligned}
$$

<u>main topic</u> S : compact regular surface, with orientation N ← always assume this today

What is $\iint_S K\,dA$?

<u>example</u> 1)

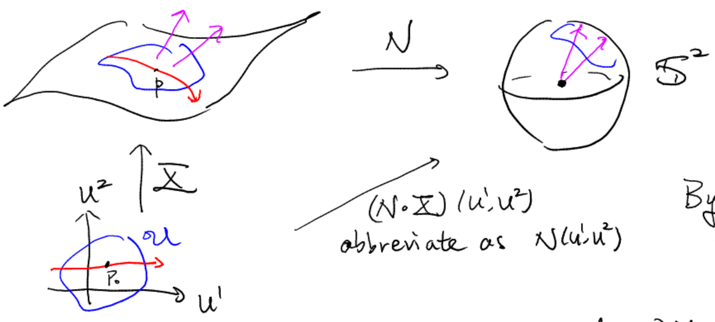


compute $\Rightarrow \iint_{\text{torus}} K\,dA = 0$

2) sphere of radius R. $K = \frac{1}{R^2}$

$\iint K\,dA = \frac{1}{R^2}\,\text{Area} = \frac{1}{R^2}(4\pi R^2) = 4\pi$ ~ independent of R

<u>discussion</u> (from the point of view of Gauss map)



Suppose $K(p) \neq 0$ $= \det(DN|_p)$

i.e. $DN|_p$ is invertible

$\Rightarrow D(N\circ X)|_{p_0}$ is injective

By IFT (shrink $\mathcal{U}$ if necessary), $N\circ X$ is a coordinate chart of $\mathbb{S}^2$

In this coordinate chart. $dA_{\mathbb{S}^2} = \left|\frac{\partial N}{\partial u^1} \times \frac{\partial N}{\partial u^2}\right| du^1 du^2$

recall $\begin{cases} \frac{\partial N}{\partial u^1} = a_1^1 \frac{\partial X}{\partial u^1} + a_1^2 \frac{\partial X}{\partial u^2} \\ \frac{\partial N}{\partial u^2} = a_2^1 \frac{\partial X}{\partial u^1} + a_2^2 \frac{\partial X}{\partial u^2} \end{cases} \Rightarrow \frac{\partial N}{\partial u^1} \times \frac{\partial N}{\partial u^2} = \underbrace{\det(a)}_{=K} \underbrace{\frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}}_{=N\cdot|\partial_1 X \times \partial_2 X|}$

$dA_{\mathbb{S}^2} = |K|\, dA_S$

Hence, $K\,dA_S$ ~ "signed" area of image of S under the Gauss map.

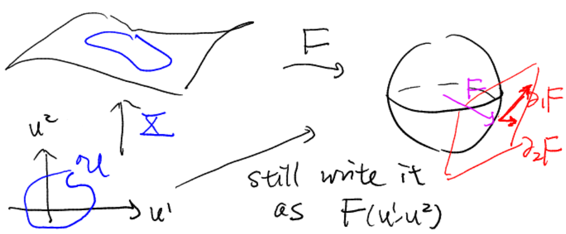
For convenience, we assume $\left\langle \frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}, N \right\rangle > 0$

$\begin{cases} K>0 & \text{if } \left\langle \frac{\partial N}{\partial u^1} \times \frac{\partial N}{\partial u^2}, N \right\rangle > 0 \\ K<0 & \text{if } \left\langle \frac{\partial N}{\partial u^1} \times \frac{\partial N}{\partial u^2}, N \right\rangle < 0 \end{cases}$

$K = \dfrac{\left\langle \frac{\partial N}{\partial u^1} \times \frac{\partial N}{\partial u^2}, N \right\rangle}{|\partial_1 X \times \partial_2 X|}$ (if $\langle \partial_1 X \times \partial_2 X, N\rangle > 0$)

§1 degree of a map to $\mathbb{S}^2$

1° <u>defn</u> In general, consider $F: S \to \mathbb{S}^2$

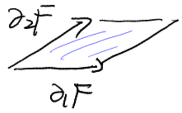


By switching u^1, u^2, we assume

$\langle \partial_1 X \times \partial_2 X, N\rangle > 0$ — (+)

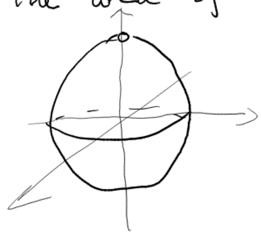
Let $\delta_F(p) = \dfrac{\langle \partial_1 F \times \partial_2 F, F\rangle}{|\partial_1 X \times \partial_2 X|}$ — (*)

Define the degree of F, $\deg(F)$ by $\underbrace{\frac{1}{4\pi}}_{\text{area}(S^2)}\iint_S \mathcal{J}_F(p)\, dA$

rmk
- Since $|F|^2 = 1$, $\partial_1 F, \partial_2 F \perp F$
  $\langle \partial_1 F \times \partial_2 F, F\rangle$ is the signed area of 
- $\boxed{\text{HW}}$ Check $\mathcal{J}_F(p)$ is well-defined, as long as the coordinate chart satisfies $(+)$
- For the Gauss map N, $\mathcal{J}_N(p) = K(p) \Rightarrow \iint_S K\,dA = 4\pi \deg(N)$

question Is $\deg(F)$ always an integer?
Intuitively, it counts "how many times" the image of F wraps around S^2.

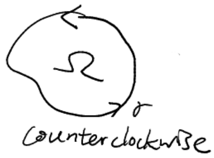
2° The area of S^2: $(\xi^1, \xi^2) \mapsto \frac{1}{1+|\xi|^2}\left(2\xi^1, -2\xi^2, -1+|\xi|^2\right)$



at $(0,0)$ $\partial_1 \Upsilon = (2,0,0)$, $\partial_2 \Upsilon = (0,-2,0)$
outer normal $N = \Upsilon = (0,0,-1)$
$\Rightarrow \langle \partial_1\Upsilon \times \partial_2 \Upsilon, N\rangle > 0$.
$|\partial_1 \Upsilon \times \partial_2 \Upsilon| = \sqrt{\det(g)} = \frac{4}{(1+|\xi|^2)^2}$ $(= g_{11} = g_{22},\ 0 = g_{12})$

$\Rightarrow \text{Area}(S^2) = \iint_{\mathbb{R}^2} \frac{4}{(1+x^2+y^2)^2}\, dx\, dy$ (write $\xi^1 = x$, $\xi^2 = y$)

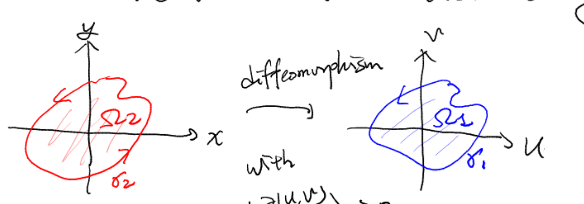
Green's theorem $\int_{\partial\Omega} \langle (P,Q), \gamma'\rangle\, dt = \iint_\Omega (Q_x - P_y)\, dx\, dy$



If $P = -y$, $Q = x$ $\Rightarrow$ $Q_x - P_y = 2$
If $P = \frac{-2y}{1+x^2+y^2}$, $Q = \frac{2x}{1+x^2+y^2}$ $\Rightarrow$ $Q_x - P_y = \frac{4}{(1+x^2+y^2)^2}$ —— (d)

Hence, $\iint_{x^2+y^2 \le R^2} \frac{4}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{2\pi} \left\langle \left(\frac{-2R\sin t}{1+R^2}, \frac{2R\cos t}{1+R^2}\right), (-R\sin t, R\cos t)\right\rangle dt$ $\quad \gamma = (R\cos t, R\sin t)$
$= \frac{R^2}{1+R^2} 4\pi \longrightarrow 4\pi$ as $R \to \infty$. ※

3° Green's formula under change of variable



$\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) > 0$

$\Leftarrow$ counterclockwise to counterclockwise
(for ∂, $T \to N$: point toward the region)
$\underset{(t_1,t_2)}{T} \to \underset{(-t_2,t_1)}{N}$

Green's formula on Ω_1:
$\iint_{\Omega_1} (Q_u - P_v)\, du\, dv = \int_{\partial\Omega_1} (P,Q)\cdot d\vec{\gamma_1}$ $\quad \gamma' \, dt$

LHS $= \iint_{\Omega_2} (Q_u - P_v)\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\, dy = \boxed{\iint_{\Omega_2} (Q_u - P_v)\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) dx\, dy}$

RHS $= \int_{\partial\Omega_1} \left(P(u,v)\, u' + Q(u,v)\, v'\right) dt$ $\quad \gamma_2 : (x(t), y(t)) \Rightarrow \gamma_1 : (u(x(t), y(t)), v(x(t), y(t)))$
$= \int_{\partial\Omega_2} \left(P\left(\frac{\partial u}{\partial x}x' + \frac{\partial u}{\partial y}y'\right) + Q\left(\frac{\partial v}{\partial x}x' + \frac{\partial v}{\partial y}y'\right)\right) dt$
$= \boxed{\int_{\partial\Omega_2} \left(P\frac{\partial u}{\partial x} + Q\frac{\partial v}{\partial x},\ P\frac{\partial u}{\partial y} + Q\frac{\partial v}{\partial y}\right)\cdot d\vec{\gamma_2}}$

One can check that $\frac{\partial}{\partial x}\left(P\frac{\partial u}{\partial y} + Q\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(P\frac{\partial u}{\partial x} + Q\frac{\partial v}{\partial x}\right) = (Q_u - P_v)\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right)$

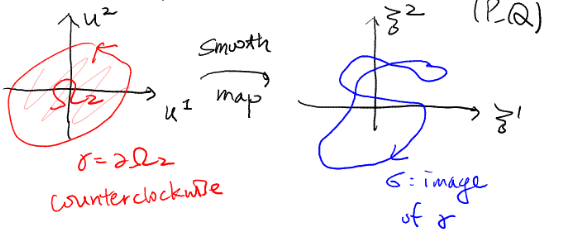
$\Rightarrow$ The red equality is just the Green's formula for the vector field

$\left(P\frac{\partial u}{\partial x}+Q\frac{\partial v}{\partial x},\ P\frac{\partial u}{\partial y}+Q\frac{\partial v}{\partial y}\right)$ on the xy-plane

It holds without any assumption on $\frac{\partial(u,v)}{\partial(x,y)}$

rmk These formal integrations & Stokes thereorems will be discussed more carefully in the last part

Summary prop



Green's theorem

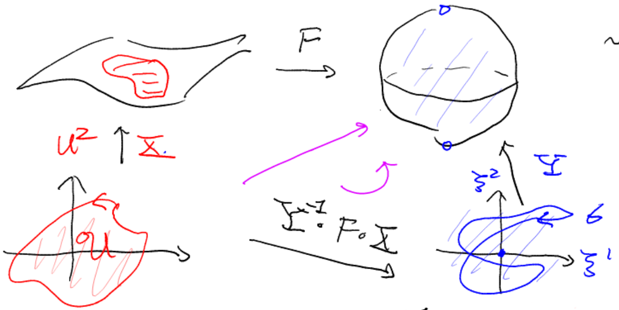
$$\iint_{\Omega_2}\left(\frac{\partial Q}{\partial \xi^1}-\frac{\partial P}{\partial \xi^2}\right)\det\left(\frac{\partial(\xi^1,\xi^2)}{\partial(u^1,u^2)}\right)du^1du^2$$

$$=\int_\gamma\left(P\frac{\partial\xi^1}{\partial u^1}+Q\frac{\partial\xi^2}{\partial u^1},\ P\frac{\partial\xi^1}{\partial u^2}+Q\frac{\partial\xi^2}{\partial u^2}\right)\cdot d\vec{r}$$

$$=\int_\sigma (P,Q)\cdot d\vec{\sigma}$$

change of variable for vector line integral

4° Back to $F: S\to S^2$. Suppose that $F(X(\mathcal{U}))\not\ni\{(0,0,1)\}$



$\rightsquigarrow$ Look at $Y^{-1}\circ F\circ X:\ \mathcal{U}\subset\mathbb{R}^2\to\mathbb{R}^2$

$$\iint_{X(\mathcal{U})}\underbrace{k_F(p)\,dA}=\ ?$$

$$\|$$

$$\underbrace{\langle\partial_1(F\circ X)\times\partial_2(F\circ X),F\rangle\,du^1du^2}$$

$$\|$$

$$\langle\left(\frac{\partial Y}{\partial\xi^1}\times\frac{\partial Y}{\partial\xi^2}\right)\Big|_{Y^{-1}\circ F\circ X},F\rangle\det\left(\frac{\partial(\xi^1,\xi^2)}{\partial(u^1,u^2)}\right)$$

$Y\big|_{Y^{-1}\circ F\circ X}$

$$F\circ X=Y\circ(Y^{-1}\circ F\circ X)$$

$$\frac{\partial}{\partial u^1}(F\circ X)=\frac{\partial Y}{\partial\xi^1}\frac{\partial\xi^1}{\partial u^1}+\frac{\partial Y}{\partial\xi^2}\frac{\partial\xi^2}{\partial u^1}$$

$$\frac{\partial}{\partial u^2}(F\circ X)=\frac{\partial Y}{\partial\xi^1}\frac{\partial\xi^1}{\partial u^2}+\frac{\partial Y}{\partial\xi^2}\frac{\partial\xi^2}{\partial u^2}$$

$$=\frac{4}{(1+(\xi^1)^2+(\xi^2)^2)^2}\det\left(\frac{\partial(\xi^1,\xi^2)}{\partial(u^1,u^2)}\right)$$

functions in (u^1,u^2)

computation in stereographic projection

Therefore, $$\iint_{X(\mathcal{U})}k_F(p)\,dA=\iint_\Omega\frac{4}{(1+(\xi^1)^2+(\xi^2)^2)^2}\det\left(\frac{\partial(\xi^1,\xi^2)}{\partial(u^1,u^2)}\right)du^1du^2$$

$$=\iint_\Omega\left(\frac{\partial}{\partial\xi^1}\left(\frac{2\xi^1}{1+|\xi|^2}\right)-\frac{\partial}{\partial\xi^2}\left(\frac{-2\xi^2}{1+|\xi|^2}\right)\right)\det\left(\frac{\partial(\xi^1,\xi^2)}{\partial(u^1,u^2)}\right)du^1du^2$$

the prop in 3°

$$=\int_{\gamma=\partial\Omega}\text{vector line integral}$$

$$=\int_{\sigma=(Y^{-1}\circ F\circ X)(\gamma(t))}\left(\frac{-2\xi^2}{1+|\xi|^2},\frac{2\xi^1}{1+|\xi|^2}\right)\cdot d\vec{\sigma}$$

Lemma $$\iint_{X(\mathcal{U})}k_F\,dA=\int_\sigma\left(\frac{-2\xi^2}{1+|\xi|^2},\frac{2\xi^1}{1+|\xi|^2}\right)\cdot d\vec{\sigma}$$

σ: image of $\partial\mathcal{U}$ under $Y^{-1}\circ F\circ X$ (counterclockwise)

5° recall !? Sard theorem : $F:\mathbb{R}^m \to \mathbb{R}^n$ smooth.

$q \in \mathbb{R}^n$ is called a singular value if $\exists\, p \in F^{-1}(q)$ such that rank $DF|_p < n$

regular value if $\forall\, p \in F^{-1}(q)$ rank $DF|_p = n$

→ The set of all singular values has measure zero

rank/discussion • un-interesting case $m < n$. rank DF = $m -$ nullity$(DF) \le m < n$

⇒ image = singular values

• It works for $F: S_1 \to S_2$ smooth map between regular surface

6° thm S : compact, regular surface with orientation N.
$F: S \to S^2$ smooth. Then $\deg(F) = \frac{1}{4\pi}\iint_S k_F \, dA$ is an integer

Moreover, if $q \in S^2$ is a regular value

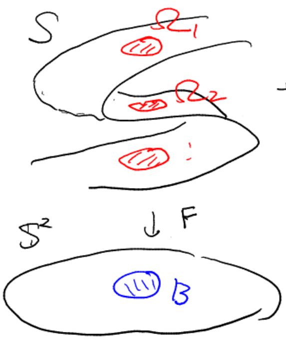
then, $\deg(F) = \sum_{p \in F^{-1}(q)} \operatorname{sgn}(k_F)$ → sign function takes value in ± 1

pf: By Sard theorem, there always exist a plethora of regular values.
Say q is a regular value. By composing with $SO(3)$, $S \xrightarrow{F} S^2 \xrightarrow{\overset{SO(3)}{\overset{\Downarrow}{B}}} S^2$

we may assume $(0,0,1)$ is a regular value

Note that $\iint_S k_F \, dA = \iint_S k_{B\circ F} \, dA$

step 1 claim. $\exists\, B$: open neighborhood of $N = (0,0,1)$ in S^2 such that $F^{-1}(B)$ is the finite union of disjoint open sets $\bigcup_{j=1}^{K} \Omega_j$.
and $F: \Omega_j \to B$ is a diffeomorphism for $j \in \{1, 2, \cdots, K\}$



pf of the claim : a) $\forall\, p \in F^{-1}(N)$ ⇒ IFT applies ⇒ $\exists\, \mathcal{U}_p \xrightarrow[\sim]{F} B_p$

Hence, $F^{-1}(N)$ is discrete (for $p' \in F^{-1}(N)$, $p' \neq p$ ⇒ $p' \notin \mathcal{U}_p$)

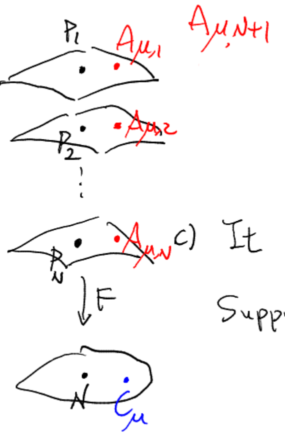
By compactness of S, $F^{-1}(N)$ is finite
Say, $\{p_1, \dots, p_K\}$

b) For each j, choose $\tilde{V}_j \subset \mathcal{U}_{p_j}$, $\tilde{V}_i \cap \tilde{V}_j = \phi$ for $i \neq j$

Let $\tilde{B} = \bigcap_{j=1}^{K} F(\tilde{V}_j)$: open nbd of p.

and let $V_j = F^{-1}(\tilde{B}) \cap \tilde{V}_j$: open nbd of p_j

⇒ V_j = disjoint. $V_j \xrightarrow{F} \tilde{B}$ diffeomorphism.



c) It remains to prove that $F^{-1}(\tilde{B}) = \bigcup_{j=1}^{K} V_j$, by shrinking $\tilde{B}$ if necessary

Suppose not. Then, $\exists\, \{C_\mu\} \in S^2 \longrightarrow N \to \#\{f^{-1}(C_\mu)\} > K$

Say, $A_{\mu j} \in V_j \cap f^{-1}(C_\mu)$ and there is some other $A_{\mu_{N+1}} \in f^{-1}(C_\mu)$

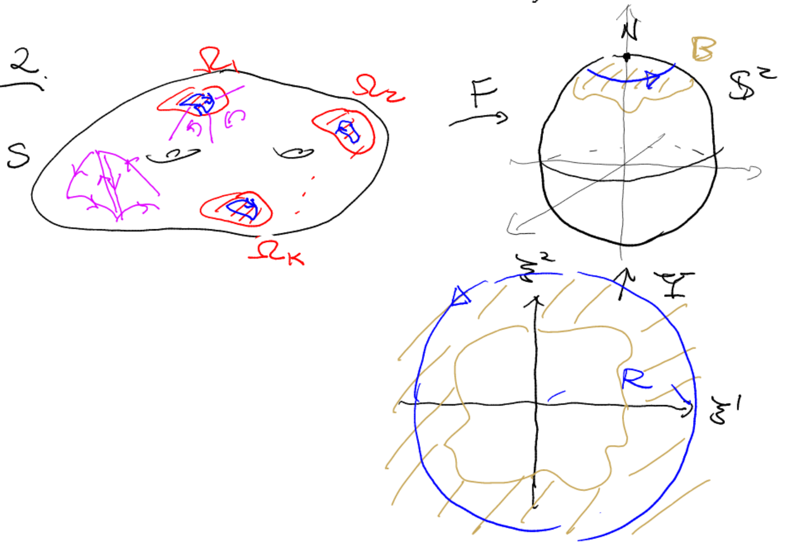
By compactness (and passing to subsequence), $A_{\mu,N+1} \xrightarrow{\mu\to\infty}$ some point in S

But $F(A_{\mu,N+1}) = C_\mu \to N \Rightarrow$ some point $\in F^{-1}(N) = \{P_1, \cdots, P_N\}$

Say, $A_{\mu,N+1} \to P_1$ as $\mu \to \infty$

Remember that # $A_{\mu,1} \to P_1$ as well. It condradicts to $V_1 \xrightarrow{F} \tilde{B}$ is a diffeomorphism #

Step 2.



For $R \gg 1$, $\Upsilon(|\xi| > R) \subset B$

$\downarrow$ as $R \to \infty$

N

From step 2. $\iint_S \mathcal{K}_F \, dA$

$= \lim_{R\to\infty} \iint_{S \setminus F^{-1}(\Upsilon(|\xi|>R))} \mathcal{K}_F \, dA$

But we can (divide $S \setminus F^{-1}(\Upsilon(|\xi|>R))$ into small regions, and) apply the lemma in 4°

$$\iint_{S\setminus F^{-1}(\Upsilon(|\xi|>R))} \mathcal{K}_F \, dA = \sum_{j=1}^{K} \pm \int_{\partial B_R} \left(\frac{-2\xi^2}{1+|\xi|^2}, \frac{2\xi^1}{1+|\xi|^2} \right) \cdot d\vec{s} = \sum \pm 4\pi \frac{R^2}{1+R^2}$$

Sign issue on S: counter clockwise w.r.t. complement, $S \setminus F^{-1}(\Upsilon(|\xi|>R))$

(Say, N = outer normal) $\Leftrightarrow$ clockwise w.r.t. $F^{-1}(\Upsilon(|\xi|>R))$, nbd of each P_j

on S^2 depend on sign of $\det\left(X_u\ X_v\ N\right)$ and $\det(F_u, F_v, F)$

$= \text{sign}\ \mathcal{K}_F(P_j)$

if $\text{sign}\,\mathcal{K}_F(P_j) = \begin{cases} +1, & \text{clockwise w.r.t. upper hemisphere} \\ -1, & \text{counter clockwise} \end{cases}$ (in terms of outer normal)

counter clockwise w.r.t lower hemi-sphere

$\to \int_{\partial B_R} \cdots \cdot d\vec{s} = \oplus\, 4\pi \frac{R^2}{1+R^2}$

By the same token, it corresponds $\ominus$ vector line integral

※

§II. applications: Gauss–Bonnet (p.t. 1) and applications

S: compact, regular surface with an orientation N

1° Suppose that $F(p,t): S \times I \to S^2$, smooth, $I = (a,b) \subset \mathbb{R}^1$

Then $\deg F(-,t): I \to \mathbb{R}^1$ is smooth in t, and takes integer value $\Rightarrow \deg F(-,t)$ is a constant

If $V(p): S \to \mathbb{R}^3 \setminus \{0\}$, nowhere vanishing vector field

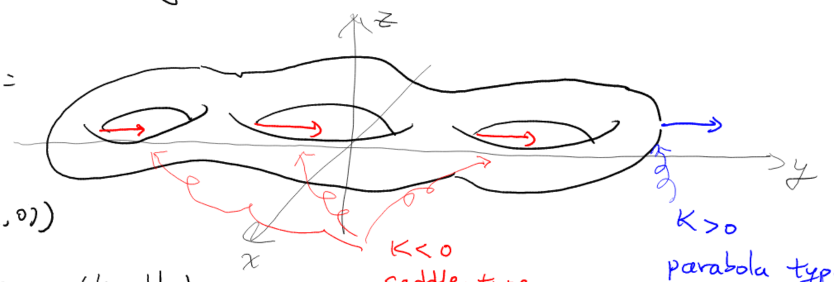
$F_V(p) = V(p)/|V(p)| : S \to S^2$.

It says that $\deg F_v(p)$ remains unchanged if we perturb v without creating zeros.

2° Gauss-Bonnet (p.f I) recall: Need an orientation on S to calculate δ_F and $\deg(F)$. When $F = N$, discussion on P.1 says that $\delta_N = K$

Hence, $\iint_S K\,dA = 4\pi \deg(N) \in 4\pi\mathbb{Z}$

- Intuitive discussion :

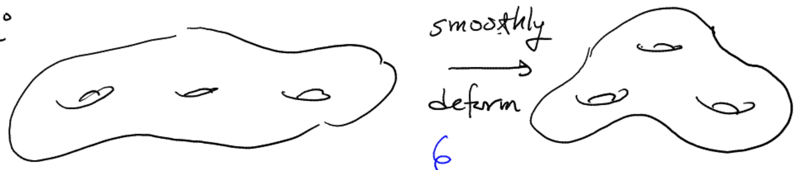


Study $N^{-1}((0,1,0))$

Adding 1 more genus (handle) $\rightsquigarrow$ create 1 more pre-image with $K<0$

$$\Rightarrow \iint_S K\,dA = 4\pi(1-g) \qquad g = \text{genus} = \#\{\text{handles}\}$$

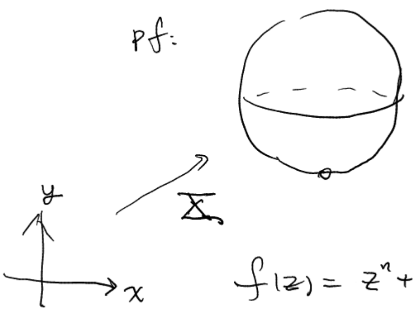
- Combine it with 1°



leaves degree(N) unchanged.

3° Fundamental theorem of algebra: Any non-constant complex polynomial, $z^n + a_{n-1}z^{n-1} + \cdots + a_0$ must have a root on $\mathbb{C}$. $n \in \mathbb{N}$

pf:



$\Phi : (x,y) \mapsto \frac{1}{1+x^2+y^2}(2x, 2y, 1-x^2-y^2)$ $\qquad \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\}$

$\Psi : (u,v) \mapsto \frac{1}{1+u^2+v^2}(2u, -2v, -1+u^2+v^2)$ $\qquad \mathbb{R}^2 \to S^2 \setminus \{(0,0,1)\}$

Write $z = x+iy$, $w = u+iv$. Then $w = \frac{1}{z}$

$f(z) = z^n + \cdots + a_0 \rightsquigarrow F = \Phi \circ f \circ \Phi^{-1} : S^2 \setminus \{(0,0,-1)\} \circlearrowleft$

claim: By setting $F(0,0,-1) = (0,0,-1)$, $F : S^2 \circlearrowleft$ smooth

Study it on $S^2 \setminus \{(0,0,+1)\} \xleftarrow{\Psi} \mathbb{R}^2$: $\underbrace{\Psi^{-1} \circ \Phi}_{\text{taking } 1/\square} \circ f \circ \underbrace{\Phi^{-1} \circ \Psi}_{z = \frac{1}{w} \,\leftarrow\, w}$

$$f \circ (\Phi^{-1} \circ \Psi)(w) = \frac{1}{w^n} + \frac{a_{n-1}}{w^{n-1}} + \cdots + a_0$$
$$= \frac{1 + a_{n-1}w + \cdots + a_0 w^n}{w^n}$$

$$\Psi^{-1} \circ \Phi \circ f \circ \Phi^{-1} \circ \Psi = \frac{w^n}{1 + a_{n-1}w + \cdots + a_0 w^n}$$: sends 0 to 0 and smooth on a nbd of 0.

But we can do it for $f(z,t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$

$\Rightarrow$ z^n and $f(z)$ gives the same degree

compute it by hand, $\deg = n \neq 0$

If $\deg \neq 0$, must be surjective ... #