Gauss's Theorema Egregium

thm Suppose that $F: S \to \tilde{S}$ is a local isometry. Then, they have the same Gaussian curvature

§I. Preparation

0° Einstein's summation convention: repeated indices → taking sum

e.g. $a_{jk} v^k$ means $\sum_{k=1}^{2} a_{jk} v^k$

S_{ii} mean $\sum_{i=1}^{2} S_{ii}$

1° [linear algebra] Quadratic form (put same vector in symmetric bilinear form)

$[S_{kl}] \in M_{n\times n}(\mathbb{R}) \rightsquigarrow v^k S_{kl} v^l$ is a quadratic form on $\{v^k\} \in \mathbb{R}^n$

not require symmetry

It is zero if $S_{kl} + S_{lk} = 0 \quad \forall k, l$

2° [Taylor's theorem] $f(x^1, \dots, x^n)$: smooth function (on a neighborhood of 0)

$$\Rightarrow f(x^1, \dots, x^n) = \underbrace{f(0) + T_j x^j + \frac{1}{2} P_{jk} x^j x^k + \frac{1}{6} Q_{jkl} x^j x^k x^l}_{\text{is uniquely determined by}} + O(|x|^4)$$

remainder

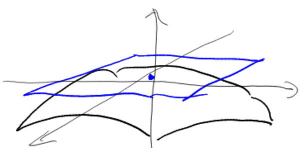
∂_j become $O(|x|^3)$

$T_j = (\partial_j f)|_0 \quad P_{jk} = (\partial_j \partial_k f)|_0 \quad Q_{jkl} = (\partial_j \partial_k \partial_l f)|_0$

3° [regular surface] i) rigid motion $x \mapsto Ax + b \quad A \in O(3),\ b \in \mathbb{R}^3$ will not change the Gaussian curvature

Exercise

ii) $p \in S$. By rigid motion, we can assume that $\begin{cases} p = \text{origin} \\ T_pS = xy\text{-plane} \end{cases}$



After that, by IFT, S near p is a graph over $\{(x, y, f(x,y)) : x^2+y^2 < \varepsilon\}$

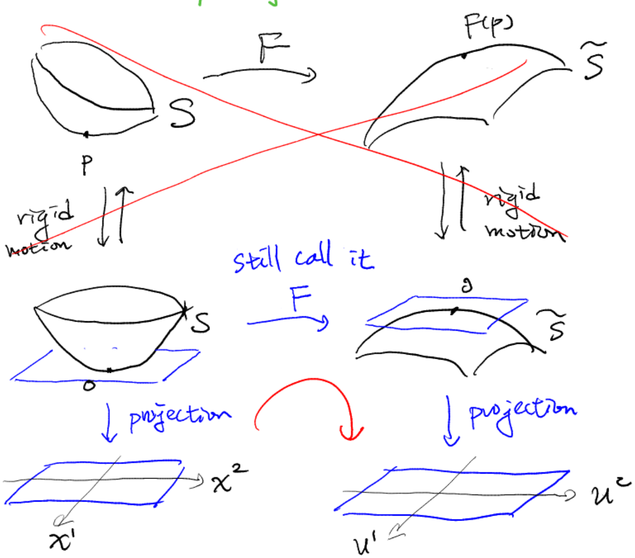
$f_x|_0 = 0 = f_y|_0 \qquad f(0) = 0$

iii) $\Rightarrow f(x,y) = \frac{1}{2}(ax^2 + \underbrace{2bxy}_{bxy+byx} + cy^2) + O((x^2+y^2)^{\frac{3}{2}})$

By $\boxed{\text{HW}}$, $K(0) = ac - b^2$

4° Kronecker delta, $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

§II the proof (not full version)



step 0° By composing with a rigid motion, we can assume $\begin{cases} 0 \in S \text{ and } 0 \in \tilde{S} \\ T_0S = xy\text{-plane} = T_0\tilde{S} \\ F(0) = 0 \end{cases}$

Since $DF|_0$: xy-plane $\circlearrowleft$ itself and preserves the inner product, $DF|_0 \in O(2)$

By composing $\begin{bmatrix} (DF|_0)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, we can assume $DF|_0 = \text{identity}$

Step 1° By §I-3°. S near 0 is $\{(x^1, x^2, f(x^1,x^2)\}$ with $\partial_j f|_0 = 0$, $j=1,2$

and $K(0) = a_{11}a_{22} - (a_{12})^2$ $\quad a_{ij} = (\partial_i\partial_j f)|_0$

Similarly, $\tilde{S}$: $\{(u^1, u^2, \tilde{f}(u^1, u^2)\}$

$\tilde{K}(0) = \tilde{a}_{11}\tilde{a}_{22} - (\tilde{a}_{12})^2$

$\boxed{\text{goal}}$ $a_{11}a_{22} - (a_{12})^2 = \tilde{a}_{11}\tilde{a}_{22} - (\tilde{a}_{12})^2$

Step 2° the metric on S, $\partial_j X = (0, \cdots, 1, \cdots 0, \partial_j f)$ — j-th; last component

$$\Rightarrow g_{ij} = \delta_{ij} + \partial_i f\,\partial_j f \qquad f = \tfrac{1}{2}a_{k\ell}x^k x^\ell + O(|x|^3)$$

$$= \delta_{ij} + (a_{ik}x^k + O(|x|^2))(a_{j\ell}x^\ell + O(|x|^2)) \qquad \partial_i f = \tfrac{1}{2}a_{i\ell}x^\ell + \tfrac{1}{2}a_{ki}x^k + O(|x|^2)$$

$$= \delta_{ij} + a_{ik}a_{j\ell}x^k x^\ell + O(|x|^3) \qquad = a_{ik}x^k + O(|x|^2)$$

Similarly, $\tilde{g}_{\alpha\beta} = \delta_{\alpha\beta} + \tilde{a}_{\alpha\mu}\tilde{a}_{\beta\nu}u^\mu u^\nu + O(|u|^3)$

Step 3° local isometry F in the coordinate chart : (u^1, u^2) are functions in (x^1, x^2)

means that $\tilde{g}_{\alpha\beta}\, du^\alpha \cdot du^\beta = \left(\tilde{g}_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}\right) dx^i \cdot dx^j$

$$= g_{ij}\, dx^i \cdot dx^j$$

$$\Leftrightarrow \tilde{g}_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} = g_{ij}(x) \quad \forall\, i,j \in \{1,2\}$$

3-equations
$\Rightarrow$ expand them

$\boxed{\text{Strategy}}$

Series expansion in $\{x^i\}$, and examine the coefficients (order 0, 1, 2)

Step 4°

$$u^\alpha = \underset{F(0)=0}{0} + \overset{\delta^\alpha_i x^i}{x^\alpha} + \tfrac{1}{2}P^\alpha_{ij}x^i x^j + \tfrac{1}{6}Q^\alpha_{ijk}x^i x^j x^k + O(|x|^4)$$

xy-plane = tangent plane.
and $DF|_0$ = identity map.

$$\Rightarrow \begin{cases} O(|u|) = O(|x|) \\ \dfrac{\partial u^\alpha}{\partial x^i} = \delta^\alpha_i + P^\alpha_{ij}x^j + \tfrac{1}{2}Q^\alpha_{ijk}x^j x^k + O(|x|^3) \end{cases}$$

symmetric here : $P^\alpha_{ij} = P^\alpha_{ji}$, $Q^\alpha_{ijk} = Q^\alpha_{jik} = \cdots$

$$\Rightarrow \delta_{ij} + a_{ik}a_{j\ell}x^k x^\ell + O(|x|^3)$$

$$= (\delta_{\alpha\beta} + \tilde{a}_{\alpha\mu}\tilde{a}_{\beta\nu}u^\mu u^\nu + O(|u|^3))\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}$$

$$= (\delta_{\alpha\beta} + \tilde{a}_{\alpha\mu}\tilde{a}_{\beta\nu}x^\mu x^\nu + O(|x|^3))$$
$$(\delta^\alpha_i + P^\alpha_{ik}x^k + \tfrac{1}{2}Q^\alpha_{ik\ell}x^k x^\ell + O(|x|^3))$$
$$(\delta^\beta_j + P^\beta_{jm}x^m + \tfrac{1}{2}Q^\beta_{jmn}x^m x^n + O(|x|^3))$$

Step 5° 0-order $\quad \delta_{ij} = \delta_{\alpha\beta}\delta^\alpha_i\delta^\beta_j = \delta_{ij}$ ✓

1-order $\quad 0 = \delta_{\alpha\beta}P^\alpha_{ik}x^k\delta^\beta_j + \delta_{\alpha\beta}\delta^\alpha_i P^\beta_{jm}x^m$

$$= P^j_{ik}x^k + P^i_{jm}x^m = (P^j_{ik} + P^i_{jk})x^k$$

True for any x : small $\Rightarrow P^j_{ik} = -P^i_{jk}$

But $i,j,k \in \{1,2\}$. for P^j_{ik}, either $i=j$, or $i=k$ or $j=k$

$$\left.\begin{array}{ll} i=j: & P^i_{ik} = -P^i_{ik} \Rightarrow P^i_{ik}=0 \\ i=k: & P^j_{ii} = -P^i_{ji} = -P^i_{ij} = 0 \\ j=k: & P^j_{ij} = P^j_{ji} = 0 \end{array}\right\} \Rightarrow \quad P^k_{ij}=0\,! \ \forall i,j,k$$

(uses 2D)

step 6° 2-order

$$a_{ik}a_{jl}x^kx^l = \tilde{a}_{\alpha\mu}\tilde{a}_{\beta\nu}x^\mu x^\nu \delta^\alpha_i\delta^\beta_j + \frac{1}{2}\delta_{\alpha\beta}Q^\alpha_{ikl}x^kx^l\delta^\beta_j + \frac{1}{2}\delta_{\alpha\beta}\delta^\alpha_i Q^\beta_{jmn}x^mx^n$$

$$= \tilde{a}_{ik}\tilde{a}_{jl}x^kx^l + \frac{1}{2}\left(Q^j_{ikl} + Q^i_{jkl}\right)x^kx^l$$

$$\Rightarrow \left(a_{ik}a_{jl} - \tilde{a}_{ik}\tilde{a}_{jl} - \frac{1}{2}\left(Q^j_{ikl} + Q^i_{jkl}\right)\right)x^kx^l = 0$$

For each (i,j), the quadratic form is zero.

i) $(i,j)=(1,1)$. $\left(a_{1k}a_{1l} - \tilde{a}_{1k}\tilde{a}_{1l} - \frac{1}{2}\left(Q^1_{1kl} + Q^1_{1kl}\right)\right)x^kx^l = 0$

By §I-1° for $(k,l)=(2,2)$

$$\Rightarrow (a_{12})^2 - (\tilde{a}_{12})^2 = Q^1_{122} \quad \text{——} \ ①$$

Similarly, $(i,j)=(2,2)$. then $(k,l)=(1,1)$

$$\Rightarrow (a_{21})^2 - (\tilde{a}_{21})^2 = Q^2_{211} \quad \text{——} \ ②$$

ii) $(i,j)=(1,2)$ $\left(a_{1k}a_{2l} - \tilde{a}_{1k}\tilde{a}_{2l} - \frac{1}{2}\left(Q^2_{1kl} + Q^1_{2kl}\right)\right)x^kx^l = 0$

By §I-1° for $(k,l)=(1,2)$

$$a_{11}a_{22} - \tilde{a}_{11}\tilde{a}_{22} - \frac{1}{2}\left(Q^2_{112} + Q^1_{212}\right)$$

$$= -(a_{12})^2 + (\tilde{a}_{12})^2 + \frac{1}{2}\left(Q^2_{121} + Q^1_{221}\right)$$

$$\Rightarrow \left(a_{11}a_{22} + (a_{12})^2\right) - \left(\tilde{a}_{11}\tilde{a}_{22} + (\tilde{a}_{12})^2\right) = Q^2_{211} + Q^1_{122} \quad \text{——} \ ③$$

iii) ③ − ① − ② : $a_{11}a_{22} - (a_{12})^2 = \tilde{a}_{11}\tilde{a}_{22} - (\tilde{a}_{12})^2$ #