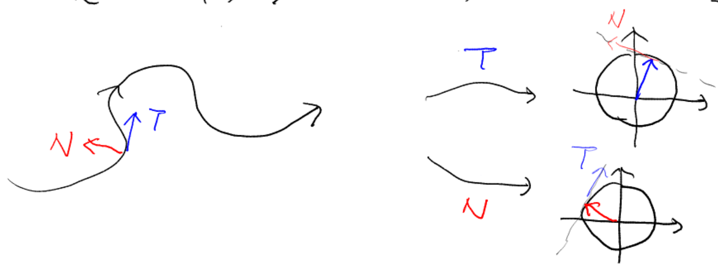
What does it mean by "a surface is curved"? ch. 3 & 7 of [MR]

<u>recall</u> plane curve $\gamma(s)$ both T & N are 1^{st} derivatives

$\kappa = \langle T', N\rangle = -\langle T, N'\rangle$ <u>recall</u> $\begin{cases} T' \perp T \Rightarrow T' \parallel N \\ N' \perp N \Rightarrow N' \parallel T \end{cases}$



Basically, curvature is the derivative of $T: \gamma \to S^1$
or $N: \gamma \to S^1$

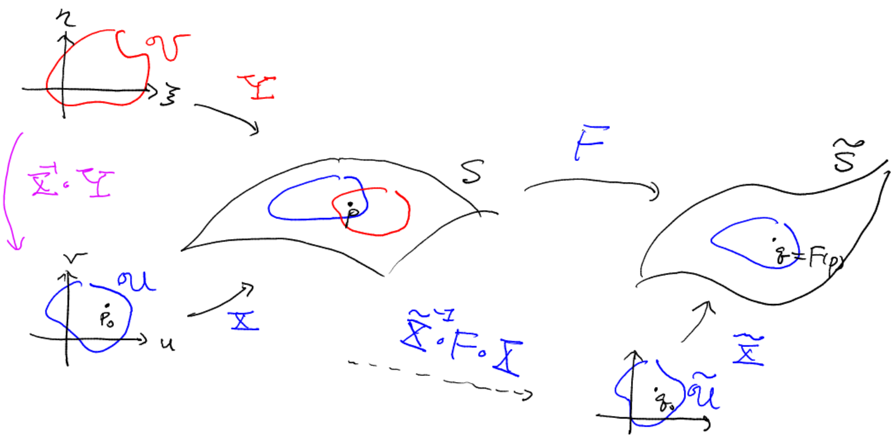
<u>recall</u> tangent plane & derivative of a map

<u>defn</u> $T_pS = \text{image}(D\Sigma|_{p_0})$
$= \text{span}\left\{ \frac{\partial \Sigma}{\partial u}, \frac{\partial \Sigma}{\partial v} \right\}$
well-defined $= \text{span}\left\{ \frac{\partial \Upsilon}{\partial \xi}, \frac{\partial \Upsilon}{\partial \eta} \right\}$

$\Upsilon = \Sigma \circ (\Sigma^{-1} \circ \Upsilon)$

$\Rightarrow D\Upsilon = D\Sigma \cdot D(\Sigma^{-1} \circ \Upsilon)$

$$\begin{bmatrix} \frac{\partial \Upsilon}{\partial \xi} & \frac{\partial \Upsilon}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Sigma}{\partial u} & \frac{\partial \Sigma}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{bmatrix}$$



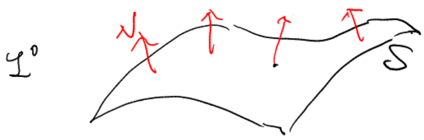
<u>defn</u> the <u>differential</u> of F at p, $DF|_p : T_pS \longrightarrow T_{F(p)}\tilde{S}$ by

$$\begin{array}{ccc} T_pS & & T_{F(p)}\tilde{S} \\ \big\downarrow {\scriptstyle (D\Sigma|_{p_0})^{-1}} & & \big\uparrow {\scriptstyle D\tilde{\Sigma}|_{q_0}} \\ \mathbb{R}^2 & \xrightarrow{D(\tilde{\Sigma}^{-1} \circ F \circ \Sigma)|_{p_0}} & \mathbb{R}^2 \end{array}$$

$\boxed{\text{check}}$ By chain rule, $DF|_p$ is a well-defined linear map.

§I. the Gauss map.

Now, let S be a regular surface with an orientation N &
(also contructed by 1^{st} order derivatives)

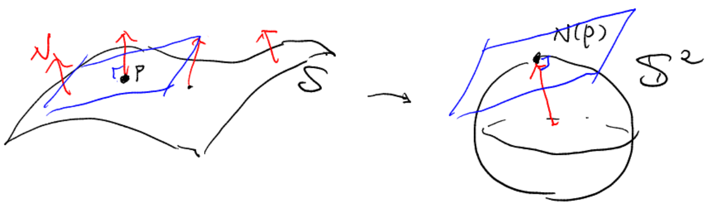


N: unit-normed and smooth

$\Rightarrow N : S \to \mathbb{S}^2$ is a smooth map.

<u>defn</u> This is called the Gauss map.

Analogous to the plane curve, its derivative shall tell us how S is curved. (~ rate of change of unit normal)



From the discussion last time,

$DN|_p : T_pS \to T_{N(p)}\mathbb{S}^2$

! <u>But</u> both T_pS and $T_{N(p)}\mathbb{S}^2$ are the 2-plane orthogonal to $N(p)$

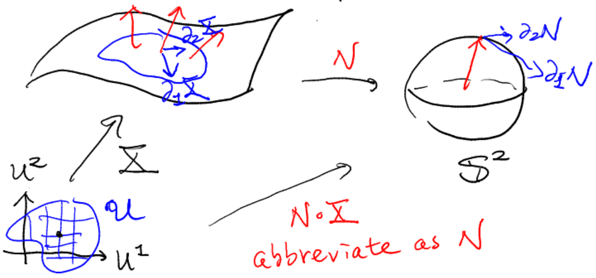
<u>upshot</u> $DN|_p$ is a linear transform on the same (abstract) vector space,
$\forall p \in S$. $DN|_p : T_pS \to T_pS = T_{N(p)}S^2$

To get some precise measurement, consider its determinant, trace or eigenvalues.

<u>Q</u> Is it diagonalizable?

<u>A</u> Yes! In fact, it is symmetric.

2° study $DN|_p$. choose a coordinate chart $(\mathcal{U}, \mathbb{X})$



Denote $\frac{\partial}{\partial u^j}$ by ∂_j

$$(DN)(\partial_j \mathbb{X}) = \partial_j N$$

<u>rmk</u> Since $\langle N, N\rangle = 1$

$$\Rightarrow \langle \partial_j N, N\rangle = 0 \Rightarrow \partial_j N \perp N$$

$$\{\partial_j N\}_{j=1}^2 \in T_pS = \operatorname{span}\{\partial_j \mathbb{X}\}_{j=1}^2$$

$\leftarrow$ NOT necessarily independent

i) $\langle \partial_j N, \partial_k \mathbb{X}\rangle = \partial_j \underbrace{\langle \overset{\text{normal}}{N}, \overset{\text{tangent}}{\partial_k \mathbb{X}}\rangle}_{=0} - \langle N, \partial_j\partial_k \mathbb{X}\rangle$

$$= -\langle N, \partial_k \partial_j \mathbb{X}\rangle = \langle \partial_k N, \partial_j \mathbb{X}\rangle$$

Or $\langle (DN)(\partial_j \mathbb{X}), \partial_k \mathbb{X}\rangle = \langle \partial_j \mathbb{X}, (DN)(\partial_k \mathbb{X})\rangle$

Since $\{\partial_j \mathbb{X}\}$ is a basis, $\langle (DN)_p(V), W\rangle = \langle V, (DN)_p W\rangle$

Hence, $DN_p : T_pS \circlearrowleft$ is self-adjoint

(in terms of an orthonormal basis, the 2×2 matrix is symmetric)

ii) trace & determinant?

$(DN)(\partial_j \mathbb{X}) = \partial_j N = \sum_{i=1}^2 a_j^i \, \partial_i \mathbb{X}$ solve a_j^i?

$$\langle \partial_j N, \partial_k \mathbb{X}\rangle = \sum_{i=1}^2 a_j^i \langle \partial_i \mathbb{X}, \partial_k \mathbb{X}\rangle$$

a needs not to be symmetric

$$\Rightarrow \begin{bmatrix} \langle \partial_1 N, \partial_1 \mathbb{X}\rangle & \langle \partial_1 N, \partial_2 \mathbb{X}\rangle \\ \langle \partial_2 N, \partial_1 \mathbb{X}\rangle & \langle \partial_2 N, \partial_2 \mathbb{X}\rangle \end{bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{bmatrix} \begin{bmatrix} \langle \partial_1 \mathbb{X}, \partial_1 \mathbb{X}\rangle & \langle \partial_1 \mathbb{X}, \partial_2 \mathbb{X}\rangle \\ \langle \partial_2 \mathbb{X}, \partial_1 \mathbb{X}\rangle & \langle \partial_2 \mathbb{X}, \partial_2 \mathbb{X}\rangle \end{bmatrix}$$

<u>notation</u> $h_{jk} = -\langle \partial_j N, \partial_k \mathbb{X}\rangle = \langle N, \partial_j \partial_k \mathbb{X}\rangle$ symmetric by i)

$g_{jk} = \langle \partial_j \mathbb{X}, \partial_k \mathbb{X}\rangle$ symmetric by definition.

iii) g_{jk} and h_{jk} are <u>locally defined</u> functions on $\mathcal{U}$ or $\mathbb{X}(\mathcal{U})$

will discuss more later

<u>But</u> $-\frac{1}{2}\operatorname{tr}([a]) = -\frac{1}{2}\operatorname{tr}([h][g]^{-1}) = -\frac{1}{2}\operatorname{tr}(DN)$

$\det([a]) = \det([h][g]^{-1}) = \det(DN)$

are defined on S

main topic in the 1st part

<u>defn</u> $K(p) := \det(DN|_p)$: the <u>Gaussian</u> <u>curvature</u> of S 高斯曲率

analogous to curvature of plane curve

$H(p) := -\frac{1}{2}\operatorname{tr}(DN|_p)$: the <u>mean</u> <u>curvature</u> of S (平)均曲率

§Ⅱ. examples

1° z-plane. say $z=0$

$$\underset{1\ \ 2}{(x,y)} \overset{X}{\longmapsto} (x,y,0)$$

choose upward $N=(0,0,1) \Rightarrow DN \equiv 0$

$\Rightarrow [g_{ij}] = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ & $[h_{ij}] = 0$

$\Rightarrow K = 0 = H$

2° sphere of radius R $\quad x^2+y^2+z^2=R^2$

$$(u^1,u^2) \overset{X}{\longmapsto} \frac{R}{1+|u|^2}(2u^1, 2u^2, 1-|u|^2)$$

choose outward $N = \frac{1}{R}X \Rightarrow [h_{ij}] = -\frac{1}{R}[g_{ij}]$

$\Rightarrow [h][g]^{-1} = -\frac{1}{R}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\Rightarrow K = \frac{1}{R^2}, \quad H = +\frac{1}{R}$ (Smaller radius, larger curvature)

$\boxed{\text{check}}$ $\quad g_{11} = g_{22} = \frac{4R^2}{(1+|u|^2)^2}, \quad g_{12}=0$

0° (surface of revolution) rotate $(\alpha(t)^{7^\circ}, \beta(t))$ around the z-axis

i) $\underset{1\ \ 2}{(t,\theta)} \overset{X}{\longmapsto} (\alpha\cos\theta, \alpha\sin\theta, \beta)$

$$\begin{cases} X_t = (\alpha'\cos\theta, \alpha'\sin\theta, \beta') \\ X_\theta = (-\alpha\sin\theta, \alpha\cos\theta, 0)\end{cases} \Rightarrow X_t \times X_\theta = \alpha(-\beta'\cos\theta, -\beta'\sin\theta, \alpha')$$

Choose $N = \dfrac{(-\beta'\cos\theta, -\beta'\sin\theta, \alpha')}{\sqrt{(\alpha')^2+(\beta')^2}}$

ii) g_{jk}: $\quad g_{11} = (\alpha')^2+(\beta')^2, \quad g_{12}=0 \quad g_{22}=\alpha^2$

iii) $h_{jk} = \langle N, \partial_j\partial_k X\rangle$

$$\begin{cases} X_{tt} = (\alpha''\cos\theta, \alpha''\sin\theta, \beta'') \\ X_{t\theta} = (-\alpha'\sin\theta, \alpha'\cos\theta, 0) \\ X_{\theta\theta} = (-\alpha\cos\theta, -\alpha\sin\theta, 0)\end{cases} \Rightarrow \begin{cases} h_{11} = \dfrac{1}{\sqrt{(\alpha')^2+(\beta')^2}}(-\alpha''\beta'+\alpha'\beta'') \\ h_{12} = 0 \\ h_{22} = \dfrac{1}{\sqrt{(\alpha')^2+(\beta')^2}}(\alpha\beta')\end{cases}$$

3° cylinder of radius 1 : $\alpha = 1, \ \beta = t$ in 0°

$\Rightarrow [g] = \begin{bmatrix}1&0\\0&1\end{bmatrix}, [h] = \begin{bmatrix}0&0\\0&1\end{bmatrix} \Rightarrow K = 0, \quad H = \frac{-1}{2}$

4° Beltrami's pseudo-sphere : $\alpha(t) = e^t, \quad \beta(t) = \int_0^t \sqrt{1-e^{2p}}\,dp \qquad t\in(-\infty,0)$

$\alpha' = e^t, \quad \beta' = \sqrt{1-e^{2t}}$

$\Rightarrow (\alpha')^2+(\beta')^2 = 1$

$\Rightarrow [g] = \begin{bmatrix}1&0\\0&e^{2t}\end{bmatrix}$

$\alpha'' = \alpha' = e^t, \quad \beta'\beta'' = -\alpha'\alpha'' \Rightarrow \beta'' = \dfrac{-e^{2t}}{\sqrt{1-e^{2t}}}$

$\Rightarrow [h] = \begin{bmatrix}\dfrac{-e^t}{\sqrt{1-e^{2t}}} & 0 \\ 0 & e^t\sqrt{1-e^{2t}}\end{bmatrix}$

$\Rightarrow K = \det([h])\cdot\det([g]^{-1}) = -e^{2t}\cdot e^{-2t} = -1$

$H =$ some function in t

§ III. first and second fundamental form

What are g_{ij} and h_{ij} ?

1° $g_{ij}(p) = \langle \partial_i \mathbb{X}|_p , \partial_j \mathbb{X}|_p \rangle$: the coefficient of $\underbrace{\text{the inner product } \langle\, ,\, \rangle \text{ on } T_pS}$

a positive definite, symmetric bilinear form on $\mathbb{R}^2 \cong T_pS$ for each $p \in S$

For $V, W \in T_pS$, write $\begin{cases} V = \sum_{i=1}^{2} V^i \partial_i \mathbb{X}|_p \\ W = \sum_{i=1}^{2} W^i \partial_i \mathbb{X}|_p \end{cases}$ $\Rightarrow \langle V, W \rangle = \sum_{i,j=1}^{2} g_{ij}(p) V^i W^j$

2° Similarly, $h_{ij}(p) = \langle -DN(\partial_i \mathbb{X}|_p), \partial_j \mathbb{X}|_p \rangle$: the coefficient of the <u>symmetric</u> bilinear form on $\mathbb{R}^2 \cong T_pS$ for each $p \in S$

↳ from §I-2°-ii) : $\langle DN(V), W \rangle = \langle V, (DN)(W) \rangle$

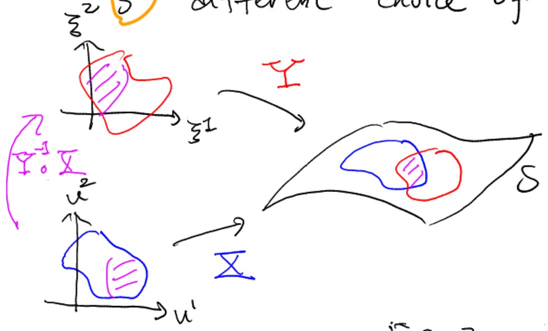
As above, $\langle DN(V), W \rangle = -\sum_{i,j=1}^{2} h_{ij}(p) V^i W^j$

<u>recap</u> g_{ij} tells us the "distance" and "angle" for curves on S

h_{ij} tells us how the rate of change of the normal

classical approach

3° different choice of coordinate (chart) :



relation between g_{ij} and $\tilde{g}_{ij}$, h_{ij} and $\tilde{h}_{ij}$?

$$\mathbb{X} = \Upsilon \circ (\Upsilon^{-1} \circ \mathbb{X})$$

$$\Rightarrow \begin{bmatrix} \frac{\partial \mathbb{X}}{\partial u^1} & \frac{\partial \mathbb{X}}{\partial u^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Upsilon}{\partial \xi^1} & \frac{\partial \Upsilon}{\partial \xi^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix}$$

$$[g_{ij}] = \begin{bmatrix} \left(\frac{\partial \mathbb{X}}{\partial u^1}\right)^T \\ \left(\frac{\partial \mathbb{X}}{\partial u^2}\right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbb{X}}{\partial u^1} & \frac{\partial \mathbb{X}}{\partial u^2} \end{bmatrix}$$

$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \in$ kernel

$\Rightarrow \sum_{j=1}^{2} v^j \frac{\partial \mathbb{X}}{\partial u^j} \perp \frac{\partial \mathbb{X}}{\partial u^1}, \frac{\partial \mathbb{X}}{\partial u^2}$

$\Rightarrow v^1 = 0 = v^2$

$$= \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^2}{\partial u^1} \\ \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial \Upsilon}{\partial \xi^1}\right)^T \\ \left(\frac{\partial \Upsilon}{\partial \xi^2}\right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial \Upsilon}{\partial \xi^1} & \frac{\partial \Upsilon}{\partial \xi^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix}$$

$$\overset{①}{=} \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^2}{\partial u^1} \\ \frac{\partial \xi^1}{\partial u^2} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix} \begin{bmatrix} \tilde{g}_{kl} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix}$$

$$\Rightarrow \underbrace{g_{ij} = \sum_{k,l=1}^{2} \frac{\partial \xi^k}{\partial u^i} \tilde{g}_{kl} \frac{\partial \xi^l}{\partial u^j}} \quad \text{—— } (\star 1)$$

Similarly, $\begin{bmatrix} \frac{\partial N}{\partial u^1} & \frac{\partial N}{\partial u^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial N}{\partial \xi^1} & \frac{\partial N}{\partial \xi^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} \\ \frac{\partial \xi^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} \end{bmatrix}$

$\Rightarrow [h_{ij}] = -\begin{bmatrix} \left(\frac{\partial N}{\partial u^1}\right)^T \\ \left(\frac{\partial N}{\partial u^2}\right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbb{X}}{\partial u^1} & \frac{\partial \mathbb{X}}{\partial u^2} \end{bmatrix}$ would satisfy the same rule:

$$\Rightarrow \underbrace{h_{ij} = \sum_{k,l=1}^{2} \frac{\partial \xi^k}{\partial u^i} \tilde{h}_{kl} \frac{\partial \xi^l}{\partial u^j}} \quad \text{—— } (\star 2)$$

<u>Summary</u>

$[g] = S[\tilde{g}]S^T$

$[h] = S[\tilde{h}]S^T$

$\Rightarrow [h][g]^{-1} = S[\tilde{h}]\cancel{S^T(S^T)^{-1}}[\tilde{g}]^{-1}S^{-1}$

$\Rightarrow$ trace$([h][g]^{-1})$ & det$([h][g]^{-1})$ does not depend on the choice of coordinate chart

4° modern notation

<u>defn</u> Denote $\langle U, V\rangle$ by $I(U,V)$, and $\langle (DN)(U), V\rangle$ by $\mathbb{I}(U,V)$

They are called the first and second fundamental form of S, respectively. (for each p, a symmetric bilinear form on T_pS)

We usually write them as $I = \sum_{i,j=1}^{2} g_{ij}\, du^i \cdot du^j$ and $\mathbb{I} = \sum_{i,j=1}^{2} h_{ij}\, du^i \cdot du^j$

<u>notation</u> $du^i \cdot du^j$ is the ~~symmetric~~ bilinear form

with $(du^i \cdot du^j)(\partial_k X, \partial_l X) = \begin{cases} 1 & \text{if } i=k,\ j=l \\ 0 & \text{otherwise} \end{cases}$

define it on basis

<u>rmk</u> $I(U,V) = I\left(\sum_k U^k \partial_k X, \sum_l V^l \partial_l X\right) = \sum_{i,j} g_{ij} U^i V^j$

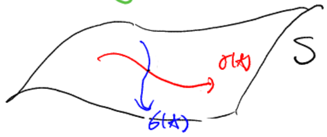
<u>advantage</u> $I = \sum_{i,j} g_{ij}\, dx^i \cdot dx^j$

$$= \sum_{i,j,k,l} \tilde{g}_{kl} \frac{\partial \xi^k}{\partial x^i} \frac{\partial \xi^l}{\partial x^j} dx^i \cdot dx^j = \sum_{k,l} \tilde{g}_{kl}\, d\xi^k \cdot d\xi^l$$

Same rule holds for $\mathbb{I}$.

§IV. Isometry and Gaussian curvature

0°



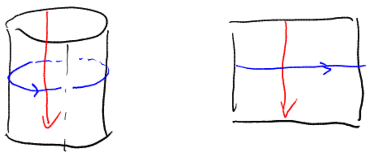
$\gamma(t)$: a curve on the surface

- arc-length $= \int |\gamma'(t)|\, dt = \int \sqrt{I(\gamma'(t), \gamma'(t))}\, dt$
- $\cos(\text{angle}) = \dfrac{I(\gamma'(0), \sigma'(0))}{|\gamma'(0)|\ |\sigma'(0)|}$

<u>Recall</u> plane $[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $[h_{ij}] = 0$, $K=0$, $H=0$

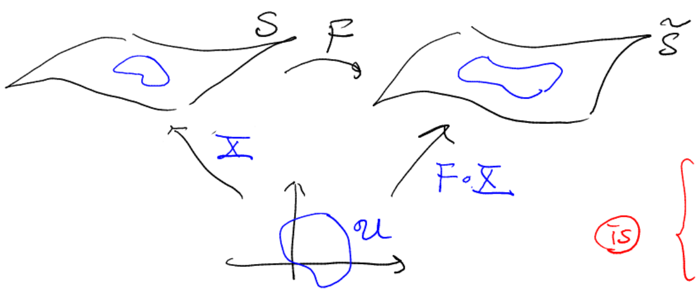
cylinder $[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $[h_{ij}] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $K=0$, $H=\frac{1}{2}$

- They have same g : the same (local) geometry of curves.



- A related question: Think $(\mathcal{U}, X)$ as drawing the "map" for S.
Can we draw a map, which carries the true notion for distance and angles of curves on S?
Mathematically, find $(\mathcal{U}, X)$ with $[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

1° <u>defn</u>



$F: S \to \tilde{S}$ is called a local isometry if

iS $\begin{cases} \langle DF|_p(U), DF|_p(V)\rangle_{T_{F(p)}\tilde{S}} = \langle U, V\rangle_{T_pS} \\ \forall p \in S,\ U, V \in T_pS \end{cases}$

discussion

In terms of coordinate chart. ⑮ $\Rightarrow$ $DF|_p$ must be bijective

By IFT. $F \circ \mathbb{X}$ is also a coordinate for $\tilde{S}$ (by shrinking $\mathcal{U}$)

⑮ $\Rightarrow$ g_{ij} of S and $\tilde{S}$ are the same.

example $F: \mathbb{R}^2 \to$ cylinder of radius 1

$(x, y) \mapsto (\cos y, \sin y, x)$

2° Gauss's Theorema Egregium (latin: Remarkable Theorem)

thm Suppose that $F: S \to \tilde{S}$ is a local isometry

Then, they have the same Gaussian curvature!

Namely, $\forall p \in S$, $K_S(p) = K_{\tilde{S}}(F(p))$

proof: NEXT TIME.

consequence / discussion

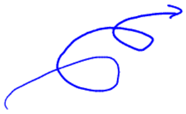
i) Gaussian curvature indeed has nothing to do with the second fundamental form. II.

The full-version proof says that $K =$ algebraic expression of $\partial^2 g_{ij}$, ∂g_{ij}, g_{ij}

ii) This does not happen for curve: We can always reparametrize a curve such that $|\gamma'(s)| = 1$

$\xrightarrow{s}$ $\mathbb{R}^1 \longrightarrow$

straight line



$\gamma(s)$

In other words, $\mathrm{I} = ds \cdot ds \; (= ds^2)$

The curvature of curve DOES depend on how the normal changes

iii) map problem: Can we draw a map of the earth which preserves the concept of length and angles?

Mathematically, construct $\mathcal{U} \subset \mathbb{R}^2 \xrightarrow{F} S^2$, which is a local isometry.

But $K_{\mathbb{R}^2} \equiv 0$, $K_{S^2} = 1$ (or $(\text{radius})^{-2}$)

It is impossible by Theorema Egregium