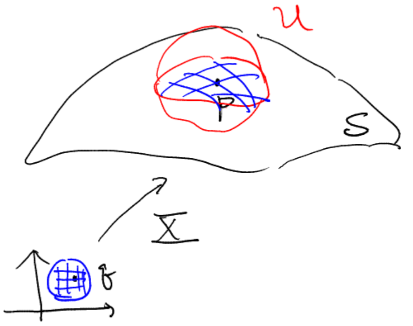
Surface in space   ch. 2 & ch. 4 of [MR]

use multi-variable calculus (and linear algebra)
to study the geometry of surface
→ concept induced by inner product
distances, angles

§I. (regular) surface

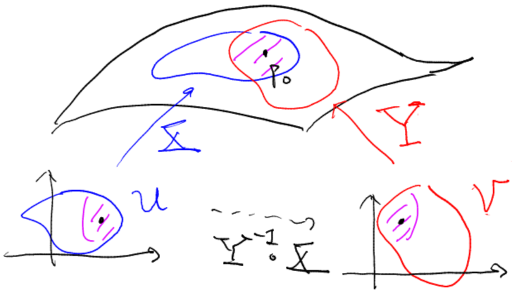
1° <u>defn</u> $S \subset \mathbb{R}^3$ is called
a (regular) surface in $\mathbb{R}^3$ (2-dim'l submanifold of $\mathbb{R}^3$)
if $\forall\ p \in S$, $\exists\ \mathcal{U}$: open set in $\mathbb{R}^2$, V: open neighborhood of p in $\mathbb{R}^3$
and smooth map $\mathbb{X}: \mathcal{U} \to V$
such that
i) $\mathbb{X}(\mathcal{U}) = S \cap V$
ii) $\mathbb{X}: \mathcal{U} \to S \cap V$ is a homeomorphism
iii) $D\mathbb{X}|_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall\ q \in \mathcal{U}$



e.g. For a smooth map $f: \mathbb{R}^3 \to \mathbb{R}^1$,
if $t_0 \in \mathbb{R}^1$ is a regular value
(i.e. $\forall\ p \in f^{-1}(t_0)$, $Df|_p$ is surjective)
then $f^{-1}(t_0)$ is a regular surface provided
that $f^{-1}(t_0) \neq \phi$

Pf: Write $p = (a, b, c)$. Without loss of generality, assume $\partial_z f|_p \neq 0$
By IFT. $\exists\ \mathcal{U}$: open neighborhood of (a,b) in $\mathbb{R}^2$
W: open neighborhoof of c in $\mathbb{R}^1$
and $h: \mathcal{U} \to W$ smooth
such that $f^{-1}(t_0) \cap (\mathcal{U} \times W) = \{ (x, y, h(x,y)) \mid (x,y) \in \mathcal{U} \}$
$\mathbb{X}$ $D\mathbb{X} = \begin{bmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \end{bmatrix}$ #
injective

2° Key feature: S locally looks like open subsets of $\mathbb{R}^2$
near each $p \in S$



$\mathbb{R}^2 \supset \mathcal{U} \to S \subset \mathbb{R}^3$
<u>defn</u> We call $(\mathcal{U}, \mathbb{X})$ a <u>coordinate chart</u>
for S

Suppose that $(\mathcal{U}, \mathbb{X})$ and $(\mathcal{V}, \mathbb{Y})$ are
two coordinate charts with $\mathbb{X}(\mathcal{U}) \cap \mathbb{Y}(\mathcal{V}) \neq \phi$

Consider $(\mathbb{Y}^{-1} \circ \mathbb{X}) : \mathbb{X}^{-1}(\text{◎}) \longrightarrow \mathbb{Y}^{-1}(\text{◎})$
i) By definition, it is homeomorphism
bijective, continuous, inverse also continuous

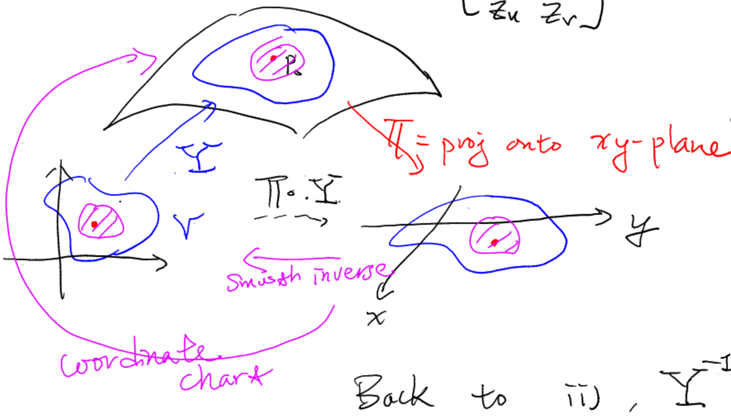
ii) $\mathbb{Y}^{-1} \circ \mathbb{X}$ is smooth

Since smoothness is a local property, it suffices to study this at any $p_0 \in$

Lemma At any $p_0 \in S$, $\exists$ some coordinate plane $(xy, yz,$ or $xz)$ such that near p_0, S is a graph over the coordinate plane, and (of course) the graph description serves as a coordinate chart.

pf: Let $Y : V \to S \subset \mathbb{R}^3$ be a coordinate chart
$(u,v) \mapsto (x,y,z)$

Since $DY = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$ is injective, we may assume w.l.o.g. that $\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\Big|_{Y^{-1}(p_0)}$ is invertible



$\rightsquigarrow$ $\pi \circ Y : V \to \mathbb{R}^2$, $D(\pi\circ Y)\big|_{Y^{-1}(p_0)}$ is invertible

By inverse function theorem
$\pi \circ Y$ (locally) has a smooth inverse
......
#

Back to ii), $Y^{-1} \circ X = (\pi \circ Y)^{-1} \circ \pi \circ X$
$\mathbb{R}^2 \leftarrow \mathbb{R}^2 \leftarrow \mathbb{R}^3 \leftarrow \mathbb{R}^2$
composition of smooth maps $\Rightarrow$ smooth

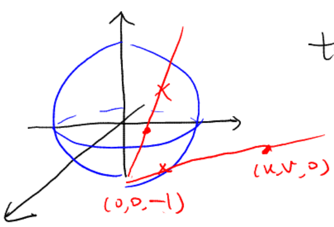
By the same token, $X^{-1} \circ Y$ is also smooth.

rmk key feature + i) + ii) will be the definition for (abstract) manifold.

3° more examples.

i) sphere. $f(x,y,z) = x^2+y^2+z^2-1 \Rightarrow 0$ is a regular value

(Only -1 is NOT a regular value
-2 is a regular value, but $f^{-1}(-2) = \emptyset$)



two standard coordinate chart: stereographic projection

- For any $(u,v) \in \mathbb{R}^2$, consider

$\boxed{\text{compute}}$ $S^2 \cap \{\text{line connecting } (0,0,-1) \ \& \ (u,v,0)\}$

$$\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

Its range is $S^2 \setminus \{(0,0,-1)\}$

- Similarly, for $(\xi,\eta) \in \mathbb{R}^2$ consider

for some purpose explain it later

$S^2 \cap \{\text{line connecting } (0,0,1) \ \& \ (\xi,-\eta,0)\}$

$$\left(\frac{2\xi}{1+\xi^2+\eta^2}, \frac{-2\eta}{1+\xi^2+\eta^2}, \frac{-1+\xi^2+\eta^2}{1+\xi^2+\eta^2}\right)$$

- $X : \mathbb{R}^2 \xrightarrow{\sim} S^2 \setminus \{(0,0,-1)\}$
$(u,v) \longmapsto \frac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2)$
$Y : \mathbb{R}^2 \xrightarrow{\sim} S^2 \setminus \{(0,0,1)\}$
$(\xi,\eta) \longmapsto \frac{1}{1+\xi^2+\eta^2}(2\xi, -2\eta, -1+\xi^2+\eta^2)$

- the transition $\Psi^{-1} \circ X : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$

  Solve (ξ, η) in terms of (u, v)

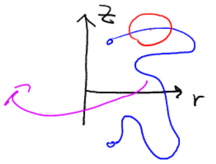
  z-component $\Rightarrow \dfrac{-1 + (\xi^2+\eta^2)}{1+(\xi^2+\eta^2)} = \dfrac{1-(u^2+v^2)}{1+(u^2+v^2)} \Rightarrow -1 - \cancel{(u^2+v^2)} + \cancel{(\xi^2+\eta^2)} + (u^2+v^2)(\xi^2+\eta^2)$
  $= 1 - \cancel{(u^2+v^2)} + \cancel{(\xi^2+\eta^2)} - (u^2+v^2)(\xi^2+\eta^2)$

  $\Rightarrow (\xi^2+\eta^2) = \dfrac{1}{u^2+v^2}$

  x-component $\Rightarrow \xi = \dfrac{1+\xi^2+\eta^2}{1+u^2+v^2}\, u = \dfrac{1+\frac{1}{u^2+v^2}}{1+u^2+v^2}\, u = \dfrac{u}{u^2+v^2}$

  y-component $\Rightarrow \eta = \dfrac{-v}{u^2+v^2}$ ※

ii) surface of revolution



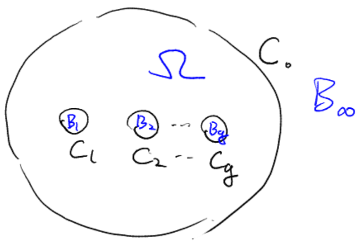
$s \mapsto \gamma(s) = (\alpha(s), \beta(s))$ $\to$ $\left\{\begin{array}{l} \cdot\ \alpha(s) > 0 \\ \cdot\ \gamma'(s) \neq 0 \\ \cdot\ \forall s_0\ \exists\, \mathcal{U}: \text{open nbd of } \gamma(s_0) \text{ and } \delta > 0 \text{ such that} \end{array}\right.$

$\Uparrow$
(a, b)

$\gamma : (s_0-\delta, s_0+\delta) \to \mathcal{U} \cap \gamma((a,b))$ is a homeomorphism.

$\Rightarrow \big(\alpha(s)\cos\theta,\ \alpha(s)\sin\theta,\ \beta(s)\big)$ is a regular surface, which is NOT compact

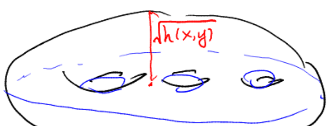
iii) surface of genus $g \geq 1$



$C_0, C_1 \cdots C_g$: circles on $\mathbb{R}^2$ as the picture

Choose $h(x,y) : \mathbb{R}^2 \to \mathbb{R}$ smooth $\to$

$\left\{\begin{array}{l} \cdot\ h < 0 \text{ on } B_1 \cup \cdots \cup B_g \cup B_\infty \\ \cdot\ h > 0 \text{ on } \Omega \\ \cdot\ h = 0, \text{ but } (h_x, h_y) \neq 0 \text{ on } C_1 \cup \cdots \cup C_g \cup C_\infty \end{array}\right.$



$\rightsquigarrow F(x,y,z) = z^2 - h(x,y)$

$\boxed{\text{check}}$ 0 is a regular value $\Rightarrow F^{-1}(0)$ is a regular surface

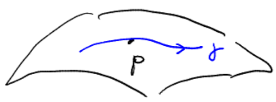
It is closed and bounded in $\mathbb{R}^3$, and thus compact

§II. tangent planes & derivative of functions

0° $\boxed{S \subset \mathbb{R}^3 \text{ a regular surface}}$. $f(x,y,z)$: smooth function

$f : S \to \mathbb{R}$ shall be "smooth", what is its derivative?

At at $p \in S$. let $\gamma(t)$: a curve on S, $\gamma(0) = p$

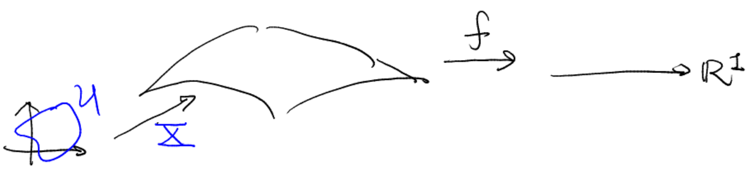


$\rightsquigarrow \left.\dfrac{d}{dt}\right|_{t=0} f(\gamma(t)) = Df\big|_p \cdot \gamma'(0)$ ------ (✝)

$\uparrow$ 1×3 matrix $[f_x\ f_y\ f_z]$ $\quad$ $\uparrow$ 3-vector

point By using coordinate chart. we are actually working with 2-vectors and linear transform on 2-dim'l space

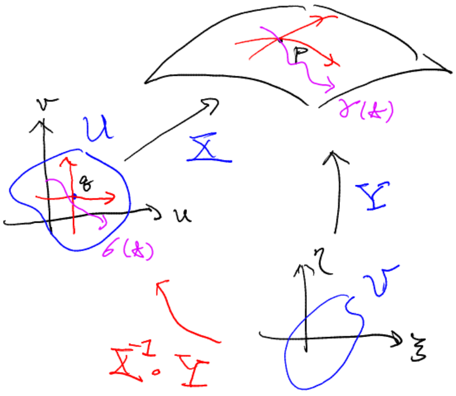
1° defn $f : S \to \mathbb{R}$ is said to be smooth



if $f \circ X$ is smooth on $\mathcal{U}$ $\forall$ coordinate chart $(\mathcal{U}, X)$

2° defn / discussion $\quad \forall p \in S$. T_pS = the tangent plane of S at p

$= \{ \gamma'(0) \mid \gamma(t) : \text{curve on } S,\ \gamma(0) = p \}$



Choose a coordinate chart for p:

$(u,v) \longmapsto X(u,v) = (x(u,v), y(u,v), z(u,v))$

$\sigma(t) = (X^{-1} \circ \gamma)(t) \Rightarrow \gamma(t) = (X \circ \sigma)(t)$

$$\Rightarrow \gamma'(0) = \frac{d}{dt}\Big|_{t=0} (X \circ \sigma)(t)$$

$$= \underbrace{DX|_q}_{3\times 2 \text{ matrix}} \cdot \underbrace{\sigma'(0)}_{2\text{-vector}} \qquad \Rightarrow T_pS \text{ is spanned by columns of } DX|_q,$$

$$\frac{\partial X}{\partial u}\Big|_q,\ \frac{\partial X}{\partial v}\Big|_q$$

(red vectors in picture)

Compare with another choice of chart (V, Y) (on the overlap region)

$$Y = X \circ (X^{-1} \circ Y)$$

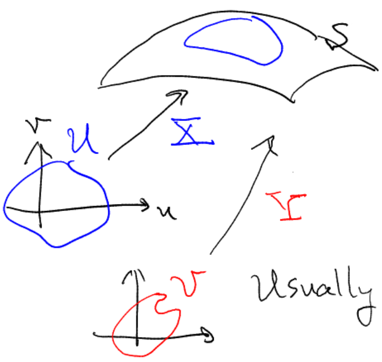
$$\underbrace{DY}_{3\times 2} = \underbrace{DX}_{3\times 2} \cdot \underbrace{D(X^{-1}\circ Y)}_{2\times 2} \qquad \begin{bmatrix} \frac{\partial Y}{\partial \xi} & \frac{\partial Y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{bmatrix}$$

Jacobian = change of basis for T_pS

3° summary / defn $\quad$ If $f : S \to \mathbb{R}$ is smooth.

$Df|_p$ is a linear transform from T_pS to $\mathbb{R}$

Instead of (+), we always use coordinate chart to do the calculation



$f(X(u,v)) : U \subset \mathbb{R}^2 \to \mathbb{R}^1$ is smooth

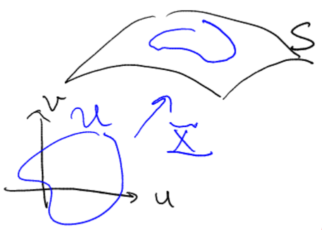
$$\begin{bmatrix} \frac{\partial f(X(u,v))}{\partial u} & \frac{\partial f(X(u,v))}{\partial v} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}}_{1\times 3} \cdot \underbrace{\begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \end{bmatrix}}_{3\times 2} \quad \text{skip this}$$

calculate it directly

Usually just writ it as $\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}$

It is convenient: $\boxed{\text{check}}$ $\begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{bmatrix}$

4° modern notation



U : easier to do computation $\xrightarrow{X}$ S : geometry picture (keep in mind)

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (equivalent to the choice of coordinate)

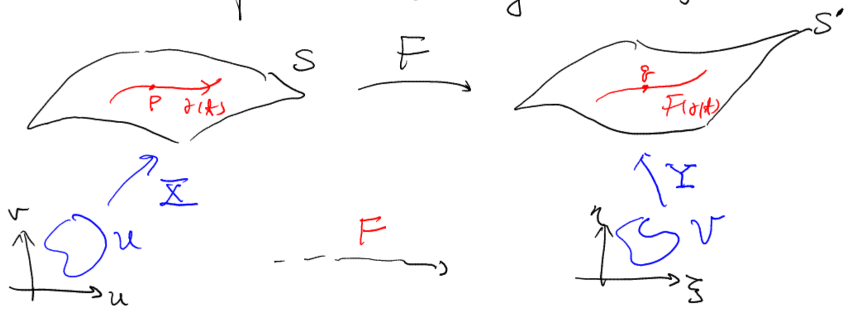
$\frac{\partial}{\partial u}$!! $\quad \frac{\partial}{\partial v}$!! : also viewed them as directional derivative operator $\longleftrightarrow$ $\frac{\partial X}{\partial u} \quad \frac{\partial X}{\partial v}$

dual basis (linear functional)

$du, dv \quad \longleftrightarrow$ the linear functional on T_pS sends $\frac{\partial X}{\partial u} \mapsto 1$, $\frac{\partial X}{\partial v} \mapsto 0$

advantage: computation is just the chain rule

5° smooth map between regular surface



<u>defn</u> $F: S \to S'$ is said to be smooth if $Y^{-1} \circ F \circ X$ is smooth for any coordinate chart

derivative? $\underbrace{\frac{d}{dt} F(\sigma(t))}_{\in\, T_q S'} = DF|_p \cdot \underbrace{\sigma'(0)}_{\in\, T_p S}$ $\Rightarrow$ DF: linear transform from T_pS to T_qS'

Based on previous discussion, DF is equivalent to $D(\underline{Y^{-1} \circ F \circ X})$

We usually abuse the notation, still denote it by F.

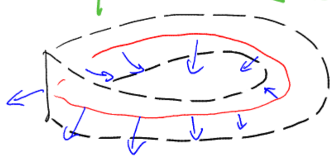
$F = (\xi(u,v), \eta(u,v))$

$$(DF)\left(\frac{\partial}{\partial u}\right) = \begin{bmatrix} \frac{\partial \xi}{\partial u} \\ \frac{\partial \eta}{\partial u} \end{bmatrix} = \frac{\partial \xi}{\partial u}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\partial \eta}{\partial u}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial \xi}{\partial u}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial u}\frac{\partial}{\partial \eta}$$

$$(DF)\left(\frac{\partial}{\partial v}\right) = \frac{\partial \xi}{\partial v}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial v}\frac{\partial}{\partial \eta}$$ (instead of 2x2 matrix this notation is easier for calculation)

§III. orientation and compact surfaces

0° Möbius band



NOT orientable

$$\begin{pmatrix} \left(1+\frac{v}{2}\cos\frac{u}{2}\right)\cos u \\ \left(1+\frac{v}{2}\cos\frac{u}{2}\right)\sin u \\ \frac{v}{2}\sin\frac{u}{2} \end{pmatrix}$$ $u \in [0, 2\pi]$, $v \in (-1, 1)$

Given a coordinate chart $(\underset{(u,v)}{\mathcal{U}}, X)$

$\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \Big/ |\text{itself}|$ is a unit normal defined on $X(\mathcal{U})$

It is of course smooth

1° <u>defn</u> A regular surface is said to be <u>orientable</u> if it admits a smooth unit normal vector field N

<u>lemma</u> $\Leftrightarrow$ $\exists\ \{(\mathcal{U}, X)\}$ covers S such that the Jacobian of any transition has positive determinant

pf: $$\begin{cases} \frac{\partial X}{\partial u} = \frac{\partial Y}{\partial \xi}\frac{\partial \xi}{\partial u} + \frac{\partial Y}{\partial \eta}\frac{\partial \eta}{\partial u} \\ \frac{\partial X}{\partial v} = \frac{\partial Y}{\partial \xi}\frac{\partial \xi}{\partial v} + \frac{\partial Y}{\partial \eta}\frac{\partial \eta}{\partial v} \end{cases} \Rightarrow \left(\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right) = \left(\frac{\partial Y}{\partial \xi} \times \frac{\partial Y}{\partial \eta}\right) \cdot \frac{\partial(\xi,\eta)}{\partial(u,v)}$$ #

<u>defn</u> a choice of N is called an <u>orietation</u> of S

2° <u>lemma</u> If S is orientable and connected, S admits exactly two orientations.

pf: $N \rightsquigarrow -N$, too.

$\tilde{N} \Rightarrow \langle N, \tilde{N} \rangle \in \pm 1$ and continuous #

3° What does a compact regular surface S look like?

(We will not need all of the properties here, but it is helpful to have the picture in mind)

- It is (diffeomorphic to) a genus g surface (classification theorem)
- The Jordan–Brouwer separation theorem:

$\mathbb{R}^3 \setminus S = \Omega_0 \amalg \Omega_\infty$ $\quad \Omega_0$ & Ω_∞ : open and connected

and $\exists R > 0$ such that $\{|x| > R\} \subset \Omega_\infty$

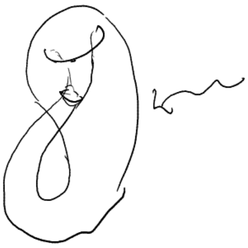
(its proof: depend on the time

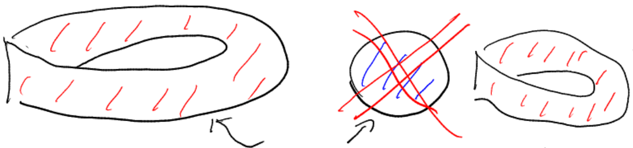
But we will see some key ingredients)

- Locally 

, call the direction pointing toward Ω_∞ the outward normal

4° rmk The Klein bottle = ~~attaching a sphere to the Möbius band~~ glue two Möbius bands together





$\Rightarrow$ non-orientable, compact surface

both boundary is abstractly a circle

It cannot be realized as a regular surface in $\mathbb{R}^3$

but okay in $\mathbb{R}^4$ (see wikipedia)