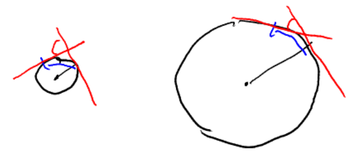
Geometry of plane and space curves   ch.1 of [MR]

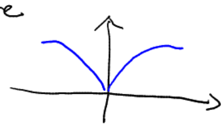
§I. plane curves



1) small circle is "more curved" than the big circle
walk along the circle for 1 unit. see how much angle the tangent line varies

2) Let $\gamma(t) = \tilde{I}$: open interval in $\mathbb{R}^1 \longrightarrow \mathbb{R}^2$ be smooth
$t \longmapsto (x(t), y(t))$ with $\gamma'(t) \neq 0 \ \forall t$

<u>rmk</u> will not discuss $t \mapsto (t^3, t^2)$ here
$x^2 = y^3$



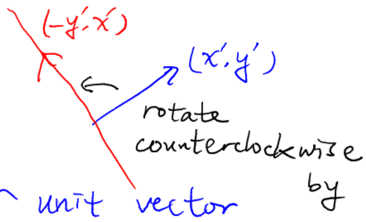
We can always introduce $s = \int_0^t |\gamma'(u)|\, du$
$=$ arc length from $\gamma(0)$ to $\gamma(t)$
as the new variable

Then, $\gamma(s) : I \longrightarrow \mathbb{R}^2$ satisfies $|\gamma'(s)| = 1 \ \forall s$
$s \longmapsto (x(s), y(s))$

use the same notation

always assume this today!

3) $\gamma'(s) = (x'(s), y'(s))$ tangent of the curve.
$\rightsquigarrow$ $\gamma''(s)$ measures how much it is curved
$\langle \gamma'(s), \gamma'(s) \rangle = \langle (x'(s), y'(s)), \text{itself} \rangle = 1$
$\frac{d}{ds} \Rightarrow 2\langle \gamma''(s), \gamma'(s) \rangle = 0$
$\Rightarrow \gamma''(s) \perp \gamma'(s) \Rightarrow \gamma''(s) \mathbin{/\!/} (-y'(s), x'(s))$ unit vector



<u>def</u> The curvature of γ is defined to be
$$K(s) = \langle (x'', y''), (-y', x') \rangle$$

<u>eg.</u> Fix $R > 0$. $\gamma(s) = (R\cos\frac{s}{R}, R\sin\frac{s}{R})$ circle of radius R
$\gamma'(s) = (-\sin\frac{s}{R}, \cos\frac{s}{R})$
$\gamma''(s) = -\frac{1}{R}(\cos\frac{s}{R}, \sin\frac{s}{R})$
$\Rightarrow K(s) = 1/R$ (smaller radius, larger curvature)

4) <u>Lemma</u> if $K(s) \equiv 0$, $\gamma(s)$ is contained in a straight line
pf: Since $(x''(s), y''(s)) = K(s)(-y'(s), x'(s))$. $\gamma''(s) = (x''(s), y''(s)) \equiv 0$
$\Rightarrow \gamma'(s) = $ constant vector $= (a, b)$
$\Rightarrow \gamma(s) = \gamma(0) + (a, b)s$ ※

5) a rigid motion is $x \longmapsto Ax + b$ for $A \in O(2)$, $b \in \mathbb{R}^2$
b: translation
A: rotation & reflection
$AA^T = I \Rightarrow \det A = \pm 1$

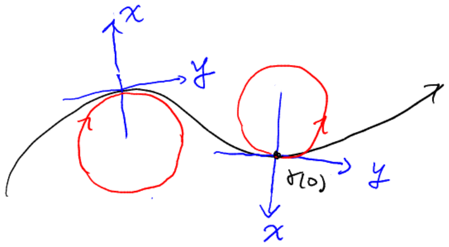
If $\det A > 0$,
γ and $A\gamma + b$ have the same curvature

If $\det A < 0$, they differ by a minus sign ()

6) Suppose that $\kappa(0) \neq 0$

After choosing suitable A and b, we may assume

$\gamma(0) =$ origin, $\gamma'(0) = (0,1)$, $\kappa(0) > 0$ $\Rightarrow$ $\gamma''(0) = \kappa_0(-1,0) = (-\kappa_0, 0)$



Consider the circle

① passing through $\gamma(0)$

② having the same tangent at $\gamma(0)$

③ with radius $1/\kappa_0$

$$\Rightarrow \quad \sigma(s) = \left(-\frac{1}{\kappa_0}, 0\right) + \frac{1}{\kappa_0}\left(\cos(\kappa_0 s), \sin(\kappa_0 s)\right)$$

$\sigma(0) = (0,0)$

$\sigma'(0) = (0,1)$

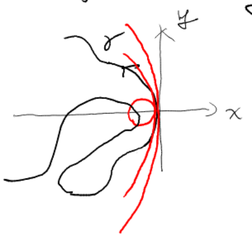
$\sigma''(0) = (-\kappa_0, 0)$ same

This circle is called the osculating circle (密切圓) at $\gamma(0)$

It is tangent to γ to 2^{nd} order (in the above sense)

$\boxed{\text{key}}$ curvature : 2^{nd} order derivative

7) If we only require ① & ②.



Consider the relation between $\gamma(s)$ and the circle

$$\left(-\frac{1}{\lambda}, 0\right) + \frac{1}{\lambda}\left(\cos(\lambda s), \sin(\lambda s)\right) \quad \left(\text{radius } \frac{1}{\lambda}\right)$$

From the picture, $\gamma(s)$: outside the circle when $1/\lambda$: small

$\gamma(s)$: inside the circle when $1/\lambda$: large

Justification :

$$f_\lambda(s) = \operatorname{dist}^2\left(\gamma(s), \left(-\frac{1}{\lambda}, 0\right)\right) = \left(x(s) + \frac{1}{\lambda}\right)^2 + y^2(s)$$

$$\left.\frac{d}{ds}\right|_{s=0} f_\lambda(s) = 2\left(x(0) + \frac{1}{\lambda}\right) \underset{0}{\underset{\parallel}{x'(0)}} + 2\underset{0}{\underset{\parallel}{y(0)}}\, \underset{1}{\underset{\parallel}{y'(0)}} = 0$$

$$\left.\frac{d^2}{ds^2}\right|_{s=0} f_\lambda(s) = 2\Big(\underset{0}{\underset{\parallel}{x(0)}} + \frac{1}{\lambda}\Big) \underset{-\kappa_0}{\underset{\parallel}{x''(0)}} + 2(\underset{0}{\underset{\parallel}{x'(0)}})^2 + 2\underset{0}{\underset{\parallel}{y(0)}}\, \underset{0}{\underset{\parallel}{y''(0)}} + 2(\underset{1}{\underset{\parallel}{y'(0)}})^2$$

$$= 2\left(1 - \frac{\kappa_0}{\lambda}\right)$$

Therefore, $2\left(1 - \frac{\kappa_0}{\lambda}\right) > 0 \iff \frac{1}{\lambda} < \frac{1}{\kappa_0}$ $\quad \left(\frac{1}{\lambda} : \text{small}\right)$

$\Rightarrow$ $f_\lambda(s)$ has local minimum at $s = 0$

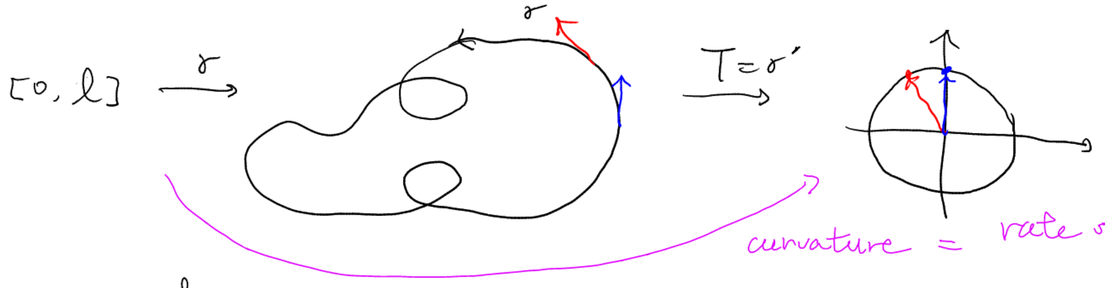
$\Rightarrow$ $\gamma(s)$ lies outside the circle ※

8) tangent $T(s) = (x'(s), y'(s))$ $\quad (x')^2 + (y')^2 = 1$

locally, define $\theta(s) = \arctan \frac{y'(s)}{x'(s)}$ multi-value

$$\tan\theta = \frac{y'}{x'} \Rightarrow \underset{\frac{(x')^2 + (y')^2}{(x')^2}}{\underset{\parallel}{\sec^2\theta}} \cdot \theta' = \frac{y''x' - x''y'}{(x')^2} \Rightarrow \theta' = \langle (x'', y''), (-y', x') \rangle$$

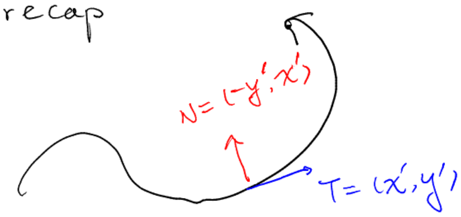
$$= \kappa$$



curvature = rate of change of the angle

$\Rightarrow \int_0^l \kappa \, ds$ = total change of angles of $T(s)$
$\in 2\pi\mathbb{Z}$ (if $T(0)=T(l)$)

9) recap



Both T and N are first order derivatives

$\langle T, N\rangle = 0$

$\Rightarrow \langle T', N\rangle + \langle T, N'\rangle = 0$

$\Rightarrow \kappa = -\langle T, N'\rangle$ ← an equivalent definition

§II. implicit function theorem

0° For a $m\times(n+m)$ matrix S, it is full rank iff surjective.
By the rank-nullity theorem, $\ker S$ is n-dim'l

1° Implicit function theorem

$F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ smooth. Suppose that $DF|_0$ is full rank
$0 \mapsto 0$

Then, $F^{-1}(0)$ near 0 is locally given by
the graph of m functions with n-variables

$(n+m)$ freedoms $\longrightarrow$ cut out m-freedoms $\longrightarrow$ n free variables

Say $DF = m\begin{bmatrix} \overset{n}{D_xF} & \vdots & \overset{m}{D_yF} \end{bmatrix}$ Then, the x-coordinates can be used as the n-variables
(D_yF ← invertible)

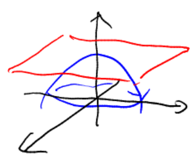
e.g. $F(x,y,z) = x^2+y^2+z^2-1$

$DF = [\,2x \quad 2y \quad 2z\,]$: full rank except the origin

@ $p=(0,0,1)$

$DF|_p = [\,0 \quad 0 \quad 2\,]$

$\ker DF|_p = \operatorname{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ $\rightsquigarrow$ the tangent plane of $F^{-1}(0)$ at p



recall Suppose that $F(p)=0$, $DF|_p$: full rank
The tangent space of $F^{-1}(0)$ at p
is $\{\, \gamma'(0) \mid \gamma(t) \in F^{-1}(0), \ \gamma(0)=p \,\}$

discussion $F(\gamma(t)) = 0 \Rightarrow$ by chain rule, $DF|_p \cdot \gamma'(0) = 0$
$\Rightarrow$ tangent space $\subset \ker DF|_p$

For "$\supset$". assume $DF|_p = m\overset{n}{[D_1F}\ \overset{m}{D_2F]}|_p$

$D_2F|_p$ is invertible

IFT $\Rightarrow$ $\exists\ g_1,\cdots,g_m$ = functions in $(x^1,x^2,\cdots,x^n)$

such that $F^{-1}(0)$ near $p=(0,\cdots,0,\cdots,0)$

$(x^1,\cdots,x^n,\ g_1(x),\cdots,g_m(x))$

For each $j\in\{1,\cdots,n\}$. consider $\gamma(t)=(0,\cdots,t,\cdots,0,\ g_1(0,\cdots,t,\cdots0)\cdots)$

$\gamma'(0)=(0,\cdots,1,\cdots,0,\ *,\cdots,*)$

$\Rightarrow$ dim(tangent space) $\geq n$ ※

§III* O(3) and SO(3)

$O(3)=\{A\in M_{3\times3}(\mathbb{R})\mid A^TA=I\}$ $\Rightarrow$ $\det A=\pm1$

$SO(3)=\{A\in O(3)\mid \det A=1\}$

0° $\langle\ ,\ \rangle$: standard inner product on $\mathbb{R}^3$

$A\in O(3)\iff \langle Au,Av\rangle=\langle u,v\rangle\quad\forall\,u,v\in\mathbb{R}^3$

(geometry : study the properties induced by distance & angle)

O(3) : preserving the "geometric properties"

1° $\boxed{\text{check}}$ $O(3)=SO(3)\amalg SO(3)\cdot\begin{bmatrix}-1&&\\&1&\\&&1\end{bmatrix}$

$M_{3\times3}(\mathbb{R})\cong\mathbb{R}^9$ $\qquad\qquad \in Sym_{3\times3}(\mathbb{R})\cong\mathbb{R}^6$

$A\longmapsto A^TA-I$: always symmetric

$\leadsto$ $F:\mathbb{R}^9\to\mathbb{R}^6$, $F^{-1}(0)=O(3)$

Can we apply IFT to this F ?

let $E^{ij}=\begin{array}{c}\\ i\end{array}\begin{array}{c} j\\ \begin{bmatrix}0&\vdots&0\\ \cdots&1&\\ 0&&0\end{bmatrix}\end{array}$ (i,j)-entry $=1$ others $=0$ $\quad i,j\in\{1,2,3\}$

2° At $I\in O(3)$ $\quad \left.\dfrac{\partial F}{\partial x^{ij}}\right|_I=$?

$\gamma(t)=I+tE^{ij}$

$F(\gamma(t))=(I+tE^{ij})^T(I+tE^{ij})-I$

$\left.\dfrac{\partial F}{\partial x^{ij}}\right|_I=\left.\dfrac{d}{dt}\right|_{t=0}\left((I+tE^{ij})^T(I+tE^{ij})-I\right)=(E^{ij})^T+E^{ij}$

One can see $DF|_I: M_{3\times3}(\mathbb{R})\to Sym_{3\times3}(\mathbb{R})$ is surjective

$\begin{bmatrix}\frac12&0&0\\0&0&0\\0&0&0\end{bmatrix}\longmapsto\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $\qquad \frac12 m\longmapsto \overset{\forall}{m}$

$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\longmapsto\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$ $\qquad m=m^T$

Hence one can use three variables to describe O(3) near I

Also, the tangent space of $O(3)$ at $\mathbb{I}$ is

$$\ker DF|_{\mathbb{I}} = \text{skew-Sym}_{3\times 3}(\mathbb{R}) \cong \mathbb{R}^3$$

3° At any other $A \in O(3)$?

Similarly, compute $\frac{d}{dt}\Big|_{t=0}\left((A + tE^{ij})^T (A + tE^{ij}) - \mathbb{I}\right)$

$$= (E^{ij})^T A + A^T E^{ij}$$

$$= (A^{-1}E^{ij}) + (A^{-1}E^{ij})^T$$

$\left(\begin{matrix} A \in O(3) \\ A^{-1} = A^T \end{matrix}\right)$

i.e. $DF|_A : M_{3\times 3}(\mathbb{R}) \longrightarrow \text{Sym}_{3\times 3}(\mathbb{R})$

$$m \longmapsto (A^{-1}m)^T + (A^{-1}m)$$

This is surjective: $\forall B$ with $B = B^T$

Take $m = \frac{1}{2} AB$

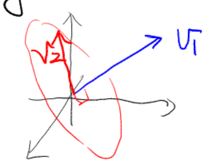
$\Rightarrow \ker DF|_A = \{ m \in M_{3\times 3}(\mathbb{R}) \mid (A^{-1}m)^T + (A^{-1}m) = 0 \}$

($\Rightarrow$ $O(3)$ is a 3-dim'l manifold (later))

4° $A \in O(3)$ $\Rightarrow$ $A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$ v_1, v_2, v_3 : orthonormal

$|v_1| = 1 \Rightarrow v_1 \in S^2 \subset \mathbb{R}^3 \rightsquigarrow$ 2-dim'l freedom

After choosing v_1, $v_2 \perp v_1$ and of unit length



$\rightarrow$ 1-dim'l freedom

$\Rightarrow v_3 = \pm v_1 \times v_2$

only 2 choices. 0-dim'l freedom

5° Now. $P(t) : (-1, 1) \longrightarrow O(3) \subset M_{3\times 3}(\mathbb{R})$

$(P(t))^T P(t) - \mathbb{I} = 0 \overset{\frac{d}{dt}}{\Longrightarrow} P^{-1}(t) P'(t)$ is skew-symmetric

$\rightsquigarrow P^{-1}(t) P'(t) : (-1, 1) \longrightarrow \text{Skew-Sym}_{3\times 3}(\mathbb{R})$

$\boxed{Q}$ Given $Q(t) : (-1, 1) \longrightarrow \text{Skew-Sym}_{3\times 3}(\mathbb{R})$

Can we find $P(t)$ such that $P^{-1}(t) P'(t) = Q(t)$?

6° $\boxed{\text{Yes}}$ $P'(t) = P(t) Q(t)$ is a linear system

If we write $P(t)$ as a 9×1 vector $X(t)$

the system takes the form $\underset{9\times 1}{\frac{d}{dt} X(t)} = \underset{9\times 9}{\tilde{Q}(t)} \underset{9\times 1}{X(t)}$

By $\boxed{\text{ODE}}$, solution always exists and is unique (on $\underline{X}(0)$)
Since RHS is homogeneous in $\underline{X}$ of degree 1
the solution exists on where $\tilde{Q}(t)$ is defined
(for general ODE, the existence is only on a small interval)

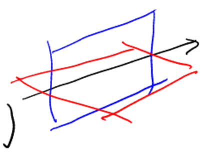
§ IV space curve : Frenet frame

$\gamma(s) : I \longrightarrow \mathbb{R}^3$

$s \longmapsto (x(s), y(s), z(s))$ with $|\gamma'(s)| = 1 \quad \forall s$

1° How far is it from a plane curve?
At each point, find the "best fit" 2-plane.
Then, compare those 2-planes

- if $\gamma(s) \subset$ straight line $\rightsquigarrow$ too many planes



$(*)$ $\boxed{\text{Assume } \gamma''(s) \neq 0}$. (very far from being straight)

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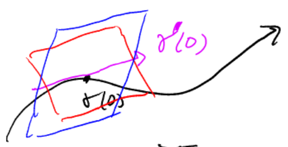


$\gamma(s)$ which plane? what is its normal?
At least, perpendicular to $\gamma'(0)$

Try Passing through $\gamma(0)$, $\gamma(s)$, containing the direction $\gamma'(0)$. Then take $s \to 0$

Normal : $V_0(s) = \gamma'(0) \times (\gamma(s) - \gamma(0))$ limit as $s \to 0$?

$V_0(0) = 0$

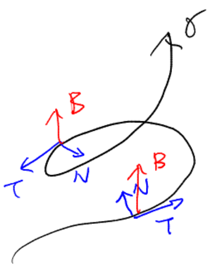
$V_0'(0) = \gamma'(0) \times \gamma'(0) = 0$

$V_0''(0) = \gamma'(0) \times \gamma''(0)$ Non zero?

- Assumption : $\langle \gamma'(s), \gamma'(s) \rangle = 1$, $\gamma''(s) \neq 0$.

$\Rightarrow \langle \gamma''(0), \gamma'(0) \rangle = 0$

$\gamma'(0) \perp \gamma''(0)$ both nonzero $\Rightarrow B_0''(0) \neq 0$



Taylor : $\gamma'(0) \times (\gamma(s) - \gamma(0)) = \frac{1}{2} (\gamma'(0) \times \gamma''(0)) s^2 + O(s^3)$ for s: small

the plane is spanned by $\gamma'(0)$ & $\gamma''(0)$

L'Hôpital $\Rightarrow \lim_{s \to 0} \dfrac{\gamma'(0) \times (\gamma(s) - \gamma(0))}{|\text{ same }|} = \dfrac{\gamma'(0) \times \gamma''(0)}{|\text{ same }|}$

2° Frenet frame : Assume $(*)$

Let $T(s) = \gamma'(s)$, $N(s) = \dfrac{\gamma''(s)}{|\gamma''(s)|} = \dfrac{T'}{\kappa}$, normal

$B(s) = T(s) \times N(s)$ bi-normal

- Similar to plane curve, define curvature

$\kappa(s) = |T'(s)| = |\gamma''(s)| > 0$

- For 1°. look at $B'(s)$ use T, N, B as basis

$$\langle B', T\rangle = \langle \underbrace{T'}_{\parallel N} \times N, T\rangle + \underbrace{\langle T \times N', T\rangle}_{0} = 0$$

$$\langle B', N\rangle =: \tau \quad \text{torsion of } \gamma$$

$$\langle B, B\rangle = 1 \Rightarrow \langle B', B\rangle = 0$$

- $T' = \kappa N$, $B' = \tau N$

$N' = ?$

$$\langle N, N\rangle = 1 \Rightarrow \langle N', N\rangle = 0$$
$$\langle N, T\rangle = 0 \Rightarrow \langle N', T\rangle = -\langle N, T'\rangle = -\kappa$$
$$\langle N, B\rangle = 0 \Rightarrow \langle N', B\rangle = -\langle N, B'\rangle = -\tau$$

$$\Rightarrow N' = -\kappa T - \tau B$$

- $$\underset{\substack{1\times 3\\(3)}}{[T\ N\ B]'} = \underset{\substack{1\times 3\\(3)}}{[T\ N\ B]}\underset{3\times 3}{\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}} \qquad (**)$$

3° $\gamma(s) : I \to \mathbb{R}^3$ with $(*)$

$\rightsquigarrow P(s) = [T\ N\ B] : I \to SO(3)$ satisfying $(**)$

- $\gamma(s)$ is determined by $P(s)$ up to translation

$$\gamma(s) = \int T(s)\,ds + (\text{translation})$$

- From §III. 6°

$P(s)$ is determined by $\kappa(s)$ & $\tau(s)$ up to the choice of $P(0)$

More precisely, if $P'(s) = P(s)\begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}$

and $P(0) = \mathbb{I}$

$\Rightarrow A\cdot P(s)$ is solution $\forall A \in SO(3)$

<u>Thm</u> a space curve with $(*)$ is determined by curvature and torsion up to direct rigid motion

$x \mapsto Ax + b \quad A \in O(3)$

$\det A > 0$

<u>rmk</u> similar thm holds for plane curve in fact. that is easier

<u>prop</u> if $\tau \equiv 0$, γ is a plane curve

pf: $\tau \equiv 0 \Rightarrow B$: constant vector

$$\frac{d}{ds}\langle \gamma, B\rangle = \langle T, B\rangle = 0 \Rightarrow \langle \gamma, B\rangle = \text{const} \quad \divideontimes$$