Mayer–Vietoris sequence [BT, §I.2 and §I.5]

§I. some homology algebra (for vector space over $\mathbb{R}$)

1° <u>defn</u> $0 \to V_0 \xrightarrow{T_1} V_1 \xrightarrow{T_2} \cdots \to V_{k-1} \xrightarrow{T_k} V_k \xrightarrow{T_{k+1}} V_{k+1} \to \cdots$

V_j : vector spaces $/\mathbb{R}$. T_j : linear transforms.

It is called an exact sequence if $\ker T_{k+1} = \operatorname{im} T_k \quad \forall k$

2° <u>lemma</u> Suppose there is an exact sequence

$\to V_{k-2} \xrightarrow{T_{k-1}} V_{k-1} \xrightarrow{T_k} V_k \xrightarrow{T_{k+1}} V_{k+1} \xrightarrow{T_{k+2}} V_{k+2} \to \cdots$ with V_{k-1}, V_{k+1} : finite dim'l

Then, V_k is finite dim'l as well, and $V_k \cong \underbrace{\operatorname{coker} T_{k-1}}_{\substack{\| \\ V_{k-1}/\operatorname{im} T_{k-1}}} \oplus \ker T_{k+2}$

$V_{k-1}/\operatorname{im} T_{k-1}$: complement of image.

pf: i) Since $\dim V_{k-1} < \infty$ and $\ker T_k = \operatorname{im} T_{k-1}$,

$\dim V_{k-1}/\operatorname{im} T_{k-1} < \infty$. and $\operatorname{coker} T_{k-1} = V_{k-1}/\operatorname{im} T_{k-1} \xrightarrow{[T_k]} V_k$ is injective

Moreover, its image is the same as image T_k, which is also the same as $\ker T_{k+1}$

ii) Since $T_{k+1}(V_k) = \ker(T_{k+2})$ and $\dim V_{k+1} < \infty$

$V_{k+1} \supset \ker T_{k+2} \cong V_k/\ker T_{k+1}$, which is finite dim'l

iii) Hence, $V_k \cong V_k/\ker T_{k+1} \oplus \ker T_{k+1}$

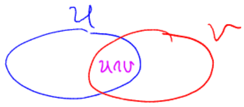
$\cong \underbrace{\ker T_{k+2}}_{(ii)} \oplus \underbrace{V_{k-1}/\operatorname{im} T_{k-1}}_{(i)}$ ⌘

3° <u>rmk</u> This may not be true for $\mathbb{Z}$-module

$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ but $\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2$

injective surjective

§II Mayer–Vietoris sequence



<u>goal</u> $M = U \cup V$ U & V : open subset of M (hence manifolds as well)

↖ manifold w/o boundary

compute $H^k_{dR}(M)$ from $H^k_{dR}(U)$, $H^k_{dR}(V)$ and $H^k_{dR}(U \cap V)$

1° short exact sequence of differential forms

- $\eta \in \Omega^k(M)$. ⟵ $\eta|_U$, $\eta|_V$ determines η
- Given $\eta_1 \in \Omega^k(U)$, $\eta_2 \in \Omega^k(V)$: if $\eta_1|_{U\cap V} = \eta_2|_{U\cap V}$

  then $\exists\, \eta \in \Omega^k(M) \to \eta|_U = \eta_1$, $\eta|_V = \eta_2$

In fact, there is a (short) exact sequence

$$0 \to \Omega^k(M) \xrightarrow{r} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{s} \Omega^k(U\cap V) \to 0$$

$$\eta \longmapsto (\eta|_U,\ \eta|_V)$$

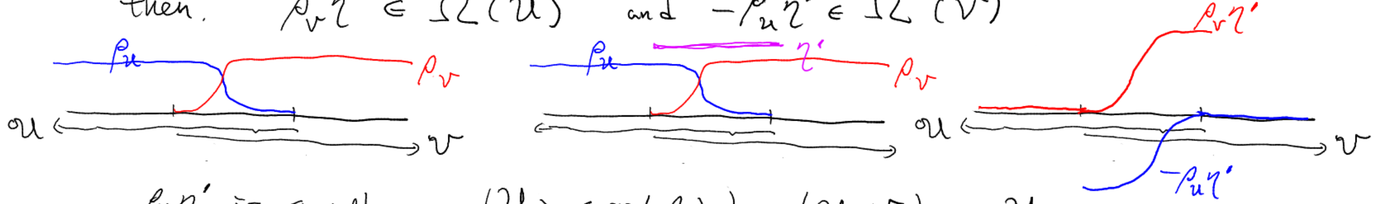
$$(\eta_1,\ \eta_2) \longmapsto \eta_1|_{U\cap V} - \eta_2|_{U\cap V}$$

check exactness : i) $\longrightarrow$ is clearly injective

ii) if $\eta_1 = \eta_2$ on $\mathcal{U}\cap\mathcal{V}$, we can let $\eta = \begin{cases} \eta_1|_{\mathcal{U}} \\ \eta_2|_{\mathcal{V}} \end{cases}$

iii) $\forall\ \eta' \in \Omega^k(\mathcal{U}\cap\mathcal{V})$, let $\rho_{\mathcal{U}}, \rho_{\mathcal{V}}$: partition of unity subordinate to $\{\mathcal{U},\mathcal{V}\}$

then. $\rho_{\mathcal{V}}\eta' \in \Omega^k(\mathcal{U})$ and $-\rho_{\mathcal{U}}\eta' \in \Omega^k(\mathcal{V})$



$\rho_{\mathcal{V}}\eta'$ is smooth on $(\mathcal{U}\setminus \mathrm{supp}(\rho_{\mathcal{V}})) \cup (\mathcal{U}\cap\mathcal{V}) = \mathcal{U}$

and $\rho_{\mathcal{V}}\eta' - (-\rho_{\mathcal{U}}\eta')\big|_{\mathcal{U}\cap\mathcal{V}} = (\rho_{\mathcal{U}}+\rho_{\mathcal{V}})\eta' = \eta'$

2° prop $M = \mathcal{U}\cup\mathcal{V}$ induces the following (long) exact sequence

$$\to H^n(M) \to H^n(\mathcal{U}) \oplus H^n(\mathcal{V}) \to H^n(\mathcal{U}\cap\mathcal{V}) \to 0$$

$$\cdots$$

$$\to H^k(M) \to H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \to H^k(\mathcal{U}\cap\mathcal{V})$$

$$\xrightarrow{s} H^{k-1}(\mathcal{U}) \oplus H^{k-1}(\mathcal{V}) \xrightarrow{r} H^{k-1}(\mathcal{U}\cap\mathcal{V}) \xrightarrow{d^*}$$

$$\cdots$$

$$0 \to H^0(M) \to H^0(\mathcal{U}) \oplus H^0(\mathcal{V}) \to H^0(\mathcal{U}\cap\mathcal{V})$$

pf: i) Since $\mathcal{U}\cap\mathcal{V} \subset \mathcal{U}, \mathcal{V} \subset M$. $H^k(M) \to H^k(\mathcal{U}) \oplus H^k(\mathcal{V})$ and $H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \to H^k(\mathcal{U}\cap\mathcal{V})$ is given by ι^* / restriction (with subtraction)

ii) d^*: the map from $H^{k-1}(\mathcal{U}\cap\mathcal{V})$ to $H^k(M)$

$$\begin{array}{ccccccccc}
 & & \tilde{\eta} & & d(\rho_{\mathcal{V}}\eta') \quad -d(\rho_{\mathcal{U}}\eta') & & 0 & & \\
0 \to & & \Omega^k(M) & \xrightarrow{r} & \Omega^k(\mathcal{U}) \oplus \Omega^k(\mathcal{V}) & \xrightarrow{s} & \Omega^k(\mathcal{U}\cap\mathcal{V}) & \to 0 \\
 & & d\uparrow & & d\uparrow \qquad \uparrow d & & \uparrow d & & \\
0 \to & & \Omega^{k-1}(M) & \xrightarrow{r} & \Omega^{k-1}(\mathcal{U}) \oplus \Omega^{k-1}(\mathcal{V}) & \xrightarrow{s} & \Omega^{k-1}(\mathcal{U}\cap\mathcal{V}) & \to 0 \\
 & & & & \rho_{\mathcal{V}}\eta' \qquad -\rho_{\mathcal{U}}\eta' & & \overset{\Downarrow}{\eta'} & &
\end{array}$$

$[\eta'] \in H^{k-1}(\mathcal{U}\cap\mathcal{V})$, $d\eta' = 0$

$\Rightarrow (\rho_{\mathcal{V}}\eta', -\rho_{\mathcal{U}}\eta') \in \Omega^{k-1}(\mathcal{U}) \oplus \Omega^{k-1}(\mathcal{V})$ $\quad s(\rho_{\mathcal{V}}\eta', \rho_{\mathcal{U}}\eta') = \eta'$

$\Rightarrow (d(\rho_{\mathcal{V}}\eta'), -d(\rho_{\mathcal{U}}\eta')) \in \Omega^k(\mathcal{U}) \oplus \Omega^k(\mathcal{V}) \xmapsto{s} d\eta' = 0$

$\boxed{\text{check}}$ diagram commute

$\Rightarrow$ Hence. $\exists\ \tilde{\eta} \in \Omega^k(M)$.

$\to \tilde{\eta}|_{\mathcal{U}} = d(\rho_{\mathcal{V}}\eta')$ and $\tilde{\eta}|_{\mathcal{V}} = -d(\rho_{\mathcal{U}}\eta')$

From these expressions. $d\tilde{\eta} = 0$. define $d^*[\eta'] = [\tilde{\eta}] \in H^k_{dR}(M)$

$\boxed{\text{Caveat}}$ $\tilde{\eta}$ is exact on $\mathcal{U}$ and $\mathcal{V}$.

But need not to be exact on $M = \mathcal{U}\cap\mathcal{V}$

Otherwise d^* is always a trivial map.

iii) <u>check</u> well-definedness of d^*

Two choices are made in the construction: $\begin{cases}\text{representative of } [\eta'] \\ s^{-1}\eta'\end{cases}$

• For $s^{-1}\eta'$, if $s(\eta_1,\eta_2)=\eta'$

$$\Rightarrow \exists\, \theta \in \underset{\text{global!}}{\Omega^{k-1}(M)}, \Rightarrow \begin{cases}\eta_1 = \rho_v\eta' + \theta|_{U\cap V}\\ \eta_2 = \rho_u\eta' + \theta|_{U\cap V}\end{cases} \Rightarrow [\tilde\eta] = [\tilde\eta + d\theta]$$

• If $[\eta'] = [d\phi'] \in H^{k-1}_{dR}(U\cap V) \Rightarrow [\tilde\eta] \overset{?}{=} 0$

$$\begin{array}{ccccccccc}
 & & & & d(\rho_v\phi') & & -d(\rho_u\phi') & & \eta' \\
0 & \to & \Omega^{k-1}(M) & \to & \Omega^{k-1}(U) & \oplus & \Omega^{k-1}(V) & \to & \Omega^{k-1}(U\cap V) \to 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d \\
0 & \to & \Omega^{k-2}(M) & \to & \Omega^{k-2}(U) & \oplus & \Omega^{k-2}(V) & \to & \Omega^{k-2}(U\cap V) \to 0 \\
 & & & & \rho_v\phi' & & -\rho_u\phi' & & \phi'
\end{array}$$

$$s\big(d(\rho_v\phi'), -d(\rho_u\phi')\big) = \big(d(\rho_v\phi') + d(\rho_u\phi')\big)\big|_{U\cap V} = d\phi'\big|_{U\cap V} = \eta'$$

$$\Rightarrow \tilde\eta\,|_U = d(d(\rho_v\phi')) = 0, \quad \tilde\eta\,|_V = -d(d(\rho_u\phi')) = 0 \quad \Rightarrow [\tilde\eta] = 0$$

iv) exactness: $\to H^{k-1}(U\cap V) \xrightarrow{d^*} H^k(M) \xrightarrow{r} H^k(U)\oplus H^k(V) \to$

$\ker r \overset{?}{\supseteq} \operatorname{im} d^*$. From the construction, $\operatorname{im} d^* \subset \ker r$

$\ker r \overset{?}{\subseteq} \operatorname{im} d^*$

$$\begin{array}{cccccccc}
 & & \xi \in & & & & & \\
 & 0 \to & \Omega^k(M) & \to & \Omega^k(U) & \oplus & \Omega^k(V) & \\
 & & & & & & & \\
0 \to & & \Omega^{k-1}(M) & \to & \Omega^{k-1}(U) & \oplus & \Omega^{k-1}(V) & \to \Omega^{k-1}(U\cap V) \to 0 \\
 & & & & \eta_1 & & \eta_2 & \eta = s(\eta_1,\eta_2)
\end{array}$$

Given $[\xi] \in H^k_{dR}(M)$ with $r([\xi]) = 0$

$\Rightarrow \xi|_U = d\eta_1, \quad \xi|_V = d\eta_2$

Let $\eta = (\eta_1 - \eta_2)|_{U\cap V}$. Then, $d^*[\eta] = [\xi]$

v) exactness: $H^{k-1}(U)\oplus H^{k-1}(V) \xrightarrow{s} H^{k-1}(U\cap V) \xrightarrow{d^*} H^k(M)$

$\operatorname{im} s \subset \ker d^*$: $([\eta_1],[\eta_2]) \in H^{k-1}(U)\oplus H^{k-1}(V)$, $d\eta_1 = 0 = d\eta_2$

$$\begin{array}{ccccccc}
0 \to & \Omega^k(M) & \to & \Omega^k(U) & \oplus & \Omega^k(V) & \\
 & 0 & \leftarrow & 0 & & 0 & \\
 & & & \updownarrow & & \updownarrow & \\
0 \to & \Omega^{k-1}(M) & \to & \Omega^{k-1}(U) & \oplus & \Omega^{k-1}(V) & \to \Omega^{k-1}(U\cap V) \to 0 \\
 & & & \eta_1 & & \eta_2 & \overset{s}{\mapsto}\ \eta_1 - \eta_2
\end{array}$$

$\ker d^* \subset \operatorname{im} s$

$$\begin{array}{ccccccc}
0 \to & \Omega^k(M) & \to & \Omega^k(U) & \oplus & \Omega^k(V) & \\
 & \tilde\eta & & (d\eta_1 & , & d\eta_2) & 0 \\
 & \updownarrow & & & & & \updownarrow \\
0 \to & \Omega^{k-1}(M) & \to & \Omega^{k-1}(U) & \oplus & \Omega^{k-1}(V) & \to \Omega^{k-1}(U\cap V) \to 0 \\
 & \theta & & (\eta_1 & , & \eta_2) & \eta
\end{array}$$

$d^*([\eta]) = 0$ in $H^k(M) \Rightarrow \exists\, \theta \in \Omega^{k-1}(M) \Rightarrow \tilde\eta = d\theta$

Then. $d(\eta_1 - \theta) = d\eta_1 - \tilde\eta|_U = 0 \Rightarrow [\eta_1 - \theta] \in H^{k-1}(U)$

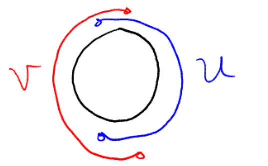
$d(\eta_2 - \theta) = d\eta_2 - \tilde\eta|_V = 0 \Rightarrow [\eta_2 - \theta] \in H^{k-1}(V)$

and $s(\eta_1 - \theta, \eta_2 - \theta) = s(\eta_1,\eta_2) = \eta \Rightarrow \ker d^* \subset \operatorname{im} s$

vi) exactness of $H^k(M) \to H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) \to H^k(\mathcal{U}\cap\mathcal{V})$
is easier (does not involve d^*) $\boxed{\text{DIY}}$ ※

§III some examples

1° $H^*_{dR}(S^1)$



$\mathcal{U} \cong \mathbb{R}^1$, $\mathcal{V} \cong \mathbb{R}^1$, $\mathcal{U}\cap\mathcal{V} \cong$ two copies of $\mathbb{R}^1$

$H^1(S^1) = d^*(H^0(\mathcal{U}\cap\mathcal{V})) \cong H^0(\mathcal{U}\cap\mathcal{V})/_{\mathrm{Im}\, S} \cong \mathbb{R}^1$

$\to H^1(S^1) \to 0$

$0 \to \underbrace{H^0(S^1)}_{\{\text{constant functions}\} = \mathbb{R}^1} \to \underbrace{H^0(\mathcal{U}) \oplus H^0(\mathcal{V})}_{\mathbb{R}^1\oplus\mathbb{R}^1} \xrightarrow{S} \underbrace{H^0(\mathcal{U}\cap\mathcal{V})}_{\mathbb{R}^1\oplus\mathbb{R}^1}$

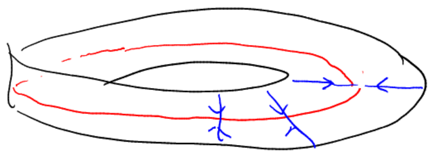
image $= \mathbb{R}\langle(1,1)\rangle$
coker $\cong \mathbb{R}\langle(1,-1)\rangle$

2° cylinder $S^1 \times \mathbb{R}^1 \underset{\iota}{\overset{\pi}{\rightleftarrows}} S^1$ $\qquad \pi(e^{i\theta}, s) = e^{i\theta}$
$\iota(e^{i\theta}) = (e^{i\theta}, 0)$

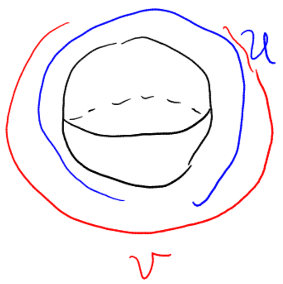
$\pi \circ \iota = \mathbb{1}_{S^1}$, $\quad \iota \circ \pi$: homotopic to $\mathbb{1}_{S^1\times\mathbb{R}^1}$

$\Rightarrow H^k_{dR}(S^1 \times \mathbb{R}^1) \cong H^k_{dR}(S^1)$

<u>rmk</u> same argument works for Möbius band (w/o boundary



3° $H^*_{dR}(S^{n>1})$



$\mathcal{U} = S^n \setminus \{\text{south pole}\} \cong \mathbb{R}^n$
$\mathcal{V} = S^n \setminus \{\text{north pole}\} \cong \mathbb{R}^n$

$\mathcal{U}\cap\mathcal{V} = S^n \setminus \{S, N\}$
$\cong S^{n-1} \times \mathbb{R}^1$

$\to H^n(S^n) \to 0$

$H^{n-1}(S^n) \to H^{n-1}(\mathcal{U}) \oplus H^{n-1}(\mathcal{V}) \to \overset{\mathbb{R}^1}{H^{n-1}(\mathcal{U}\cap\mathcal{V})}$

$\to H^1(S^n) \to 0 \qquad\qquad 0$

$0 \to H^0(S^n) \to \underbrace{H^0(\mathcal{U})}_{\mathbb{R}^1} \oplus \underbrace{H^0(\mathcal{V})}_{\mathbb{R}^1} \xrightarrow{\text{surjective}} \underbrace{H^0(\mathcal{U}\cap\mathcal{V})}_{\mathbb{R}^1}$

$\Rightarrow H^n(S^n) \cong \mathbb{R}^1$, $\quad H^k(S^n) = 0$, $\quad k \in \{2, 3, \dots, n-1\}$

$H^1(S^n) = 0$, $\quad H^0(S^n) \cong \mathbb{R}^1$

§IV. about closed, oriented manifold (sketching the keys)

M^n : closed (compact, boundaryless), connected, oriented manifold.

$\boxed{Q}$ Why is $H^k_{dR}(M)$ always finite dimℓ ?
Why is $\int \cdot \wedge \cdot : H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}^1$ non-degenerate?

1° M : closed $\Rightarrow$ $\exists$ finite good cover ([BT, theorem 5.1])

namely. $\{\mathcal{U}_\ell\}_{\ell=1}^{L}$: finite open cover

$\ni$ any non-empty $\mathcal{U}_{j_1} \cap \cdots \cap \mathcal{U}_{j_k}$ is diffeomorphic to $\mathbb{R}^n$

e.g.

<u>key prop</u> [BT, prop 5.3.1]

If a manifold (need not to be closed) has finite good cover, H^*_{dR} are all finite dim'l

pf: 0) Induction on L + Mayer-Vietoris + lemma in §1.2.

1) $(\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k) \cap \mathcal{U}_{k+1}$ has a good cover consisting of $\{\mathcal{U}_j \cap \mathcal{U}_{k+1}\}_{j=1}^{k}$ ... #

2° Poincaré duality

- In algebraic topology, one usually considers the "homology theory" as well, but there is no such gadget in differential forms formulation.
- Instead, Bott & Tu introduced the "compactly supported" de Rham theory.

$$\Omega^k_c(M) = \{\text{smooth } k\text{-form with compact support}\}$$

$$H^k_c(M) = \ker d|_{\Omega^k_c} \Big/ d\Omega^{k-1}_c$$

e.g. $\mathbb{R}^1$, $\Omega^0_c = \{\text{compact supported } C^\infty \text{ functions}\}$

$\Rightarrow \ker d = 0 \quad \Rightarrow \quad H^0_c(\mathbb{R}^1) = 0$

$\Omega^1_c = \{h(x)\,dx : \operatorname{supp}(h) = \text{compact}\}$

$h(x)\,dx = df$ for some $f \in \Omega^0_c \iff \int_{-\infty}^{\infty} h(x)\,dx = 0$

$$\Rightarrow \quad \Omega^1_c \longrightarrow \mathbb{R}^1, \quad h(x)dx \mapsto \int_{-\infty}^{\infty} h(x)\,dx \qquad \ker = d\Omega^0_c \quad \Rightarrow \quad H^1_c(\mathbb{R}^1) \cong \mathbb{R}^1$$

Note that $\int - \wedge - : H^k(\mathbb{R}^1) \times H^{1-k}_c(\mathbb{R}^1) \longrightarrow \mathbb{R}^1$ is non-degenerate

- H^k_c satisfies a slightly different Mayer-Vietoris sequence

$$\to H^k_c(\mathcal{U}\cap\mathcal{V}) \longrightarrow H^k_c(\mathcal{U}) \oplus H^k_c(\mathcal{V}) \to H^k_c(\mathcal{U}\cup\mathcal{V}) \xrightarrow{d^*} H^{k+1}_c(\mathcal{U}\cap\mathcal{V}) \to H^{k+1}_c(\mathcal{U}) \oplus H^{k+1}_c(\mathcal{V})$$

(short exact)

d^* : $\quad 0 \leftarrow \Omega^k_c(\mathcal{U}\cup\mathcal{V}) \leftarrow \Omega^k_c(\mathcal{U}) \oplus \Omega^k_c(\mathcal{V}) \leftarrow \Omega^k_c(\mathcal{U}\cap\mathcal{V}) \leftarrow 0$

$(\eta, \eta) \longleftarrow\!\shortmid \eta$

$\eta_1 - \eta_2 \longleftarrow\!\shortmid (\eta_1, \eta_2)$

Given $\xi \in \Omega^k_c(\mathcal{U}\cup\mathcal{V})$ with $d\xi = 0$ $\rightsquigarrow$ ξ come from $(\rho_u\xi, -\rho_v\xi)$

Taking d : $\quad 0 \leftarrow (d(\rho_u\xi), -d(\rho_v\xi)) \longleftarrow d^*\xi$

$0 \in \Omega^{k+1}_c(\mathcal{U}\cup\mathcal{V})$

· pairing with the Mayer-Vietories of H^*_{dR}

$$\begin{array}{cccccccccccc}
\to H_c^{n-k}(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & H_c^{n-k}(\mathcal{U}) \oplus H_c^{n-k}(\mathcal{V}) & \longrightarrow & H_c^{n-k}(\mathcal{U}\cup\mathcal{V}) & \xrightarrow{d^*} & H_c^{n-k+1}(\mathcal{U}\cap\mathcal{V}) & \longrightarrow & H_c^{n-k+1}(\mathcal{U}) \oplus H_c^{n-k+1}(\mathcal{V}) \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
\leftarrow H^k(\mathcal{U}\cap\mathcal{V}) & \leftarrow & H^k(\mathcal{U}) \oplus H^k(\mathcal{V}) & \leftarrow & H^k(\mathcal{U}\cup\mathcal{V}) & \xleftarrow{d^*} & H^{k-1}(\mathcal{U}\cap\mathcal{V}) & \leftarrow & H^{k-1}(\mathcal{U}) \oplus H^{k-1}(\mathcal{V})
\end{array}$$

$\int - \wedge -$

By algebra: if these four parts are non-degenerate (and finite dim), so is the middle part.

$\Rightarrow$ $H^k(M) \otimes H_c^{n-k}(M) \to \mathbb{R}^1$: non-degenerate

if M : compact. $H_c^{n-k}(M) \equiv H^{n-k}(M)$