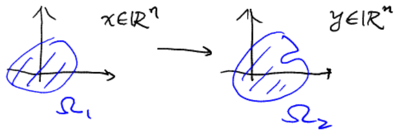
Stokes theorem and Poincaré lemma [BT. §I.3 and §I.4]

recall $y = y(x)$ diffeomorphism



$$\int_{\Omega_2} f(y)\, dy^1\cdots dy^n = \int_{\Omega_1} f(y(x)) \left|\det \frac{\partial(y^1,\cdots,y^n)}{\partial(x^1,\cdots,x^n)}\right| dx^1\cdots dx^n$$

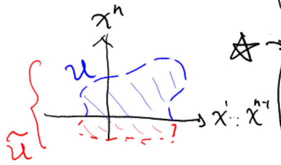
Riemann (or Lebesgue) integral

§I. orientation and boundary

! not in the sense of topological space

1° defn M : manifold with boundary : paracompact and Hausdorff.

- $\forall p \in M$. $\exists$ open neighborhood that is homeomorphic to an open subset of $\mathbb{H}^n = \{(x^1,\cdots,x^n)\in\mathbb{R}^n \mid x^n \geq 0\}$
- tranisition function are smooth

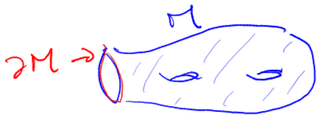


☆ (a function / map f defined on on open subset set $\mathcal{U}$ of $\mathbb{H}^n$ is said to be smooth if $\exists\ \tilde{\mathcal{U}}$: open in $\mathbb{R}^n$, $\tilde{\mathcal{U}}\cap\mathbb{H}^n = \mathcal{U}$. $\exists\ \tilde{f}$: smooth on $\tilde{\mathcal{U}}$. $\ni$ $\tilde{f}|_{\mathcal{U}} = f$)

- interior points $= \{ p \in M \mid \exists$ open nbd homeomorphic to open sets in $\mathbb{R}^n\}$

$\partial M = M \setminus \{$interior points$\}$

2° lemma M^n: manifold w/ boundary $\Rightarrow$ ∂M is a manifold w/o boundary of dimension $n-1$



pf: $\boxed{\text{DIY}}$

3° a differential form is said to be smooth if ☆ (i.e. locally $\exists$ smooth extension)

recall $\boxed{\text{HW}}$ M is orientable if $\exists\ \omega \in \Omega^n(M)$ nowhere zero

$\Leftrightarrow$ $\exists$ coordinate cover $\ni$ det (Jacobian of transition) > 0

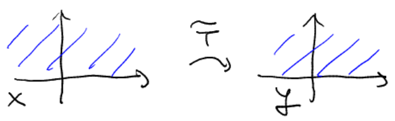
lemma $\tilde{T}$: $\mathbb{H}^n \circlearrowleft$ diffeomorphism and $\det(\mathrm{Jac}(\tilde{T})) > 0$

Then, $T = \tilde{T}|_{\mathbb{R}^{n-1}\times\{0\}}$ is also a diffeomorphism and $\det(\mathrm{Jac}(T)) > 0$

pf. Denote $\mathbb{R}^{n-1}\times\{0\}$ by $\partial\mathbb{H}^n$.

By IFT, $\tilde{T} : \mathbb{H}^n \setminus \partial\mathbb{H}^n \circlearrowleft$. Hence, $\tilde{T} : \partial\mathbb{H}^n \circlearrowleft$

In terms of coordinate, $\tilde{T}$ is given by $y = y(x)$



defined on some open set in $\mathbb{R}^n$ containing $\mathbb{H}^n$.

and
$$\begin{cases} y^n(x^1,\cdots,x^{n-1},0) = 0 \\ \det\left[\dfrac{\partial(y^1,\cdots,y^n)}{\partial(x^1,\cdots,x^n)}\right] > 0 \end{cases}$$

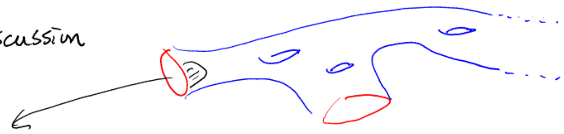
On $\partial\mathbb{H}^n \ni (x^1,\cdots,x^{n-1},0)$,
$$\det(\mathrm{Jac}(\tilde{T})) = \det\left[\begin{array}{ccc|c} & \mathrm{Jac}(T) & & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline \dfrac{\partial y^n}{\partial x^1}\Big|_{\partial\mathbb{H}^n} = 0 & \cdots & 0 & \dfrac{\partial y^n}{\partial x^n}\Big|_{\partial\mathbb{H}^n} \end{array}\right] > 0$$

Hence, $\det(\mathrm{Jac}(T)) \cdot \dfrac{\partial y^n}{\partial x^n}\Big|_{\partial\mathbb{H}^n} > 0$

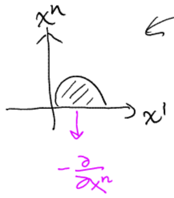
Since $\tilde{T}$ maps $\mathbb{H}^n$ to $\mathbb{H}^n$, $\left.\frac{\partial y^n}{\partial x^n}\right|_{\partial \mathbb{H}^n} > 0 \Rightarrow \det(\mathrm{Jac}(T)) > 0$ ※

Cor If M is orientable, then ∂M is also orientable.

discussion



On a connected, oriented manifold, there are two equivalent classes of orientations



Suppose that M is orientable w/ a choice of (equivalent class of) orientation ω. Let N be a connected component of ∂M

On a coordinate chart. $\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n \qquad f(x) \neq 0$

defn Define the induced (class of) orientation by

outward direction $\iota\left(-\frac{\partial}{\partial x^n}\right)\omega = \underline{(-1)^n f(x)\, dx^1 \wedge \cdots \wedge dx^{n-1}}$

From the proof of the lemma, the equivalent class is well-defined

§II. integration and Stokes theorem

1° $f : N^n \to M^m$ a smooth map between manifolds

$\eta \in \Omega^k(M) \rightsquigarrow f^*(\eta) \in \Omega^k(N)$

$$\left(\begin{array}{l} \forall p \in N \quad T_pN \xrightarrow{f_*} T_{F(p)}M \\ \rightsquigarrow T_p^*N \xleftarrow{f^*} T^*_{F(p)}M \rightsquigarrow \Lambda^k(T_p^*N) \xleftarrow{f^*} \Lambda^k(T^*_{F(p)}M) \\ \forall v_1, \cdots, v_k \in T_pN. \quad (f^*(\eta))(v_1, \cdots, v_k) = \eta|_{F(p)}(f_* v_1, \cdots, f_* v_k) \end{array}\right)$$

$\boxed{\text{check}}$ $\overset{\in N}{d} f^*(\eta) = f^*(\overset{\in M}{d}\eta)$

The proof is the same as the well-definedness of d

2° defn M^n w/ orientation, $\eta \in \Omega^n(M)$ w/ compact support

Define $\int_M \eta$ as follows:

Let $\{\rho_\alpha\}_\alpha$: partition of unity, $\mathrm{supp}\, \rho_\alpha \subset$ some coordinate chart

$\Rightarrow \rho_\alpha \eta$: still has compact support.

$\Rightarrow \rho_\alpha \eta = f_\alpha(x)\, dx^1 \wedge \cdots \wedge dx^n$ compact support

use the coordinate $\ni dx^1 \wedge \cdots \wedge dx^n$ is (equivalent to) the orientation

$\Rightarrow$ Define $\int_M \rho_\alpha \eta$ by $\int_{\mathbb{R}^n} f_\alpha(x)\, dx^1 \cdots dx^n$ Riemann integral!

Finally $\int_M \eta := \sum_\alpha \int_M \rho_\alpha \eta$

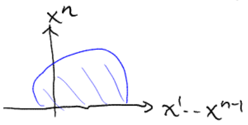
rmk
- change of variable formula for integration $\Rightarrow \int_M \rho_\alpha \eta$ does not depend on the coordinate
- By the common refinement argument, $\int_M \eta$ is independent of the choice of $\{\rho_\alpha\}_\alpha$
- reversing the orientation $\Rightarrow$ flip the sign of $\int_M \eta$

3° <u>thm</u> (Stokes) M^n: manifold w/ boundary and orientation. $\eta \in \Omega^{n-1}(M)$ w/ compact support. Then $\int_M d\eta = \int_{\partial M} \eta$

pf: i) $\eta = \sum_\alpha \rho_\alpha \eta$ Since $\operatorname{supp}\eta$ is compact, this is a finite sum

$d\eta = \sum_\alpha d(\rho_\alpha \eta)$ $\quad \int_M d\eta = \sum_\alpha \int_M d(\rho_\alpha\eta) \overset{?}{=} \sum_\alpha \int_{\partial M} \rho_\alpha \eta$ It suffices to prove it for each $\rho_\alpha \eta$

ii) Assume $\operatorname{supp}(\eta) \subset$ some chart



- By $x^1 \to -x^1$, we may assume $dx^1 \wedge \cdots \wedge dx^n$ is the orientation.
  $\Rightarrow$ The induced orientation on $\mathbb{R}^{n-1} \times \{0\}$ is $(-1)^n dx^1 \wedge \cdots \wedge dx^n$
- $\eta = \sum_{j=1}^{n} (-1)^{j-1} \eta_j(x)\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$
  $\hookrightarrow$ smooth with compact support

iii) $d\eta = \left(\sum_j \frac{\partial \eta_j}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n$

$x^n = 0$ on $\partial M \Rightarrow dx^n = 0$ on ∂M

$\int_M d\eta = \int_0^\infty \left(\int_{\mathbb{R}^{n-1}} \left(\sum_j \frac{\partial \eta_j}{\partial x^j}\right) dx^1 \cdots dx^{n-1} \right) dx^n$, $\quad \eta|_{\partial M} = \eta_n(x) \left((-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1}\right)$

$\Rightarrow \int_{\partial M} \eta = -\int_{\mathbb{R}^{n-1}} \eta_n(x)\, dx^1 \cdots dx^{n-1}$

Riemann integrals

For $j \neq n$ $\quad \int_{\mathbb{R}^{n-1}} \frac{\partial \eta_j}{\partial x^j} dx^1 \cdots dx^{n-1} = \int_{\mathbb{R}^{n-2}} \left(\int_{-\infty}^{\infty} \frac{\partial \eta_j}{\partial x^j} dx^j \right) dx^1 \cdots \widehat{dx^j} \cdots dx^{n-1} = 0$

$= \eta_j(\infty) - \eta_j(-\infty) = 0$ by compact supp.

For $j = n$ $\quad \int_0^\infty \frac{\partial \eta_n}{\partial x^n} dx^n = \cancel{\eta_n(\infty)}^{\,0} - \eta_n(0)$

$\Rightarrow \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\partial \eta_n}{\partial x^n} dx^n\, dx^1 \cdots dx^{n-1} = -\int_{\mathbb{R}^{n-1}} \eta_n\, dx^1 \cdots dx^{n-1}$ ※

4° <u>rmk</u> $\boxed{\text{HW}}$

- M = curve in $\mathbb{R}^3$. η = function $\Rightarrow$ fundamental theorem of vector line integral
- M = surface ∂M = curve, η = 1-form $\Rightarrow$ (classical) Stokes theorem
- M = domain in $\mathbb{R}^3$. ∂M = surface, η = 2-form $\Rightarrow$ divergence theorem

LHS: only differential forms and orientation $\quad$ RHS: need inner product $\Rightarrow$ normal, area, etc.

5° Now, shift the focus to de Rham cohomology.

$\boxed{\text{Assumption}}$ M: compact and boundaryless, connected with an orientation
usually call closed manifold

<u>lemma</u> $\int_M$ descends to a map on $H^n_{dR}(M) \longrightarrow \mathbb{R}$

pf: $H^n_{dR}(M) = \Omega^n(M) / d\Omega^{n-1}(M)$

$\quad\quad\quad \Downarrow$ (n-form) $\quad$ ((n-1)-form)

$[\eta] = [\eta + d\theta] \Rightarrow \int_M \eta \overset{?}{=} \int_M \eta + d\xi$

$\Leftrightarrow \int_M d\xi = \int_{\partial M = \phi} \xi = 0$ ※

lemma $\int \cdot \wedge \cdot$ descends to a map on $H^k_{dR}(M) \times H^{n-k}_{dR}(M) \longrightarrow \mathbb{R}$

pf: $H^k_{dR} = \text{closed } k\text{-forms } (\ker(d|_{\Omega^k})) \big/ \text{exact } k\text{-forms } (d\Omega^{k-1})$

$\Downarrow$

$[\eta + d\xi] = [\eta]$, also $[\tilde{\eta} + d\tilde{\xi}] = [\tilde{\eta}] \in H^{n-k}_{dR}$

$$\int_M (\eta + d\xi) \wedge (\tilde{\eta} + d\tilde{\xi}) \overset{?}{=} \int_M \eta \wedge \xi$$

$$\Leftarrow \quad 0 = \int d\xi \wedge \tilde{\eta} = \int \eta \wedge (d\tilde{\xi}) = \int d\xi \wedge d\tilde{\xi}$$

$$\underset{k-1}{d}(\xi \wedge \underset{n-k}{\tilde{\eta}}) = d\xi \wedge \tilde{\eta} + (-1)^{k-1} \xi \wedge \cancel{d\tilde{\eta}}^{\,0}$$ Since $\tilde{\eta} \in \ker(\tilde{d})$

$$\Rightarrow \quad 0 = \int_{\partial M = \phi} \xi \wedge \tilde{\eta} = \int_M d(\xi \wedge \tilde{\eta}) = \int_M d\xi \wedge \tilde{\eta} \quad \cancel{\times}$$

thm (Poincaré duality in de Rham cohomology) Under $\boxed{\text{assumption}}$
the above pairing $H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$ is non-degenerate
Hence. $H^{n-k}_{dR}(M) \cong (H^k_{dR}(M))^*$, $\Rightarrow H^n_{dR}(M) \cong (H^0_{dR}(M))^* \cong \mathbb{R}$

[sketch the (one) proof next week] constant functions

§ II. homotopy and Poincaré lemma

goal $H^k_{dR}(\mathbb{R}^n) = \mathbb{R}$ if $k=0$. and trivial for other k

0° M, N : manifold w/o boundary (for simplicity)
$f: M \to N \Rightarrow f^*: \Omega^k(N) \to \Omega^k(M)$
Since $f^* d = d f^* \Rightarrow f^*$ descends to $H^k_{dR}(N) \xrightarrow{f^*} H^k_{dR}(M)$

$$[\eta + d\xi] \longmapsto [f^*\eta + d f^*\xi]$$

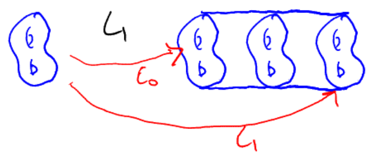
still d-closed
well-definedness

I° defn $f_0, f_1 : M \to N$ are said to be homotopic
if $\exists\ F: M \times [0,1] \to N$ such that $\begin{cases} F(-,0) = f_0 \\ F(-,0) = f_1 \end{cases}$

prop If f_0 and f_1 are homotopic,
then $f_0^* = f_1^*$ as a map from $H^k_{dR}(N)$ to $H^k_{dR}(M)$

pf: $\iota_0(p) = (p, 0)$
$\iota_1(p) = (p, 1)$

$$\begin{tikzcd} M \arrow[r, shift left, "\iota_0"] \arrow[r, shift right, "\iota_1"'] & M \times [0,1] \arrow[r, "F"] & N \end{tikzcd}$$



$f_0 = F \circ \iota_0$
$f_1 = F \circ \iota_1$

$\iota_0^* : \Omega^k(M \times [0,1]) \to \Omega^k(M)$: restriction on $M \times \{0\}$

$\eta \in \Omega^k(M)$ with $d\eta = 0$

$\Rightarrow\ F^*\eta = \underbrace{\alpha(t)}_{\text{only } dx^j,\ \text{no } dt} + dt\wedge\beta(t)$ where $\alpha(t)\in\Omega^k(M)$ and $\beta(t)\in\Omega^{k-1}(M)$

1-parameter family of k & $k-1$ forms

$\alpha(t) = \sum_{i_1,\dots,i_k} S_{i_1\cdots i_k}(x,t)\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ in coordinate

$$0 = F^* d\eta = dF^*\eta$$
$$= d_M\alpha(t) + dt\wedge\frac{\partial\alpha(t)}{\partial t} - dt\wedge d_M\beta(t)$$

$$\Rightarrow \begin{cases} d_M\alpha(t) = 0 \\ \dfrac{\partial\alpha(t)}{\partial t} = d_M\beta(t)\end{cases}$$

Note that $f_0^*\eta = (F\circ\iota_0)^*(\eta) = \iota_0^*(F^*\eta)$
$= \alpha(0) \in \Omega^k(M)$

$f_1^*\eta = \alpha(1)$

$$\Rightarrow\ \alpha(1)-\alpha(0) = \int_0^1 d_M\beta(t)\,dt$$
$$= d_M\Big(\int_0^1 \beta(t)\,dt\Big)$$

a path of $(k-1)$-forms

$\Rightarrow\ [f_0^*\eta] = [f_0^*\eta] \in H^k_{dR}(M)$ ※

2° lemma (Poincaré) $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$, $H^{k\neq 0}_{dR}(\mathbb{R}^n) = 0$

pf: $n=1$: by single variable calculus.

$n>1$

$$\mathbb{R}^{n-1}\times\mathbb{R}^1 \underset{S}{\overset{\pi}{\rightleftarrows}} \mathbb{R}^{n-1}$$

$\pi(x^1,\dots,x^{n-1},x^n) = (x^1,\dots,x^{n-1})$

$S(x^1,\dots,x^{n-1}) = (x^1,\dots,x^{n-1},0)$

$\pi\circ S = \mathbb{1}_{\mathbb{R}^{n-1}} \Rightarrow S^*\circ\pi^* = \mathbb{1}$ on $H^k_{dR}(\mathbb{R}^{n-1})$

It remains to check $\pi^*\circ S^* = \mathbb{1}$ on $H^k_{dR}(\mathbb{R}^n)$

$S\circ\pi : \mathbb{R}^n \to \mathbb{R}^n$
$(x^1,\dots,x^{n-1},x^n) \mapsto (x^1,\dots,x^{n-1},0)$

Clearly, $S\circ\pi$ is homotopic to $\mathbb{1}$ by $(1-t)(S\circ\pi)+t\,\mathbb{1}$

$(x^1,\dots,x^{n-1},x^n)\mapsto(x^1,\dots,x^{n-1},tx^n)$

By prop, $(\pi\circ S)^* = S^*\circ\pi^*$
$= \mathbb{1}$ on $H^k_{dR}(\mathbb{R}^n)$ ※

3° remark on the argument in [BT, §I.4]

Some homology algebra: $\mu_0,\mu_1 : \Omega^j \to \tilde\Omega^j$ commute with d.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0 \arrow[r, "d"] \arrow[d, shift right, "\mu_0"'] \arrow[d, shift left, "\mu_1"] & \cdots \arrow[r] & \Omega^{j-1} \arrow[r, "d"] & \Omega^j \arrow[r, "d"] \arrow[dl, "K"'] \arrow[d, shift right, "\mu_0"'] \arrow[d, shift left, "\mu_1"] & \Omega^{j+1} \arrow[r] \arrow[dl, "K"'] & \cdots \\
0 \arrow[r] & \tilde\Omega^0 \arrow[r, "d"] & \cdots \arrow[r] & \tilde\Omega^{j-1} \arrow[r, "d"] & \tilde\Omega^j \arrow[r, "d"] & \tilde\Omega^{j+1} \arrow[r] & \cdots
\end{tikzcd}$$

μ_0 is "chain homotopic" to μ_1 if $\exists K : \Omega^j \to \tilde\Omega^{j-1}$
$\Rightarrow \mu_0-\mu_1 = \pm dK \pm Kd$

lemma chain homotipic $\Rightarrow$ $\mu_0 = \mu_1$ on H^j to $\tilde H^j$

Compare with the prop: $\mu_0 = f_0^*$, $\mu_1 = f_0^*$. only apply to ker d
$\Rightarrow f_0^* - f_1^* = \pm dK$

crystalised as a homological algebra lemma

$\Rightarrow$ K can be constructed explicitly if there is a homotopy in the level of manifold.