exterior algebra and differential forms

§0. prelude

$\boxed{Q}$ What can be integrated over a manifold? function?
What is Green / Stokes theorem? (integration by parts)

1° recall 3×3 determinant can be regarded as a tri-linear function on $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$: $(u,v,w)\mapsto \det\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix}$

Besides tri-linearity, it is also "alternative":

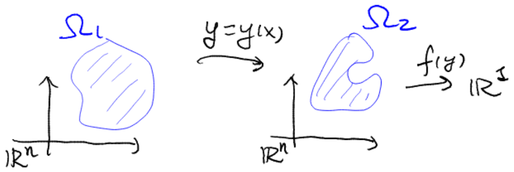
$$\det(u,v,w) = -\det(v,u,w) = -\det(u,w,v) = \cdots$$

$\rightsquigarrow$ gives the volume of the parallelepiped spanned by u, v, w

2° recall determinant of 2×2-minor $\mathbb{R}^3\times\mathbb{R}^3 \ni u, v \mapsto \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$

- it is a alternative bilinear function
- gives the area of the parallelogram. (projection onto the xy-plane)

3° recall change of variable formula



$$\int_{\Omega_2} f(y)\, dy^1\cdots dy^n \quad \longleftarrow \text{Riemann (or Lebesgue) integral}$$

$$= \int_{\Omega_1} f(y(x)) \left|\det\left[\frac{\partial y}{\partial x}\right]\right| dx^1\cdots dx^n$$

§I. some linear algebra

$V \cong \mathbb{R}^n$. n-dim'l vector space over $\mathbb{R}$

Let's use the notation in geometry: let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be a basis of V

$$V = \{ \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \mid a_i \in \mathbb{R}\}$$

a_i: scalar, not yet function

$\rightsquigarrow \{dx^i\}_{i=1}^n$ be the dual basis of V^*

1° $S: \underbrace{V\times V\times\cdots\times V}_{k} \to \mathbb{R}$ is said to be multi-linear if it is linear in each argument

i.e. $S(v_1,\cdots, av_j+\tilde{v}_j,\cdots) = aS(v_1,\cdots,v_j,\cdots) + S(v_1,\cdots,\tilde{v}_j,\cdots)$

$\Rightarrow \mathcal{J}^k(V^*) = \{$ multi-linear functions $\}$ is a vector space of $\dim = n^k$

It has the following basis: $dx^{i_1}\otimes\cdots\otimes dx^{i_k}$

where $(dx^{i_1}\otimes\cdots\otimes dx^{i_k})\left(\frac{\partial}{\partial x^{j_1}},\cdots,\frac{\partial}{\partial x^{j_k}}\right) = \delta^{i_1}_{j_1}\cdots\cdots\delta^{i_k}_{j_k}$

rmk In general, if S: multi-linear functions on $\overbrace{V\times V\times\cdots\times V}^{k}$

$\tilde{S}$: multi-linear functions on $\underbrace{V\times V\times\cdots\times V}_{l}$

$\Rightarrow (S\otimes\tilde{S})(v_1,\cdots,v_k,v_{k+1},\cdots,v_{k+l}) = S(v_1,\cdots,v_k)\cdot\tilde{S}(v_{k+1},\cdots,v_{k+l})$

2° If $V \xrightarrow{f} W$ a linear transform

$\Rightarrow$ $V^* \xleftarrow{f^*} W^*$ $\rightsquigarrow$ $f^*\alpha : V \to \mathbb{R}$ defined by $(f^*\alpha)(v) = \alpha(f(v))$

$\alpha \in W^*$

$\boxed{\text{check}}$ In terms of basis and dual basis representative, $[f^*] = $ transpose of $[f]$

Similarly, $\mathcal{T}^k(V^*) \xleftarrow{f^*} \mathcal{T}^k(W^*)$, $S \in \mathcal{T}^k(W^*)$

defined by $(f^*S)(v_1, \cdots, v_k) = S(f(v_1), \cdots, f(v_k))$

$\boxed{\text{check}}$ $(f^*S) \otimes (f^*\tilde{S}) = f^*(S \otimes \tilde{S})$

3° defn. Denote by $\Lambda^k(V^*) \subset \mathcal{T}^k(V^*)$ the alternative ones, namely

$\omega \in \Lambda^k(V^*)$ if $\omega(v_1, \cdots v_i, \cdots, v_j, \cdots, v_k) = - S(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$

$\forall v_1, \cdots, v_k \in V$ and $i \neq j \in \{1, \cdots, k\}$

- For any $S \in \mathcal{T}^k(V^*)$, let $alt(S) \in \Lambda^k(V^*)$ defined by

$$alt(S)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \; S(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

S_k: symmetric group of degree k

$\operatorname{sgn}\sigma$: $+1$: even permutation, -1 : odd permutation

e.g. $k=2$: $alt(S)(v_1, v_2) = \frac{1}{2}(S(v_1, v_2) - S(v_2, v_1))$

$k=3$:
$$alt(S)(v_1, v_2, v_3) = \frac{1}{6}\begin{pmatrix} S(v_1, v_2, v_3) + S(v_3, v_1, v_2) + S(v_2, v_3, v_1) \\ -S(v_1, v_3, v_2) - S(v_3, v_2, v_1) - S(v_2, v_1, v_3) \end{pmatrix}$$

rmk $\Lambda^k(V^*)$ is a vector space of $\dim = \binom{n}{k}$

4° defn (wedge product) for $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$

Define $\omega \wedge \eta \in \Lambda^{k+l}(V^*)$ by $\frac{(k+l)!}{k!\,l!} alt(\omega \otimes \eta)$

lemma for any $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^l(V^*)$, $\theta \in \Lambda^m(V^*)$

$$\begin{cases} \text{i)} \; \omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \\ \text{ii)} \; (f^*\omega) \wedge (f^*\eta) = f^*(\omega \wedge \eta) \\ \text{iii)} \; (\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!} alt(\omega \otimes \eta \otimes \theta) \end{cases}$$

pf: (only check the last one, the first two are easier)

claim
- alt acts on $\Lambda^k(V^*)$ as the identity (no changes) $\leftarrow$ follows directly from the construction
- if $alt(\tilde{S}) = 0 \Rightarrow alt(\tilde{S} \otimes S) = 0 = alt(S \otimes \tilde{S})$
- $alt(\omega \otimes \eta \otimes \theta) = alt(alt(\omega \otimes \eta) \otimes \theta)$

$\tilde{S} \in \mathcal{T}^k(V^*)$, $S \in \mathcal{T}^l(V^*)$ with $alt(\tilde{S}) = 0$

$G = \{\sigma \in S_{k+l} \mid \sigma(k+j) = k+j \text{ for } j = 1, \cdots, l\} \leq S_{k+l}$ (subgroup) $\quad G \cong S_k$

By algebra $S_{k+l} = \bigcup$ co-sets of $G = \bigcup_{[\gamma] \in G \backslash S_{k+l}} G \cdot \gamma$

$$alt(\tilde{S} \otimes S)(v_1, \cdots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{[\gamma] \in G \backslash S_{k+l}} \sum_{g \in G} \operatorname{sgn}(g \cdot \gamma) (\tilde{S} \otimes S)(v_{(g\cdot\gamma)(1)}, \cdots, v_{(g\cdot\gamma)(k+l)})$$

Fix σ. consider $\sum_{g\in G} \mathrm{sgn}(g)(\tilde{S}\otimes S)(v_{(g\cdot\sigma)(1)},\cdots,v_{(g\cdot\sigma)(k+l)})$ $\qquad w_j = v_{\sigma(j)}$

$$= \sum_{g\in G}\mathrm{sgn}(g)(\tilde{S}\otimes S)(w_{g(1)},\cdots w_{g(k)}, w_{g(k+1)},\cdots, w_{g(k+l)})$$ by def of $\otimes$ and G

$$= \sum_{g\in G}\mathrm{sgn}(g)\,\tilde{S}(w_{g(1)},\cdots,w_{g(k)})\, S(w_{k+1},\cdots, w_{k+l})$$

$=0$ since $\mathrm{alt}(\tilde{S})=0$

Since $\mathrm{alt}^2=\mathrm{alt}$ $\Rightarrow$ $\mathrm{alt}(\underbrace{\omega\otimes\eta-\mathrm{alt}(\omega\otimes\eta)}_{=\tilde{S}})=0$

$\Rightarrow$ $\mathrm{alt}((\omega\otimes\eta-\mathrm{alt}(\omega\otimes\eta))\otimes\theta)=0$

i.e. $\mathrm{alt}(\omega\otimes\eta\otimes\theta)=\mathrm{alt}(\mathrm{alt}(\omega\otimes\eta)\otimes\theta)$

Hence $(\omega\wedge\eta)\wedge\theta = \frac{(k+l+m)!}{(k+l)!\,m!}\mathrm{alt}((\omega\wedge\eta)\otimes\theta)$

$$= \frac{(k+l+m)!}{\cancel{(k+l)!}\,m!}\frac{\cancel{(k+l)!}}{k!\,l!}\underbrace{\mathrm{alt}(\mathrm{alt}(\omega\otimes\eta)\otimes\theta)}_{=\mathrm{alt}(\omega\otimes\eta\otimes\theta)} = \omega\wedge(\eta\wedge\theta)$$ ※

5° • <u>contraction</u> Given any $v\in V$, and $S\in\Lambda^k(V^*)$

$S(v,-,\cdots,-):\underbrace{V\times\cdots\times V}_{k-1}\to\mathbb{R}$ is still alternative

Denote it by $\iota_v S\in\Lambda^{k-1}(V^*)$

iota or $v\lrcorner S$

• basis for $\Lambda^k(V^*)\cong\mathbb{R}^{\binom{n}{k}}$: $\{dx^{i_1}\wedge\cdots\wedge dx^{i_k}\}_{i_1<i_2<\cdots<i_k}$

<u>e.g.</u> $\underbrace{(dx^1\wedge dx^2)}_{=-dx^2\wedge dx^1}(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}) = \frac{2!}{1!\,1!}\mathrm{alt}(dx^1\otimes dx^2)(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})$

$$=\frac{2!}{1!\,1!}\frac{1}{2!}\left((dx^1\otimes dx^2)(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})-(dx^1\otimes dx^2)(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^j})\right)$$

$$=\delta^1_j\delta^2_k-\delta^1_k\delta^2_j=\begin{cases}1 & (j,k)=(1,2)\\ -1 & (j,k)=(2,1)\\ 0 & \text{otherwise}\end{cases}$$

In general, $(dx^{i_1}\wedge\cdots\wedge dx^{i_k})(\frac{\partial}{\partial x^{j_1}},\cdots,\frac{\partial}{\partial x^{j_k}})=1$

if $(j_1,\cdots,j_k)$ is an even permutation of $(i_1,\cdots,i_k)$

<u>e.g.</u> $\Lambda^n(V^*)=\mathbb{R}\langle dx^1\wedge\cdots\wedge dx^n\rangle$

$dx^1\wedge\cdots\wedge dx^n : V\times\cdots\times V\to\mathbb{R}^1$

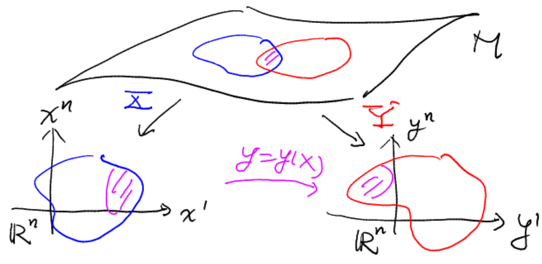
$(v_1,\cdots,v_n)\longmapsto$ express them in terms of the basis $\{\frac{\partial}{\partial x^j}\}$ and calculate the determinant

§II. differential form

1° <u>defn</u> M: smooth manifold (possibly with boundary)

a smooth k-form on M is an assignment $p\to\Lambda^k(T_p^*M)$, and is smooth in the sense of coordinate

<u>discussion</u>



By $(\mathcal{U},X)$, a k-form η is $\sum_{i_1<\cdots<i_k}\alpha_{i_1\cdots i_k}(x)\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

By (V, Ψ), it is $\sum_{j_1<\cdots<j_k} \beta_{j_1\cdots j_k}(y)\, dy^{j_1}\wedge\cdots\wedge dy^{j_k}$

<u>recall</u> $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto y(x)$ $\rightsquigarrow$ $T_x\mathbb{R}^n \cong \mathbb{R}^n \longrightarrow \mathbb{R}^n \cong T_y\mathbb{R}^n$

given by $\frac{\partial y}{\partial x}$: the differential of the map.

$\rightsquigarrow$ $(T_x\mathbb{R}^n)^* \cong \mathbb{R}^n \longleftarrow \mathbb{R}^n \cong (T_y\mathbb{R}^n)^*$

$$\sum_i \frac{\partial y^j}{\partial x^i}\, dx^i \longleftarrow dy^j$$

Hence, $\beta_{j_1\cdots j_k}(y)\, dy^{j_1}\wedge\cdots\wedge dy^{j_k} = \beta_{j_1\cdots j_k}(y(x))\, \frac{\partial y^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_k}}{\partial x^{i_k}}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

⚠ all order of $(i_1,\cdots,i_k)$

<u>e.g.</u> $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x^1,x^2,x^3) \mapsto (x^2, x^1, x^3+x^1x^2) = (y^1, y^2, y^3)$

$dy^1 = dx^2$
$dy^2 = dx^1$
$dy^3 = dx^3 + x^1dx^2 + x^2dx^1$

$\Rightarrow$ $dy^1\wedge dy^2 = dx^2\wedge dx^1 = -dx^1\wedge dx^2$

$dy^1\wedge dy^3 = dx^2\wedge dx^3 + x^2 dx^2\wedge dx^1$
$= dx^2\wedge dx^3 - x^2 dx^1\wedge dx^2$

<u>rmk</u> sometimes, we will write $\frac{1}{k!}\sum_{i_1\cdots i_k} \tilde{\alpha}_{i_1\cdots i_k}(x)\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ ← all order

by requiring $\tilde{\alpha}_{i_1\cdots j_\mu\cdots i_\nu\cdots i_k} = -\tilde{\alpha}_{i_1\cdots i_\nu\cdots j_\mu\cdots i_k}$

$\sum_{i_1<\cdots<i_k} (\tilde{\alpha}_{i_1\cdots i_k}(x))\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

2° <u>defn</u> There is a naturally defined operator $d: \Omega^k(M) \to \Omega^{k+1}(M)$ (the exterior derivative; $\Omega^k(M)$: all smooth k-forms)
which in coordinate is given by

$$d\Big(\sum_{i_1\cdots i_k} \alpha_{i_1\cdots i_k}(x)\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}\Big) = \sum_{i, i_1,\cdots,i_k} \frac{\partial \alpha}{\partial x^i}\, dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

<u>lemma</u> i) d is well-defined
ii) $d^2: \Omega^k(M) \to \Omega^{k+2}(M)$ is identically zero.

<u>e.g.</u> In $\mathbb{R}^3$

$\Omega^0 = \{f\}$ $\quad df = \sum_j \frac{\partial f}{\partial x^j}\, dx^j$

$\Omega^1 = \{\alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3\}$ $\quad \left(\frac{\partial\alpha_3}{\partial x^2} - \frac{\partial \alpha_2}{\partial x^3}\right) dx^2\wedge dx^3 + \cdots$

$\Omega^2 = \{\beta_1 dx^2\wedge dx^3 + \beta_2 dx^3\wedge dx^1 + \beta_3 dx^1\wedge dx^2\}$ $\quad \sum_j \frac{\partial \beta_j}{\partial x^j}\, dx^1\wedge dx^2\wedge dx^3$

$\Omega^3 = \{h\, dx^1\wedge dx^2\wedge dx^3\}$

$df \sim \nabla f$, $\quad d\alpha \sim \mathrm{curl}(\alpha)$, $\quad d\beta \sim \mathrm{div}(\beta)$

ii): $\mathrm{curl}\circ\mathrm{grad} = 0$ & $\mathrm{div}\circ\mathrm{curl} = 0$

Pf of the lemma: For i), it suffices to check that d commutes with change of coordinate

For simplicity, only focus on $\beta(y)\, dy^1\wedge\cdots\wedge dy^k$

$\boxed{\beta(y(x))\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^k}{\partial x^{i_k}}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}}$ ←

$\downarrow d$

$\frac{\partial\beta}{\partial y^j}\, dy^j\wedge dy^1\wedge\cdots\wedge dy^k$

change of coordinate in $\bigcirc$ $= \left(\frac{\partial B}{\partial y^j}\frac{\partial y^j}{\partial x^i}\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^k}{\partial x^{i_k}}\right)dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$

d in $\square$

$$= \frac{\partial}{\partial x^i}\left(B(y(x))\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^k}{\partial x^{i_k}}\right)dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= \frac{\partial B}{\partial y^j}\frac{\partial y^j}{\partial x^i}\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^k}{\partial x^{i_k}}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$\cancel{+ B\underbrace{\frac{\partial^2 y^1}{\partial x^i\partial x^{i_1}}}_{\text{sym in }(i,i_1)}\frac{\partial y^2}{\partial x^{i_2}}\cdots\frac{\partial y^k}{\partial x^{i_k}}\underbrace{dx^i\wedge dx^{i_1}}_{\text{skew-sym in }(i,i_1)}\wedge\cdots\wedge dx^{i_k}} + \cdots$$

0

For ii), similarly focus on $\alpha(x)\,dx^1\wedge\cdots\wedge dx^k$

$$d(\alpha\, dx^1\wedge\cdots\wedge dx^k) = \frac{\partial\alpha}{\partial x^\mu}dx^\mu\wedge dx^1\wedge\cdots\wedge dx^k$$

$$d^2(\alpha\, dx^1\wedge\cdots\wedge dx^k) = \frac{\partial^2\alpha}{\partial x^\nu\partial x^\mu}dx^\nu\wedge dx^\mu\wedge dx^1\wedge\cdots\wedge dx^k = 0$$

by the same token

※

3° $0\to\Omega^0(M)\xrightarrow{d}\Omega^1(M)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^j(M)\xrightarrow{d}\cdots\cdots\xrightarrow{d}\Omega^n(M)\longrightarrow 0$

$\Omega^0(M) = \{\text{smooth functions}\}$

$d^2=0 \Rightarrow d(\Omega^{j-1}(M))\subset \ker(d|_{\Omega^j(M)})$

<u>defn</u> The j-th de Rham cohomology of M, $H^j_{dR}(M) := \ker(d|_{\Omega^j}) \big/ d\Omega^{j-1}$

$\ker(d|_{\Omega^j})$: (d-)closed j-form; $d\Omega^{j-1}$: exact j-form

$\boxed{Q}$ What are they? How to compute them?

e.g. $M=\mathbb{R}^1$ $\quad 0\to\Omega^0(M)\xrightarrow{d}\Omega^1(M)\to 0$

$f \qquad \alpha(x)\,dx$

Since, $df = f'(x)\,dx$, $\quad df=0 \iff f = $ constant function

Hence, $H^0_{dR}(M)\cong\mathbb{R}^1$

For any $\alpha(x)\,dx$, define $h(x) = \int_0^x \alpha(t)\,dt \Rightarrow dh = \alpha(x)\,dx$

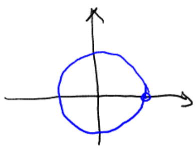
Hence, $H^1_{dR}(M) = 0$

<u>lemma</u> M: connected manifold. $\Rightarrow H^0_{dR}(M)\cong\mathbb{R}$

pf: On each $(\mathcal{U},X)$, $df = \frac{\partial f}{\partial x^j}dx^j = 0 \Rightarrow f = $ constant on $\mathcal{U}$ ※

e.g. For $\mathbb{S}^1$, $H^1_{dR}(\mathbb{S}^1) = ?$

$\mathbb{S}^1 = \{(\cos\theta,\sin\theta)\subset\mathbb{R}^2\}$



$\theta\in\mathbb{R}^1 \longrightarrow (\cos\theta,\sin\theta)\in\mathbb{S}^1$

$\rightsquigarrow$ function on $\mathbb{S}^1$ = 2π-periodic function on $\mathbb{S}^1$

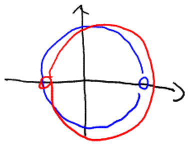
Moreover, $\theta\in(0,2\pi)\longrightarrow(\cos\theta,\sin\theta)$ serves as a coordinate chart

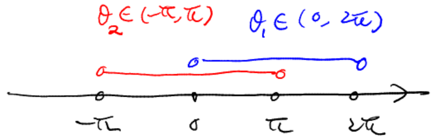
$d(f(\theta)) = f'(\theta)\,d\theta \rightsquigarrow \int_0^{2\pi} f'(\theta)\,d\theta$

$= f(2\pi) - f(0) = 0$

$\Rightarrow$ The "integration" of any exact 1-form vanishes.

Claim $d\theta$ is a well-defined 1-form





When $\theta_1 \in (\pi, 2\pi)$

$\theta_2 = \theta_1 - 2\pi$

$\Rightarrow d\theta_2 = d\theta_1$

When $\theta_1 \in (0, \pi)$, $\theta_2 = \theta_1$

$\Rightarrow d\theta_2 = d\theta_1$ on the overlap region. ※

But $\int_0^{2\pi} d\theta = 2\pi \neq 0$, $d\theta$ is a closed, non-exact 1-form

This is a commonly used notation
Do NOT suggest θ is globally defined

For any $\alpha(\theta)\, d\theta$, (2π-periodic) let $c = \frac{1}{2\pi}\int_0^{2\pi} \alpha(\theta)\, d\theta$

$\Rightarrow \int_0^{\theta} (\alpha(\phi) - c)\, d\phi$ is 2π-periodic

$\Rightarrow [\alpha(\theta)\, d\theta] = [c\, d\theta]$ in $H^1_{dR}(S^1)$ $\Rightarrow H^1_{dR}(S^1) \cong \mathbb{R}^1$ ※

$\rightsquigarrow$ for general study of $H^k_{dR}(M)$, need to consider integration

lying in some sense, $\Omega^k(M)$ is invented to give a rigorous treatment of integration on manifolds