root system and Dynkin diagram of special unitary group

§I. representation of torus

$T^n = S^1 \times \cdots \times S^1$. consider $T^n \curvearrowright \mathbb{R}^N$, preserving the standard inner product

$\Leftrightarrow \langle g\cdot x, g\cdot y\rangle = \langle x, y\rangle \Leftrightarrow T^n \longrightarrow O(N)$

1° $S^1 \curvearrowright \mathbb{R}^N$ Extend to $\mathbb{R}^N \otimes \mathbb{C} = \mathbb{C}^N$ by $g\cdot(z x) = z\,(g\cdot x)$

Also, let $\langle z x, w y\rangle = z\bar{w}\,\langle x, y\rangle$ (extend to a Hermitian inner product)

$\rightsquigarrow$ $S^1 \curvearrowright \mathbb{C}^N$ is unitary $\rightsquigarrow$ $\mathbb{R}^1 \curvearrowright \mathbb{C}^N$ is skew-Hermitian

$e^{2\pi i t}$ $\qquad v \mapsto \frac{d}{dt}\Big|_{t=0} e^{2\pi i t}\cdot v = A(v)$

$S^1 \longrightarrow U(N) \rightsquigarrow \mathbb{R}^1 \cong T_1 S^1 \longrightarrow T_I U(N) \cong \text{skew-Herm}(N)$

$1 \longmapsto A$

$\Rightarrow$ Since $A + A^* = 0$, A : diagonalizable by unitary basis. $\{v_1, \cdots, v_N\}$, and $\lambda_j \in i\mathbb{R}$

2° exponential map on $S^1 \ni \{e^{is}\}$ $\qquad i\mathbb{R} \cong T_1 S^1 \longrightarrow S^1$

$\quad\;\; \|$ unit-normed complex numbers $\qquad is \longmapsto \exp(is) = 1 + is + \frac{1}{2!}(is)^2 + \cdots$

Consider $i\mathbb{R} \xrightarrow{\sim} \mathbb{R}$; choose another identification of $T_1 S^1$

$is \longrightarrow t = \frac{s}{2\pi}$ $\qquad \exp(t) = \exp(2\pi i t)$

$T_1 S^1$ has "lattice" $\leftarrow$ mapped to 1 under exp.

under the t-identification, integers $= \mathbb{Z} \subset \mathbb{R}$

3° $A v_j = 2\pi i \lambda_j\, v_j$ $\rightsquigarrow$ $\exp(tA)\, v_j = e^{2\pi i \lambda_j t} v_j$

eigenvalue

$\qquad\qquad \| $

$\qquad e^{2\pi i t}\cdot v_j$

$$\begin{array}{ccc} T_1 S^1 & \xrightarrow{\rho_*} & T_I U(N) \\ \downarrow \exp & \circlearrowleft & \downarrow \exp \\ S^1 & \xrightarrow{\rho} & U(N) \end{array}$$

$\Rightarrow$ When $t \in \mathbb{Z}$, the action is the identity $\Rightarrow \lambda_j \in \mathbb{Z}$.

rmk/e.g. $\quad S^1 \curvearrowright \mathbb{R}^3$

$$e^{it} \longrightarrow \begin{bmatrix} \cos(\ell t) & \sin(\ell t) & 0 \\ -\sin(\ell t) & \cos(\ell t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \lambda = \ell, -\ell, 0.$$

4° For $T^n \curvearrowright \mathbb{R}^N$. again identify $T_1 T^n$ with $\mathbb{R}^n$

such that lattice $= \mathbb{Z}^n$

Since T^n is abelian, its Lie algebra has $[\,,\,] \equiv 0$

Hence, the image of $T_1 T^n$ in skew-Herm(N) has vanishing matrix bracket

By linear algebra, they can be diagonalized unitarily simultaneously.

i.e. $\exists\, v_1, \cdots, v_N \in \mathbb{C}^N$, $\ni \rho(A)\cdot v_j = 2\pi i\,\lambda_j(A)\, v_j$ $\qquad T_1 T^n \xrightarrow{\rho_*} \text{skew-Herm}(N)$

$\{\lambda_j\}_{j=1}^N$: linear functional on $T_1 T^n \cong \mathbb{R}^n$ $\qquad T^n \xrightarrow{\rho} U(N)$

taking integer value on $\mathbb{Z}^n$

§II. $G = SU(n)$

$G = \{ S \in M(n\times n; \mathbb{C}) \mid S^*S = I,\ \det S = 1 \}$ $\dim G = n^2-1$

$\mathfrak{g} = \{ U \in M(n\times n; \mathbb{C}) \mid U^* + U = 0,\ tr(U) = 0 \}$

$$\left(S(t): S(0) = I,\ S'(0) = U \Rightarrow \left.\frac{d}{dt}\right|_{t=0} (S^*S) = 0 = U^* + U \quad \left.\frac{d}{dt}\right|_{t=0} \det(S) = 0 = tr(U) \right)$$

1° G contains a torus $T^{n-1} = diag(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})$ with $e^{2\pi i \sum_j \theta_j} = 1$

$\boxed{\text{check}}$ T^{n-1} is maximal i.e. $\nexists\ T^n \subset G$ $\Leftrightarrow$ max dim of abelian Lie subalgebra in $\mathfrak{g}$ is $n-1$

$\boxed{\text{goal}}$ study $T^{n-1} \overset{Ad}{\curvearrowright} \mathbb{R}^{n^2-1} \cong \mathfrak{g}$, or $T_1 T^{n-1} \cong \mathbb{R}^{n-1} \overset{ad}{\curvearrowright} \mathbb{R}^{n^2-1} \cong \mathfrak{g}$

2° inner product on $\mathfrak{g} = \mathbb{R}^{n^2-1}$

For $U, V \in \mathfrak{g}$, $\langle U, V\rangle = Re\, tr(U^*V)$ defines an inner product (linear algebra)

$= -Re\, tr(UV) = -tr(UV)$ in the Hermitian case

This is the one commonly used in matrix calculation. It turns out to be equivalent to the Killing form.

Indeed, $B(U,V) = 2n\, tr(UV) = -\frac{1}{2n}\langle U, V\rangle$: negative-definite

$SU(n) \overset{ad}{\curvearrowright} \mathfrak{g}$: irreducible $\Rightarrow$ any two $SU(n)$-invariant bilinear forms are proportional to each other.

rmk $U(n) \overset{Ad}{\curvearrowright} \mathfrak{u}(n) = \text{skew-Hermian}(n)$ is NOT irreducible

$\cup$

$\{diag(is, is, \dots, is)\} \cong \mathbb{R}^1$: 1-dim'l invariant subspace

Now, $T^{n-1} \subset SU(n)$, identify $T_1 T^{n-1} \cong \mathfrak{t} = \{ (\theta_1, \dots, \theta_n) \in \mathbb{R}^n \mid \sum_{j=1}^n \theta_j = 0 \} \subset \mathbb{R}^n$

$\downarrow \exp$

$(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})$ lattice $= \mathbb{Z}^n \cap \mathfrak{t}$

Then, $\rho_*(\theta_1, \dots, \theta_n) = 2\pi i\, diag(\theta_1, \dots, \theta_n) \in \mathfrak{g}$

Now, use the following inner product on $\mathfrak{g}$

$\langle U, V\rangle = \frac{-1}{4\pi^2} tr(UV)$ rescaling by $1/4\pi^2$

advantage. $\langle\ ,\ \rangle$ takes integer value on lattice

3° Study $\left(T^{n-1} \overset{Ad}{\curvearrowright} \mathfrak{g} \cong \mathbb{R}^{n^2-1} \rightsquigarrow \right)$ $\mathfrak{t} \overset{ad}{\curvearrowright} \mathfrak{g} \cong \mathbb{R}^{n^2-1}$

$\mathfrak{g}$ = traceless, skew-Hermitian matrices $\overset{i}{\curvearrowright}$ traceless, Hermitian matrices

$\mathfrak{g} \otimes \mathbb{C}$ can be naturally identified with traceless matrices

$$Z = X + iY \in \mathfrak{g} \oplus i\mathfrak{g} \qquad \begin{cases} X = \frac{1}{2}(Z - Z^*) \\ iY = \frac{1}{2}(Z + Z^*) \end{cases}$$

$Z^* = -X + iY$

$\rightsquigarrow \mathfrak{t} \overset{ad\otimes 1}{\curvearrowright} \mathfrak{g} \otimes \mathbb{C} = \{\text{traceless}\}$ is still matrix bracket

$\mathfrak{t} = 2\pi i \, \mathrm{diag}(\theta_1, \cdots, \theta_n)$

$$\mathrm{ad}_{\mathfrak{t}}(E_{kl}) = 2\pi i \left[\mathrm{diag}(\theta_1, \cdots, \theta_n), E_{kl}\right] = 2\pi i \left[\sum_j \theta_j E_{jj}, E_{kl}\right]$$
$$= 2\pi i \,(\theta_k - \theta_l) E_{kl}$$

$(\dim \mathfrak{g} = n^2 - 1)$: $\mathrm{ad}_{\mathfrak{t}}$:

	eigen values	eigen vectors	
$k \neq l$	$2\pi i(\theta_k - \theta_l)$	E_{kl}	$n(n-1)$
$k > 1$	0	$E_{11} - E_{kk}$	$n-1$

$4°$ as discussed in §I-4°. (n^2-1)-eigenvalues are (n^2-1)-linear functional on $\mathfrak{t}$) roots

From 3°. they are $\underset{\sum \theta_j = 0}{(\theta_1, \cdots, \theta_n)} \longmapsto \theta_k - \theta_l \quad k \neq l$

By using $\langle U, V \rangle = \frac{-1}{4\pi^2} \mathrm{tr}(UV)$ on $2\pi i \, \mathrm{diag}(\theta_1, \cdots, \theta_n)$,
the inner product on $\mathfrak{t} \subset \mathbb{R}^n$ is the one induced by the standard inner product on $\mathbb{R}^n$.

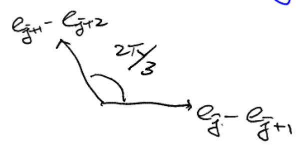
Then $\{\text{linear functional on } \mathfrak{t}\} \longleftrightarrow \{\text{inner product with some element in } \mathfrak{t}\}$
$(\theta_1, \cdots, \theta_n) \mapsto \theta_k - \theta_l \longleftrightarrow$ inner product with $e_k - e_l$

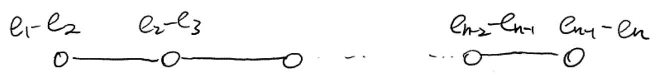
Note that $\{e_j - e_{j+1}\}_{j=1}^{n-1}$ span all the eigenvalues (roots)

If $k < l$, $e_k - e_l = \sum_{j=k}^{l-1} (e_j - e_{j+1})$

They $\{e_j - e_{j+1}\}_{j=1}^{n-1}$ are called "simple roots" (simplest eigenvalues of $\mathfrak{t} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{g}$)

$|e_j - e_{j+1}| = \sqrt{2} \qquad \langle e_j - e_{j+1}, e_{j+1} - e_{j+2} \rangle = -1 \Rightarrow$



$\rightsquigarrow$ 

Dynkin diagram. A_{n-1}

$\hookrightarrow$ $4\cos^2(\text{angle}) = 1$

$4\cos^2(\frac{\pi}{2}) = 0$: do not connect

<u>one key</u> in the classification: $4\cos^2(\text{angle}) \in \{0, 1, 2, 3\}$