introduction to Lie group.

Lie group : geometric object which describes continuous symmetry

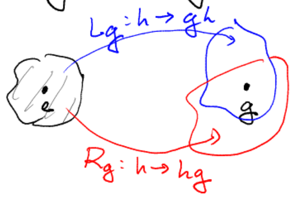
§I. basics

<u>defn</u> a Lie group G is a group equippep with a structure of (smooth) manifold. such that $\begin{cases} g \mapsto g^{-1} & (G \to G) \\ (g,h) \mapsto gh & (G \times G \to G) \end{cases}$ are smooth maps.

<u>rmk</u> We will usually denote the identity element by $\mathbf{e}$ or I
formal discussion   matrix group

G is not only locally Euclidean.

$\forall g \in G$.



left multiplication by g. $L_g \in \mathrm{Aut}(G)$, provides a diffeomorphism between a nbd of $\mathbf{e}$ and a nbd of g

i.e. there is a "canonical" way to identify open nbds of different points. ($L_h L_g^{-1} : \mathcal{U}_g \to \mathcal{U}_h$

<u>eg.</u> • $Gl(n;\mathbb{R}) \subset M(n\times n;\mathbb{R}) \cong \mathbb{R}^{n^2}$

det $\neq 0$ $\Rightarrow$ open subset in $\mathbb{R}^{n^2}$. $\Rightarrow$ manifold of dim n^2

By linear algebra, group operations are smooth

• $O(n)$: submanifold of $Gl(n;\mathbb{R})$

group operation : induced from $Gl(n;\mathbb{R})$ $\Rightarrow$ also smooth

• We will call these examples (subgroup and submanifold of $Gl(n;\mathbb{R}/\mathbb{C})$) the matrix groups.

§II. left-invariant vector field and Lie algebra

G : Lie group. (usually focus on the case when $\dim G \geq 1$)

1° $\forall V \in T_eG$. V can be extended to a smooth vector field

by $\tilde{V}|_g = \underbrace{(L_g)_*(V)}_{D L_g|_e}$

$D L_g|_e$: differential of $h \mapsto gh$ at $\mathbf{e}$

$\Rightarrow$ $T_eG \to T_gG$

$\Rightarrow$ $\tilde{V}|_{gh} = (L_{gh})_*(V)$ $T_eG \to T_{gh}G$

$= (L_g)_*(L_h)_*(V)$ $L_{gh} = L_g L_h$. $T_eG \xrightarrow{L_h} T_hG \xrightarrow{L_g} T_{gh}G$, with $L_{gh}: T_eG \to T_{gh}G$

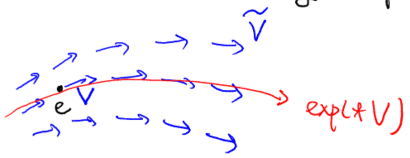
<u>defn</u> $\tilde{V} \in \Gamma(G;TG)$ is called left-invariant if $\tilde{V}|_g = (L_{gh^{-1}})_*(\tilde{V}|_h)$ $\forall g.h \in G$

Note that $\tilde{V} \longrightarrow \tilde{V}|_e \in T_eG$ is an isomorphism. $\Leftrightarrow$ $\tilde{V}|_g = L_g(\tilde{V}|_e)$ $\forall g$

2° $V \in T_eG \rightsquigarrow \tilde{V} \in \Gamma(G; TG)$ the left-invariant vector field.

<u>Lemma</u> Then, the flow generated by $\tilde{V}$, φ_t, is defined $\forall t$ (even if G is not compact)

Denote $\varphi_t(e)$ by $\exp(tV) : \mathbb{R}^1 \to G$



pf: By ODE. φ_t is defined on $(-\varepsilon, \varepsilon) \times \mathcal{U}$ for some $\varepsilon > 0$ and some nbd $\mathcal{U}$ of e

$$\frac{d}{dt}\left(h \cdot \varphi_t(g)\right) = \frac{d}{dt} L_h \circ \varphi_t(g) = (L_h)_*\left(\tilde{V}\big|_{\varphi_t(g)}\right)$$
$$= \tilde{V}\big|_{h \cdot \varphi_t(g)}$$

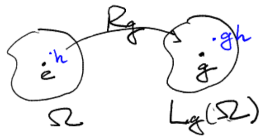
$\Leftarrow$ $\tilde{V}$ is left-invariant

By letting $\boxed{\varphi_t(h) = h \cdot \varphi_t(e)}$

$\rightsquigarrow \varphi_t : (-\varepsilon, \varepsilon) \times G \to G$: the flow generated by $\tilde{V}$

By $\varphi_{t+s} = \varphi_t \circ \varphi_s \Rightarrow \varphi_t : \mathbb{R} \times G \to G$ ※

3° $\tilde{U}, \tilde{V}$: left-invariant vector field



$$[\tilde{U}, \tilde{V}]\big|_{L_g(\mathcal{U})} = [\underbrace{(L_g)_*(\tilde{U})}_{\text{on } R_g(\Omega)}, (L_g)_*(\tilde{V})] \overset{\text{HW}}{=} (L_g)_*([\tilde{U}, \tilde{V}])$$

on Ω

$\Rightarrow [\tilde{U}, \tilde{V}]$ is also left-invariant.

<u>defn</u> a Lie algebra (over $\mathbb{R}$ in this class) is a vector space W with a bilinear map $[-,-] : W \times W \to W$ satisfying

$$\begin{cases} [X, Y] = -[Y, X] \\ [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad \text{(Jacobi)} \end{cases}$$

$\forall X, Y, Z \in W$

defn/Cor G : Lie group. $\mathfrak{g} = T_eG$ is a Lie algebra (of dim = dim G) with $[U, V] = [\tilde{U}, \tilde{V}]\big|_e$. It is called the Lie algebra of G

left-invariant extension

4° For $G = Gl(n; \mathbb{R})$: matrix group.

$\mathfrak{g} = \mathfrak{gl}(n; \mathbb{R}) \cong M(n \times n; \mathbb{R}) \cong \mathbb{R}^{n^2}$

i) $V \in \mathfrak{g}$, consider $\exp(tV) = I + tV + \frac{t^2}{2!}V^2 + \cdots + \frac{t^n}{n!}V^n + \cdots$

- as in analysis, it converges $\forall t$
- $\exp(-tV)\exp(tV) = I \Rightarrow \exp(tV)$ is invertible
- $\tilde{V}\big|_g = \frac{d}{dt}\Big|_{t=0}\left(g \cdot \exp(tV)\right) = gV \qquad \varphi_t(g) = g\exp(tV)$

ii) $[U, V] = ?$ $\qquad G$: open subsets in $\mathbb{R}^{n^2}$

$\Rightarrow g = \sum_{ij} x_{ij} E_{ij} \qquad E_{ij} = (\delta_{ij}) = 1$, others $= 0 \qquad \left(E_{ij} \leftrightarrow \frac{\partial}{\partial x_{ij}}\right)$

$$U = \sum_{ij} u_{ij} E_{ij} = \sum_{ij} u_{ij} \frac{\partial}{\partial x_{ij}} \Rightarrow \tilde{U} = gU = \sum_{ijk} x_{ik} u_{kj} E_{ij} = \sum_{ijk} x_{ik} u_{kj} \frac{\partial}{\partial x_{ij}}$$

$(E_{ij} E_{kl} = \delta_{jk} E_{il})$

$$\Rightarrow [\tilde{U}, \tilde{V}] = \left[x_{ik} U_{kj} \frac{\partial}{\partial x_{ij}},\ x_{\mu\ell} V_{\ell\nu} \frac{\partial}{\partial x_{\mu\nu}} \right]$$

@ $g = x_{ij} E_{ij}$

$$= x_{ik} U_{kj} \frac{\partial (x_{\mu\ell} V_{\ell\nu})}{\partial x_{ij}} \frac{\partial}{\partial x_{\mu\nu}} - x_{\mu\ell} V_{\ell\nu} \frac{\partial (x_{ik} U_{kj})}{\partial x_{\mu\nu}} \frac{\partial}{\partial x_{ij}}$$

$$= x_{ik} \underset{k\nu}{\cancel{U_{kj}}} \cancel{\delta^{\mu}_{i}} \cancel{\delta^{\ell}_{j}} \underset{\nu j}{\underset{\ell j}{\cancel{V_{\ell\nu}}}} \underset{ij}{\underset{i\nu}{\frac{\partial}{\partial \cancel{x_{\mu\nu}}}}} - \underset{i\ell}{\cancel{x_{\mu\ell}}} \underset{\ell k}{\cancel{V_{\ell\nu}}} \cancel{\delta^{i}_{\mu}} \cancel{\delta^{k}_{\nu}} U_{kj} \frac{\partial}{\partial x_{ij}}$$

$(j \leftrightarrow \nu)$

$$= (x_{ik} U_{k\ell} V_{\ell j} - x_{i\ell} V_{\ell k} U_{kj}) \frac{\partial}{\partial x_{ij}}$$

$$[U, V] = [\tilde{U}, \tilde{V}]\big|_{I} = (U_{i\ell} V_{\ell j} - V_{ik} U_{kj}) E_{ij}$$: coincides with matrix bracket.

I: $x_{ij} = \delta_{ij}$

(for right-invariant ones, differ by a minus sign)

5° Cor TG is always a trivial bundle

pf: $G \times \mathfrak{g} \longrightarrow TG$

$(g, V) \longmapsto (L_g)_* V = \tilde{V}$ ※

Cor S^2 cannot be a Lie group.

pf: By hairy ball, it admits no nowhere zero tangent vector field ※

rmk $\Gamma(M; TM)$ with $[\ ,\ ]$ is an ∞-dim'l Lie algebra

§ III. Adjoint and adjoint representation

1° For any $g \in G$, consider the conjugation by g: $C_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$ $\in \mathrm{Aut}(G)$

$C_g(h) = g h g^{-1}$

Clearly, $C_g(e) = e$ $\Rightarrow$ its differential

$(C_g)_*|_e : \mathfrak{g} = T_e G \circlearrowleft$

Since $(C_g)^{-1} = C_{g^{-1}}$ $\Rightarrow$ $(C_g)_*|_e$ is still invertible.

defn Denote $(C_g)_*|_e$ by Ad_g. $\mathrm{Ad}: G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \cong Gl(\dim G; \mathbb{R})$

This is called the Adjoint representation of G

2° prop $\forall\, U, V \in \mathfrak{g}$. $\frac{d}{dt}\big|_{t=0} \mathrm{Ad}_{\exp(tU)} V = [U, V]$

$\exp(tU)$: curve in G; $V \in \mathfrak{g}$; $[U, V] = [\tilde{U}, \tilde{V}]|_e$

left-invariant extension

$\tilde{U}|_g = (L_g)_*(U)$

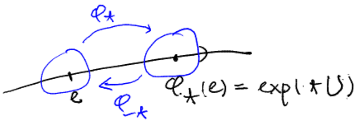
$\tilde{V}|_g = (L_g)_*(V)$

That is to say,

Lie bracket captures the infinitesimal behavior of conjugation

pf: $[\tilde{U}, \tilde{V}] = \frac{d}{dt}\big|_{t=0} (\varphi_{-t})_* (\tilde{V})$

From § II-2°, $\varphi_{-t}(g) = g \exp(-tU) = R_{\exp(-tU)}(g)$



$$(\varphi_{-t})_*(\tilde V)\big|_e = (\varphi_{-t})_*\left(\tilde V\big|_{\exp(tU)}\right) \qquad \tilde V\big|_h = (L_h)_*(V) = (L_{\exp(tU)})_*(V)$$

$$= (R_{\exp(-tU)})_* \circ (L_{\exp(tU)})_*(V)$$

$$= (C_{\exp(tU)})_*(V) = \mathrm{Ad}_{\exp(tU)}(V) \qquad ※$$

rmk For $Gl(n;\mathbb{R})$, this is straightforward.

$$\left(\begin{array}{l} \mathrm{Ad}_{\exp(tU)} V = e^{tU} V e^{-tU} \quad \text{as matrix multiplication} \\ \frac{d}{dt}\big|_{t=0} \rightsquigarrow UV - VU = [U,V] \end{array}\right)$$

3° defn $U \in \mathfrak{g} \longrightarrow \mathrm{ad}_U = \frac{d}{dt}\big|_{t=0} \mathrm{Ad}_{\exp(tU)} \in \mathrm{End}(\mathfrak{g})$

$$\mathrm{ad}_U V = [U, V]$$

This is called the adjoint representation of $\mathfrak{g}$

discussion $\mathbb{R}^{\dim G} \cong \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \cong M(\dim G \times \dim G; \mathbb{R}) \cong \mathbb{R}^{(\dim G)^2}$

$$U \longmapsto \mathrm{ad}_U$$
$$V \longmapsto \mathrm{ad}_V$$
$$[U,V] \longmapsto \mathrm{ad}_{[U,V]}$$

$$\begin{aligned} \mathrm{ad}_{[U,V]} W &= [[U,V],W] \\ &= -[[W,U],V] - [[V,W],U] \qquad \text{Jacobi identity} \\ &= [U,[V,W]] - [V,[U,W]] \\ &= \mathrm{ad}_U \circ \mathrm{ad}_V (W) - \mathrm{ad}_V \circ \mathrm{ad}_U (W) \end{aligned}$$

upshot ad: realize $(\mathfrak{g}, [\,,\,])$ as a subspace of $(\mathrm{End}(\mathfrak{g})$, commutator of linear transform$)$

4° final remark: group multiplication can almost be recovered from $[\,,\,]$

thm (Baker – Campbell – Hausdorff formula)

$\forall\, U, V \in \mathfrak{g}$

$$\exp(U)\cdot\exp(V) \text{ “}=\text{” } \exp\left(U + V + \tfrac{1}{2}[U,V] + \tfrac{1}{12}\big([U,[U,V]] + [V,[V,U]]\big) - \tfrac{1}{24}[U,[V,[U,V]]] + \left(\begin{array}{c}\text{five or more} \\ \text{brackets}\end{array}\right)\right)$$

rmk i) RHS need NOT to converge

$sl(2;\mathbb{R}) = \{\text{traceless}\}$ $\qquad U = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\exp(sU) = \begin{bmatrix} \cos(4s) & -\sin(4s) \\ \sin(4s) & \cos(4s) \end{bmatrix} \qquad \exp(tV) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\exp(U)\cdot\exp(V) = \begin{bmatrix} \cos 4 & \cos 4 - \sin 4 \\ \sin 4 & \cos 4 + \sin 4 \end{bmatrix} \qquad 2\cos 4 + \sin 4 = -2.06469\cdots$$

NOT in the image of exp.

ii) Consider matrix group.
$$\begin{cases} \exp(U) = I + U + \tfrac{1}{2}U^2 + \tfrac{1}{6}U^3 + \cdots \\ \exp(V) = I + V + \tfrac{1}{2}V^2 + \tfrac{1}{6}V^3 + \cdots \end{cases}$$

$$\exp U \cdot \exp V = \mathbb{I} + U + V + UV + \tfrac{1}{2}U^2 + \tfrac{1}{2}V^2 + \tfrac{1}{2}UV^2 + \tfrac{1}{2}U^2V + \tfrac{1}{6}U^3 + \tfrac{1}{6}V^3$$

$$= \mathbb{I} + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \cdots$$

Use $\mathcal{O}(A) = \mathcal{O}(U) = \mathcal{O}(V) \Rightarrow A = U + V + R_2 \quad (R_2 = \mathcal{O}(A^2))$

$$R_2 + \tfrac{1}{2}(U+V)^2 = UV + \tfrac{1}{2}U^2 + \tfrac{1}{2}V^2$$

$$\Rightarrow R_2 = \tfrac{1}{2}UV - \tfrac{1}{2}VU = \tfrac{1}{2}[U,V]$$

$$\Rightarrow A = U + V + \tfrac{1}{2}[U,V] + R_3 \quad (R_3 = \mathcal{O}(A^3))$$

$$R_3 + \underbrace{\tfrac{1}{4}(U+V)[U,V] + \tfrac{1}{4}[U,V](U+V)}_{\text{from } \frac{1}{2}A^2} + \underbrace{\tfrac{1}{6}(U+V)^3}_{\text{from } \frac{1}{6}A^3} = \tfrac{1}{2}UV^2 + \tfrac{1}{2}U^2V + \tfrac{1}{6}U^3 + \tfrac{1}{6}V^3$$

$$\begin{aligned}\Rightarrow R_3 = {} & \tfrac{1}{2}UV^2 + \tfrac{1}{2}U^2V + \cancel{\tfrac{1}{6}U^3} + \cancel{\tfrac{1}{6}V^3} \\ & - \tfrac{1}{4}U^2V + \cancel{\tfrac{1}{4}UVU} - \cancel{\tfrac{1}{4}VUV} + \tfrac{1}{4}V^2U - \cancel{\tfrac{1}{4}UVU} + \tfrac{1}{4}VU^2 - \tfrac{1}{4}UV^2 + \cancel{\tfrac{1}{4}VUV} \\ & - \cancel{\tfrac{1}{6}U^3} - \cancel{\tfrac{1}{6}V^3} - \tfrac{1}{6}U^2V - \tfrac{1}{6}UVU - \tfrac{1}{6}VU^2 - \tfrac{1}{6}V^2U - \tfrac{1}{6}VUV - \tfrac{1}{6}UV^2 \\ = {} & \tfrac{1}{12}\left(U^2V + UV^2 + V^2U + VU^2 - 2UVU - 2VUV\right) \\ = {} & \tfrac{1}{12}\left([U,[U,V]] + [V,[V,U]]\right)\end{aligned}$$

non-trivial part in B.C.H formula: n-th order is n^{th} commutator

§ IV. example $SL(2;\mathbb{R})$

1° $G = SL(2;\mathbb{R}) = \{ A \in M(2\times 2;\mathbb{R}) \mid \det A = 1\}$, $\quad M(2\times 2;\mathbb{R}) \cong \mathbb{R}^4 \ni \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}$

If $\det A = 1$, $A = (a_{ij})$, $\quad D(\det)\big|_A = \det(a_{ij}) \sum_{k\ell} (a^{-1})_{\ell k}\, DX_{k\ell} = \operatorname{tr}(A^{-1} DX) : \mathbb{R}^4 \to \mathbb{R}^1$

It is clearly surjective.

Hence, $SL(2;\mathbb{R})$ is a manifold, and a Lie group.

$\mathfrak{sl}(2;\mathbb{R}) = T_{\mathbb{I}} SL(2;\mathbb{R}) = \ker D(\det)\big|_{\mathbb{I}} = \{\text{traceless matrices}\}$

2° $\mathfrak{sl}(2;\mathbb{R})$ has the basis $\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix},\ f = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix},\ h = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} \right\}$

Same as $\mathfrak{gl}(n;\mathbb{R})$, bracket coincides with matrix bracket $\boxed{\text{check}}$

$$[e,f] = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} = h$$

$$[h,f] = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} = -2f$$

$$[h,e] = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0\end{bmatrix} = 2e$$

3° example of Frobenius theorem. $\tilde{e}, \tilde{f}, \tilde{h}$: left-invariant extension

- span$\{\tilde{e}, \tilde{f}\}$ is NOT involutive
- span$\{\tilde{h}, \tilde{f}\}$ is involutive

$$\exp(th) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \qquad \exp(tf) = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix}$$

$\rightsquigarrow$ Its integral through $\mathbb{I}$ is $\left\{ \begin{bmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{bmatrix} \middle| \begin{array}{l} \lambda > 0 \\ \mu \in \mathbb{R} \end{array} \right\}$

- similarly, span$\{\tilde{h}, \tilde{e}\}$ is involutive

4° adjoint representation of $sl(2;\mathbb{R})$: in terms of $\{e, f, h\}$

$$ad_e = \begin{bmatrix} & & -2 \\ & & \\ & 1 & \end{bmatrix} \qquad ad_f = \begin{bmatrix} & & \\ & & 2 \\ -1 & & \end{bmatrix} \qquad ad_h = \begin{bmatrix} 2 & & \\ & -2 & \\ & & 0 \end{bmatrix}$$

In Lie theory, one uses adjoint representation to define a symmetric bilinear form on $\mathfrak{g}$ by $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ $\longrightarrow \in End(\mathfrak{g})$

$(U, V) \mapsto tr(ad_U ad_V)$

This is called the Killing form., $B(U,V)$

(In general, G is not a matrix group, but $B(U,V)$ is well-defined)

$$ad_e = E_{32} - 2E_{13}, \quad ad_f = -E_{31} + 2E_{23}, \quad ad_h = 2E_{11} - 2E_{22}$$

$$\begin{aligned}
\Rightarrow ad_e\, ad_e &= (E_{32} - 2E_{13})(E_{32} - 2E_{13}) = -2E_{12} & tr &= 0 \\
ad_e\, ad_f &= (E_{32} - 2E_{13})(-E_{31} + 2E_{23}) = 2E_{11} + 2E_{33} & tr &= 4 \\
ad_e\, ad_h &= (E_{32} - 2E_{13})(2E_{11} - 2E_{22}) = -2E_{32} & tr &= 0 \\
ad_f\, ad_f &= (-E_{31} + 2E_{23})(-E_{31} + 2E_{23}) = -2E_{21} & tr &= 0 \\
ad_f\, ad_h &= (-E_{31} + 2E_{23})(2E_{11} - 2E_{22}) = -2E_{31} & tr &= 0 \\
ad_h\, ad_h &= (2E_{11} - 2E_{22})(2E_{11} - 2E_{22}) = 4E_{11} + 4E_{22} & tr &= 8
\end{aligned}$$

$$\Rightarrow B = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \left\{ \tfrac{1}{\sqrt{8}}(e-f), \tfrac{1}{\sqrt{8}}(e+f), \tfrac{1}{\sqrt{8}} h \right\}$$

related $\rightsquigarrow$ 1-dim'l negative signature $\rightsquigarrow$ 1-dim'l compact factor

$$SL(2;\mathbb{R}) \cong \underset{\text{compact}}{S^1} \times \underset{\text{non-compact}}{D^2}$$