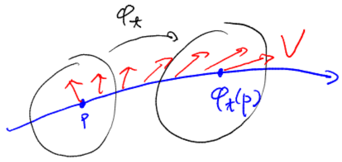
from the derivation viewpoint

## §I. Lie derivative

$$[U,V] \overset{?}{=} L_U V, \quad (L_U V)\big|_p = \lim_{t\to 0} \frac{(\varphi_{-t})_*(V|_{\varphi_t(p)}) - V|_p}{t}$$




where $\varphi_t$ : flow generated by $U$

For any $f \in C^\infty(M)$, examine $(L_U V)(f)$ at $p$.

$$\big((\varphi_{-t})_*(V|_{\varphi_t(p)})\big)(f) = V(f\circ\varphi_{-t})\big|_{\varphi_t(p)}$$


$$\lim_{t\to 0} \frac{(f\circ\varphi_{-t})(q) - f(q)}{t} = -U(f)\big|_q$$

$$\Rightarrow (f\circ\varphi_{-t})(q) - f(q) = t\, g(t,q)$$

where $g(0,q) = -U(f)\big|_q$

$$\Rightarrow \big(V(f\circ\varphi_{-t})\big)(q) = (V(f))(q) + t\big(V(g(t,-))\big)(q)$$

$$\Rightarrow \frac{1}{t}\Big(V(f\circ\varphi_{-t})\big|_{\varphi_t(p)} - V(f)\big|_p\Big) = \frac{1}{t}\Big((V(f))(\varphi_t(p)) - (V(f))(p)\Big) + V(g(t,-))(\varphi_t(p))$$

$t\to 0$: derivative of $V(f)$ along $U$ $= U(V(f))\big|_p$

$t \to 0$: $V(g(0,-))\big|_p = V(-U(f))\big|_p = -V(U(f))\big|_p$

Q.E.D.

## §II. Jacobi identity

$\forall f \in C^\infty(M)$,

$$[V,W](f) = V(W(f)) - W(V(f))$$

$$[U,[V,W]](f) = U(V(W(f))) - U(W(V(f))) - V(W(U(f))) + W(V(U(f)))$$

$$[W,[U,V]](f) = W(U(V(f))) - W(V(U(f))) - U(V(W(f))) + V(U(W(f)))$$

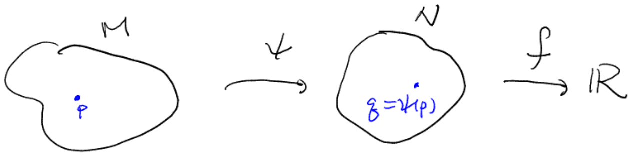
$$[V,[W,U]](f) = V(W(U(f))) - V(U(W(f))) - W(U(V(f))) + U(W(V(f)))$$

$\Rightarrow [U,[V,W]] + [W,[U,V]] + [V,[W,U]]$ acts on $C^\infty(M)$ trivially

Hence, it is zero everywhere

## § II. push-forward

$\psi : M \to N$ diffeomorphism $\Rightarrow$ $\psi_*([U,V]) \overset{?}{=} [\psi_* U, \psi_* V]$




$\forall f \in \mathcal{C}^\infty(N)$ and $U \in \Gamma(M, TM)$

$$(\psi_* U)(f)\big|_q = U(f\circ\psi)\big|_p$$

$$\Rightarrow (\psi_* U)(f) = U(f\circ\psi)\circ\psi^{-1} \in \mathcal{C}^\infty(N)$$

$$\psi_*([U,V])(f) = \big([U,V](f\circ\psi)\big)\circ\psi^{-1}$$
$$= \big(U(V(f\circ\psi)) - V(U(f\circ\psi))\big)\circ\psi^{-1}$$

$$(\psi_* V)(f) = V(f\circ\psi)\circ\psi^{-1}$$

$$\Rightarrow (\psi_* U)\big((\psi_* V)(f)\big) = U\big(V(f\circ\psi)\circ\psi^{-1}\circ\psi\big)\circ\psi^{-1} = U(V(f\circ\psi))\circ\psi^{-1}$$

$$\Rightarrow [\psi_* U, \psi_* V](f) = \big(U(V(f\circ\psi)) - V(U(f\circ\psi))\big)\circ\psi^{-1}$$

same q.e.d.