Frobenius theorem

goal "characterize" tangent bundle of a submanifold

§I. more on the vector fields

1° $V \in \Gamma(M; TM) = \{\text{vector fields}\}$

$\Leftrightarrow$ $V = v^i(x)\frac{\partial}{\partial x^i}$ in local coordinate ($v^i(x)$ = smooth)

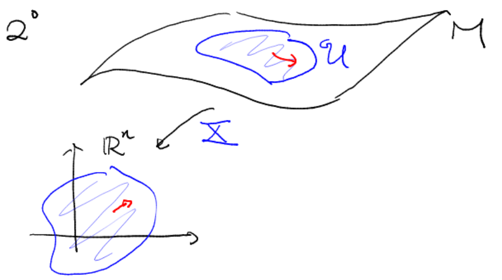
for $V = \tilde{v}^k(y)\frac{\partial}{\partial y^k}$

$$V(\cancel{f}) = v^i \frac{\partial \cancel{f}}{\partial x^i} = v^i \frac{\partial f}{\partial y^k}\frac{\partial y^k}{\partial x^i} = \tilde{v}^k \frac{\partial \cancel{f}}{\partial y^k} \Rightarrow \tilde{v}^k = v^i \frac{\partial y^k}{\partial x^i}$$

— (*)

$\Leftrightarrow$ $V : C^\infty(M) \to C^\infty(M)$

$f \longmapsto V(f)$ $\mathbb{R}$-linear and $V(fg) = V(f)g + fV(g)$

2° 

Denote X^{-1} by $F : X(\mathcal{U}) \to M \hookrightarrow \mathbb{R}^N$

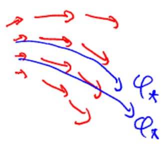
In regular surfaces $v^i \frac{\partial}{\partial x^i}$ is the image of $\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ under $DF = \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial F^N}{\partial x^1} & & \frac{\partial F^N}{\partial x^n} \end{bmatrix}_{N \times n}$

On each chart, we have $V(x) = (v^1, \cdots, v^n)$

(*) tells how the expression changes in different charts

3° <u>defn</u> Given any $V \in \Gamma(M; TM)$ (compact), it generates a one-parameter family of subgroup of diffeomorphisms by $\frac{d}{dt}\varphi_t = V$ $\varphi_0 = \mathrm{Id}$

<u>explanation</u>: • locally $\varphi_t(p) = (\varphi^1(x,t), \cdots, \varphi^n(x,t))$ $\begin{cases} \frac{d\varphi^j}{dt} = V^j(\varphi(x,t)) \\ \varphi^j(x,0) = x^j \end{cases}$



By ODE (and IFT, ...)

$\exists!$ solution for $|t| \ll 1$,

and $\varphi_t : M \to M$ diffeomorphism

• $\varphi_{t+t_0} = \varphi_t \circ \varphi_{t_0}$

In coordinate LHS = $\{\varphi^j(x, t+t_0)\}$ RHS = $\{\varphi^j(\varphi(x,t_0), t)\}$ (initial point)

$\frac{d}{dt} = \frac{d}{d(t+t_0)}$ $\to$ V^j(that point) $\frac{d}{dt}$ $\to$ V^j(that point)

Hence, both hand sides are sol'n to $\frac{d}{dt}\psi^j(p,t) = V^j(\psi(p,t))$

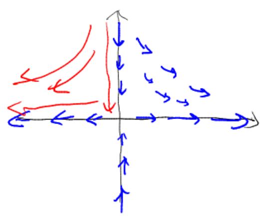
By the uniqueness of ODE, DONE. with $\psi(p,0) = \varphi(p,t_0)$

$\Rightarrow$ $\varphi : \mathbb{R}^1 \times M \to M$ can be defined $\forall t \in \mathbb{R}$

and $\varphi_{-t} \circ \varphi_t = \varphi_0 = \mathrm{Id}$ $\Rightarrow$ $\varphi_{-t} = (\varphi_t)^{-1}$

• <u>e.g.</u> $V(x,y) = (x, -y)$ on $\mathbb{R}^2$

$\frac{dx}{dt} = x$, $\frac{dy}{dt} = -y$ $\Rightarrow$ $\varphi((x_0,y_0),t) = (x_0 e^t, y_0 e^{-t})$



§II. Lie bracket

1° $U, V \in \Gamma(M; TM)$ two vector fields

$$\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$
$$f \mapsto U(V(f)) \quad \text{still a vector field / derivation?}$$

$$fg \mapsto U(V(fg)) = U\big(V(f)\,g + f\,V(g)\big)$$
$$= U(V(f))\,g + V(f)\,U(g) + U(f)\,V(g) + f\,U(V(g))$$

$\Rightarrow$ NOT a derivation

But $f \mapsto U(V(f)) - V(U(f))$ is a derivation

(the red terms cancel)

2° <u>defn</u>

$$\Gamma(M;TM) \times \Gamma(M;TM) \to \Gamma(M,TM)$$
$$(U, V) \mapsto [U,V] = UV - VU$$

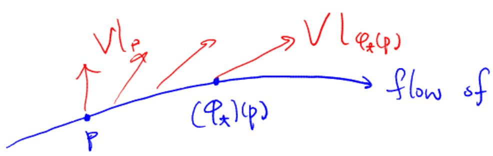
is called the Lie bracket of U and V

<u>rmk</u> in local coordinate $U = u^i(x)\frac{\partial}{\partial x^i}$, $V = v^j(x)\frac{\partial}{\partial x^j}$

$$[U,V](f) = u^i\frac{\partial}{\partial x^i}\Big(v^j\frac{\partial f}{\partial x^j}\Big) - v^i\frac{\partial}{\partial x^i}\Big(u^j\frac{\partial}{\partial x^j}\Big)$$
$$= \Big(u^i\frac{\partial v^j}{\partial x^i} - v^i\frac{\partial u^j}{\partial x^i}\Big)\Big(\frac{\partial f}{\partial x^j}\Big)$$

3° prop $[U,V]$ is the "Lie derivative" of V along the flow of U

Namely, let φ_t be the one-parameter family of diffeomorphism generated by U. Then, $[U,V] = L_U V := \frac{d}{dt}\Big|_{t=0} (D\varphi_{-t})\Big|_{\varphi_t(p)}\big(V|_{\varphi_t(p)}\big)$



$\varphi_{-t}: M \to M$

$(\varphi_t)(p) \mapsto p$

$(D\varphi_{-t})\big|_{\varphi_t(p)} : T_{\varphi_t(p)}M \to T_pM$

$V|_{\varphi_t(p)} \mapsto$ some vector

$$= \lim_{t\to 0} \frac{(D\varphi_{-t}|_{\varphi_t(p)})(V|_{\varphi_t(p)}) - V|_p}{t}$$

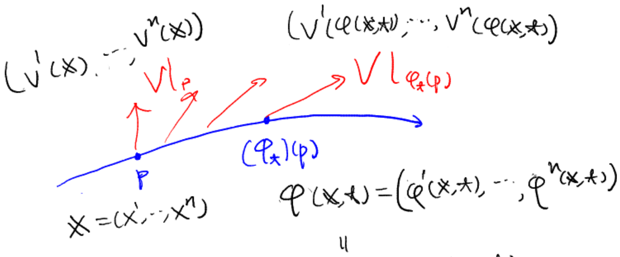
both in T_pM

Same vector space

Pf: $V = v^i(x)\frac{\partial}{\partial x^i}$ $\quad U = u^j(x)\frac{\partial}{\partial x^j}$

In coordinate, $\varphi_t(p)$ is given by $\{\varphi^j(x,t)\}$

where $\frac{d}{dt}\varphi^j(x,t) = u^j(\varphi(x,t))$, $\varphi^j(x,0) = x^j$



components of $(D\varphi_{-t})\big|_{\varphi_t(p)}\big(V|_{\varphi_t(p)}\big) = ?$

$$e_i \longmapsto \frac{\partial\psi}{\partial y^i} = \Big(\frac{\partial\psi^1}{\partial y^i}, \cdots, \frac{\partial\psi^n}{\partial y^i}\Big)$$

$$\psi(y,t) = \varphi(y,-t) \longleftarrow y = y(x,t)$$

$$v^i(\varphi(x,t))\,e_i \longmapsto v^i(\varphi(x,t))\frac{\partial\psi^j}{\partial y^i}\,e_j$$

$$e_j\text{-component of } \frac{d}{dt}\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0}\Big(v^i(\varphi(x,t))\frac{\partial\psi^j}{\partial y^i}\Big)$$

$$= \left(\frac{d}{dt} v^i(\varphi(x,t))\right)\Big|_{t=0} \frac{\partial\psi^j}{\partial y^i}\Big|_{t=0} + v^i(x)\,\frac{\partial^2\psi^j}{\partial t\,\partial y^i}\Big|_{t=0}$$

$$= \frac{\partial v^i}{\partial y^k}\Big|_{\varphi(x,0)} \frac{\partial\varphi^k}{\partial t}\Big|_{t=0} \frac{\partial\psi^j}{\partial y^i}\Big|_{t=0} + v^i(x)\,\frac{\partial^2\psi^j}{\partial t\,\partial y^i}\Big|_{t=0}$$

(i) $\frac{\partial v^i}{\partial y^k}\Big|_{\varphi(x,0)}$: $x \longmapsto y=\varphi(x,0)=x \longmapsto v \Rightarrow \frac{\partial v^i}{\partial y^k}\Big|_{\varphi(x,0)} = \frac{\partial v^i}{\partial x^k}\Big|_x$

(ii) $\frac{\partial\varphi^k}{\partial t}\Big|_{t=0}$: φ is the flow of $U \Rightarrow u^k(\varphi(x,t=0)) = u^k(x)$

(iii) $\frac{\partial\psi^j}{\partial y^i}\Big|_{t=0}$: From Taylor, $\psi^j(y,t) = y^j - t\,u^j(y) + O(t^2) \Rightarrow \delta^j_i$

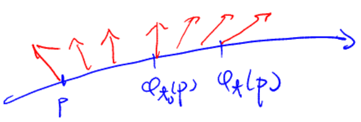
(iv) As (iii), $\frac{\partial^2\psi^j}{\partial t\,\partial y^i}\Big|_{t=0} = -\frac{\partial u^j}{\partial y^i}\Big|_{t=0}$.

Then as in (i), $y=x$ at $t=0$ $\Rightarrow -\frac{\partial u^j}{\partial x^i}$

Sum up $\frac{\partial v^j}{\partial x^k}u^k - v^i\frac{\partial u^j}{\partial x^i}$: exactly the j-th component of $[U,V]$ ※

Cor If $[U,V]\equiv 0$, then $\varphi_t\circ\psi_s = \psi_s\circ\varphi_t$
where φ_t & ψ_s are the one-parameter family of diffeomorphisms generated by U & V, respectively.

Pf: i) $[U,V]\equiv 0 \Rightarrow \frac{d}{dt}\Big|_{t=t_0} (D\varphi_{-t})(V|_{\varphi_t(p)}) = 0 \quad \forall t_0$



$$= \lim_{t\to t_0}\frac{1}{t-t_0}\left((D\varphi_{-t})(V|_{\varphi_t(p)}) - (D\varphi_{-t_0})(V|_{\varphi_{t_0}(p)})\right)$$

$t-t_0=\varepsilon$

$-t=-t_0-\varepsilon$

$\Rightarrow \varphi_{-t} = \varphi_{-t_0}\circ\varphi_{-\varepsilon}$

$\Rightarrow D\varphi_t = D\varphi_{t_0}\circ D\varphi_\varepsilon$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(D\varphi_{-t_0}(D\varphi_\varepsilon(V|_{\varphi_t(p)})) - (D\varphi_{-t_0})(V|_{\varphi_{t_0}(p)})\right)$$

$$= D\varphi_{-t_0}\left(\lim_{\varepsilon\to 0}\frac{D\varphi_\varepsilon(V|_{\varphi_t(p)}) - V|_{\varphi_{t_0}(p)}}{\varepsilon}\right)$$

$$\overset{\text{prop}}{=} D\varphi_{-t_0}\left([U,V]|_{\varphi_{t_0}(p)}\right) = 0$$

Hence $V|_{\varphi_t(p)} = (D\varphi_t)(V|_p)$

ii) Fixing $t\in\mathbb{R}$, $p\in M$. let $\gamma(s) = \varphi_t\circ\psi_s(p)$
$\tilde\gamma(s) = \psi_s\circ\varphi_t(p)$

$\gamma(0) = \varphi_t(p) = \tilde\gamma(0)$

By i), $\frac{d}{ds}\gamma(s) = (D\varphi_t)(V) = V|_{\gamma(s)}$

By definition, $\frac{d}{ds}\tilde\gamma = V|_{\tilde\gamma(s)}$

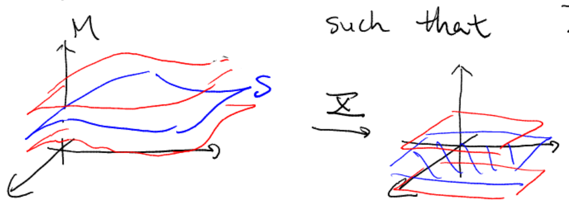
$\Rightarrow \gamma(s)$ & $\tilde\gamma(s)$ satisfy the same ODE with the same initial condition $\Rightarrow \gamma(s)\equiv\tilde\gamma(s)$ ※

upshot $[U,V]$ measures the infinitesimal non-commutativity of their flows.

§Ⅲ. Frobenius theorem

1° $S^k \subset M^n$ submanifold

By IFT, $\forall p \in S$ $\exists$ coordinate nbd of M $\{\mathcal{U}, \Phi\}$ such that $\Phi(S^k \cap \mathcal{U}) = \{x^{k+1} = x^{k+2} = \cdots = x^n = 0\}$



- locally, M is foliated by S
  i.e. $\mathcal{U} \overset{\text{diffeo}}{\cong} \mathbb{R}^k \times \mathbb{R}^{n-k}$
  each $\mathbb{R}^k \times \{q\} \overset{\text{diffeo}}{\cong} S \cap \mathcal{U}$

- $(x^1, \cdots, x^k)$ coordinate for each slice, $\{\frac{\partial}{\partial x^j}\}_{j=1}^k$ basis for $T(\mathbb{R}^k \times \{q\})$
- In general, frame (basis for $T_x(\mathbb{R}^k \times \{q\})$ $\forall x \in \mathbb{R}^k$) takes the form $\{\sum_{i=1}^k a_j^i(x^1, \cdots, x^n) \frac{\partial}{\partial x^i}\}_{j=1}^k$ where $\det(a_j^i(x)) \neq 0$

  $\Rightarrow$ their brackets $\in \text{span}\{\sum_i a_j^i(x) \frac{\partial}{\partial x^i}\}_{j=1}^k = \text{span}\{\frac{\partial}{\partial x^j}\}_{j=1}^k$

2° defn/question Fixing $k \le n$, a rank k distribution of TM is a subset $H \subset TM$ satisfying
- $H_p = H \cap T_pM$ is a k-diml subspace $\forall p$.
- It is smooth in the sense that locally, $H = \text{span}\{\sum_{\mu=1}^n a_j^\mu(x^1, \cdots, x^n) \frac{\partial}{\partial x^\mu}\}_{j=1}^k$ for a_j^μ : smooth.

$\Rightarrow$ When is H "integrable"?
(come from TS for a submanifold S)

3° discussion. From 1°, if integrable,

involutive ($\forall$ (locally defined) vector fields $U, V \in H$ $\Rightarrow [U,V] \in H$

thm [Frobenius] H is involutive iff $\forall p \in M$ $\exists$ coordinate around p $(\mathcal{U}, \Phi)$ such that $H = \text{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^k$
(namely, each $\mathbb{R}^k \times \{q\}$ is an integral of H)

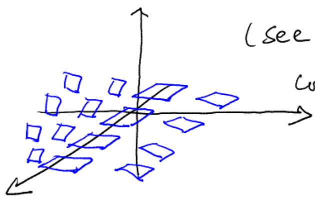
($\Leftarrow$ has been discussed above)

<u>example</u> On $\mathbb{R}^3$, consider $H = \mathrm{span}\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1, 0, y) \qquad\quad (0, 1, 0)$

$$\left[\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right] = -\frac{\partial}{\partial z} \notin H$$

$\Rightarrow$ H is NOT involutive



(see wikipedia: contact geometry)

<u>rmk</u> $U = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \rightsquigarrow \varphi_t$

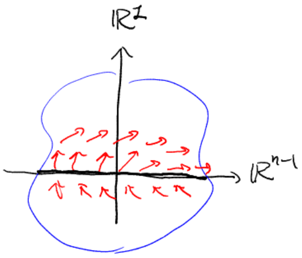
$V = \frac{\partial}{\partial y} \rightsquigarrow \psi_s$

$\varphi_t \circ \psi_s - \psi_s \circ \varphi_t = (0, 0, \text{some height})$: not connected by an integral curve generated by $\mathrm{span}\{U, V\}$.

4° <u>lemma</u> Frobenius theorem is true for rank $= 1$

pf: (every rank 1 distribution is involutive. locally $H = \mathbb{R}\langle V\rangle \Rightarrow [V, V] = 0$)

$V = \sum_{j=1}^{n} V^j(x)\frac{\partial}{\partial x^j}$ $\qquad V^j(x)$: not all zero.



For convenience, assume we work on an nbd of the origin and $V^n(0) \neq 0 \Rightarrow V^n(x) \neq 0$ for $|x| \ll 1$

Consider $(x^1, \cdots, x^{n-1}, t) \longmapsto \varphi_t(x^1, \cdots, x^{n-1}, 0) = (\varphi^1, \cdots, \varphi^n)$

where φ_t is the flow generated by V

$\left((x^1, \cdots, x^{n-1}, 0) + t(V^1(x^1 \cdots x^{n-1}, 0), \cdots, V^n(x^1 \cdots x^{n-1}, 0)) + O(t^2)\right)$

$$\Rightarrow D\varphi\big|_0 = \begin{bmatrix} 1 & & & V^1 \\ & \ddots & & \vdots \\ & & 1 & V^{n-1} \\ 0 & \cdots & 0 & V^n \end{bmatrix} \neq 0$$

By IFT, $(x^1, \cdots, x^{n-1}, t)$ is a local coordinate

By construction, $\frac{\partial}{\partial t} = V$ $\qquad \left(\frac{\partial}{\partial t} \longmapsto \frac{\partial \varphi_t}{\partial t} = V\big|_{\varphi_t(x^1, \cdots, x^{n-1}, 0)}\right)$ $\divideontimes$

5° [Proof of Frobenius] Induction on the rank:

<u>idea</u> $H = \mathrm{span}\{V_1, \cdots, V_{k-1}, V_k\} = \mathrm{span}\{\tilde{V}_1, \cdots, \tilde{V}_{k-1}, V_k\}$

replace $\qquad\qquad$ involutive $\Rightarrow$ induction

$\Rightarrow \mathbb{R}^n = \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1} \ni (x^1, \cdots, x^n)$

$\mathrm{span}\{\frac{\partial}{\partial x^j}\}_{j=1}^{k-1} = \mathrm{span}\{\tilde{V}_j\}_{j=1}^{k-1}$. Then study V_k in this coordinate

i) Apply 4° on $\mathrm{span}\{V_k\} \Rightarrow \exists\{y^1, \cdots, y^n\} \ni \frac{\partial}{\partial y^k} = V_k$

Let $f(y) = y^k$ a locally defined smooth function

<u>key</u> $V(f) = \frac{\partial}{\partial y^k} y^k = 1$

Let $\tilde{V}_j = V_j - V_j(f)\, V_k$, $j \in \{1, \cdots, k-1\}$. Then $\tilde{V}_j(f) = V_j(f) - V_j(f) \equiv 0$

ii) claim $\text{span}\{\tilde{V}_1, \dots, \tilde{V}_{k-1}\}$ is involutive

Since $\text{span}\{V_1, \dots, V_k\} = \text{span}\{\tilde{V}_1, \dots, \tilde{V}_{k-1}, V_k\}$ is involutive

$$[\tilde{V}_i, \tilde{V}_j] = h_{ij} V_k + \text{span}\{\tilde{V}_\ell\}_{\ell=1}^{k-1} \qquad h_{ij}: \text{locally defined smooth function}$$

Act on f $\Rightarrow$ $\tilde{V}_i(\underbrace{\tilde{V}_j(f)}_{\equiv 0}) - \tilde{V}_j(\underbrace{\tilde{V}_i(f)}_{\equiv 0}) = h_{ij} + (*)\underbrace{\tilde{V}_\ell(f)}_{\equiv 0} \Rightarrow h_{ij} \equiv 0$

Namely $\text{span}\{\tilde{V}_i\}_{i=1}^{k-1}$ is involutive

iii) By induction hypothesis, $\exists$ local coordinate $(x^1, \dots, x^n)$ such that $\text{span}\{\tilde{V}_i\}_{i=1}^{k-1} = \text{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^{k-1}$

Express V_k by this coordinate: $V_k = \sum_{\ell=1}^{n} S^\ell(x) \frac{\partial}{\partial x^\ell}$

known: $\left\{ \begin{array}{l} \text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k-1}}, V_k\} \text{ is involutive} \\ \tilde{V}_i(f) \equiv 0 \Rightarrow \frac{\partial}{\partial x^j} f \equiv 0 \qquad i, j \in \{1, \dots, k-1\} \end{array} \right.$

For any $j \in \{1, \dots, k-1\}$, $[\frac{\partial}{\partial x^j}, V_k] = \sum_{\ell=1}^{k-1} a_j^\ell \frac{\partial}{\partial x^\ell} + b^j V_k$

Act on f $\Rightarrow$ $\underbrace{\frac{\partial}{\partial x^j}(\underbrace{V_k(f)}_{\equiv 1})}_{0} - \underbrace{V_k(\underbrace{\frac{\partial f}{\partial x^j}}_{\equiv 0})}_{0} = \sum_{\ell=1}^{k-1} a_j^\ell \underbrace{\frac{\partial f}{\partial x^\ell}}_{\equiv 0} + b^j V_k(f)$ $\Rightarrow b^j \equiv 0$

Namely. $[\frac{\partial}{\partial x^j}, V_k] \in \text{Span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k-1}}\}$

$$= \sum_{\ell=1}^{n} \frac{\partial S^\ell}{\partial x^j} \frac{\partial}{\partial x^\ell} \Rightarrow \frac{\partial S^\ell}{\partial x^j} \equiv 0 \quad \text{for } \ell \in \{k, k+1, \dots, n\},\ j \in \{1, \dots, k-1\}$$

We can replace V_k by $\tilde{V}_k = V_k - \sum_{\ell=1}^{k-1} S^\ell \frac{\partial}{\partial x^\ell}$ $\in \text{Span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k-1}}\}$

$$= \sum_{\ell=k}^{n} S^\ell(\cancel{x^1}, \cancel{\dots}, \cancel{x^{k-1}}, x^k, \dots, x^n) \frac{\partial}{\partial x^n}$$

Hence, $H = \text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k-1}}, \tilde{V}_k\}$
where $\tilde{V}_k$ is completely on the $\mathbb{R}^{n-k+1}$ component

We can apply 4° for $\tilde{V}_k$ on $\mathbb{R}^{n-k+1} \ni (x^k, x^{k+1}, \dots, x^n)$
$\Rightarrow \exists (u^k, \dots, u^n)$ such that $\text{span}\{\frac{\partial}{\partial u^k}\} = \text{span}\{\tilde{V}_k\}$

Then, for the coordinate $((x^1, \dots, x^{k-1}), (u^k, \dots, u^n)) \in \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1}$

$$\text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k-1}}, \frac{\partial}{\partial u^k}\} = H$$

※