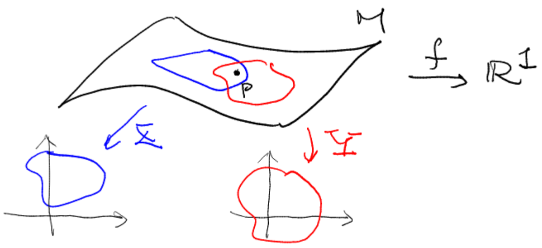
Aside : Morse function

<u>defn</u> $f : M^n \to \mathbb{R}^1$ is called a Morse function
if every critical point is non-degenerate



- p: critical if $\left.\frac{\partial f}{\partial x^i}\right|_p = 0 \quad \forall i$

  $\left(\left.\frac{\partial (f\circ \Phi^{-1})}{\partial x^i}\right|_{\Phi(p)} = 0 \right)$

- non-degenerate if $\det\left[\left.\frac{\partial^2 f}{\partial x^j \partial x^i}\right|_p \right] \neq 0$

- In general, Hessian(f) is NOT well-defined

$$\check{f}(y) := (f\circ\Psi^{-1})(y), \quad \hat{f}(x) := (f\circ\Phi^{-1})(x), \quad y(x) = (\Psi\circ\Phi^{-1})(x)$$

$$\Rightarrow \hat{f}(x) = \check{f}(y(x))$$

$$\Rightarrow \frac{\partial \hat{f}}{\partial x^i} = \sum_k \frac{\partial \check{f}}{\partial y^k}\frac{\partial y^k}{\partial x^i}$$

$$\Rightarrow \frac{\partial^2 \hat{f}}{\partial x^j \partial x^i} = \sum_{k,\ell} \frac{\partial^2 \check{f}}{\partial y^\ell \partial y^k}\frac{\partial y^\ell}{\partial x^j}\frac{\partial y^k}{\partial x^i} + \sum_k \overbrace{\frac{\partial \check{f}}{\partial y^k}}^{= 0 \text{ at critical point}} \frac{\partial^2 y^k}{\partial x^j \partial x^i}$$

$$\Rightarrow \det\left[\left.\frac{\partial^2 \hat{f}}{\partial x^j \partial x^i}\right|_{\Phi(p)} \right] = \det\left[\left.\frac{\partial^2 \check{f}}{\partial y^\ell \partial y^k}\right|_{\Psi(p)} \right] \left(\det\left[\left.\frac{\partial y}{\partial x}\right|_{\Phi(p)} \right] \right)^2$$

<u>rmk</u> in modern notation, if $df|_p = 0$

$$\Rightarrow \sum_{i,j} \left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p dx^i \otimes dx^j : T_pM \times T_pM \to \mathbb{R} \quad \text{is well-defined.}$$

§I. existence of Morse function

<u>goal</u> Sard theorem $\Rightarrow$ there are plenty of Morse functions

<u>lemma</u> $\mathcal{U} \subset \mathbb{R}^n$ open, $f : \mathcal{U} \to \mathbb{R}^1$ smooth.
Then, $f + \sum_{j=1}^n a_j x^j$ is Morse for a.e. $(a_1, \cdots, a_n) \in \mathbb{R}^n$

pf:

$$\left(\begin{array}{lll} a = (a_1, \cdots, a_n). & & f_a = f + \sum a_j x^j \\ p = \text{critical} & \Leftrightarrow & \left.\frac{\partial f_a}{\partial x^j}\right|_p = 0 = \left.\frac{\partial f}{\partial x^j}\right|_p + a_j \\ \text{non-degenerate} & \Leftrightarrow & \left.\frac{\partial^2 f_a}{\partial x^i \partial x^j}\right|_p = \left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p \text{ is invertible} \end{array} \right)$$

$\Updownarrow$

$\forall p$ w/ $g(p) = -a$

$Dg|_p$: isomorphism (surjective)

$\Updownarrow$

$-a$ is a regular value of g

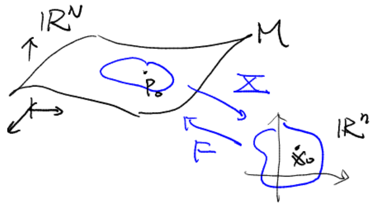
Consider $g = \left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n} \right) : \mathcal{U} \to \mathbb{R}^n$

$$Dg|_p = \left.\begin{bmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^1} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x^1 \partial x^n} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{bmatrix}\right|_p : \mathbb{R}^n \to \mathbb{R}^n$$

Apply Sard #

<u>thm</u> $f : M^n \to \mathbb{R}^1$ smooth, Whitney $M^n \xhookrightarrow{\text{embedding}} \mathbb{R}^N \ni (x^1, \cdots, x^N)$
Then, $f + \sum_{\mu=1}^N c_\mu x^\mu$ is Morse for a.e. $(c_1, \cdots, c_N) \in \mathbb{R}^N$

pf: step 0



From the assumption

$F: X(\mathcal{U}) \subset \mathbb{R}^n \to \mathbb{R}^N$ is an embedding

$\Rightarrow DF$ is injective : $N \times n$

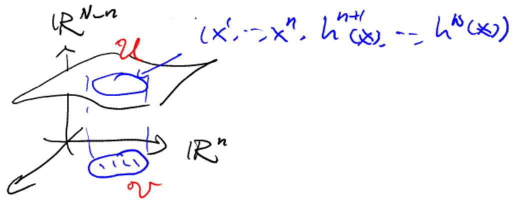
Assume the 1^{st} $n\times n$ block of $DF|_{x_0}$ is injective

$\Rightarrow$ By IFT, M near p_0 is given by $(x^1, \dots, x^n, h^{n+1}(x), \dots, h^N(x))$

Due to second countable / paracompact, M admits a countable cover of such coordinate chart

coordinates = n-coordinates in $\mathbb{R}^N$

step 1 On each coordinate chart



$\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^{N-n}$

$c = (a, b)$

In the coordinate, $f + \Sigma c_\mu x^\mu = (f(x, h(x)) + b\cdot h(x)) + a\cdot x$ on $\mathcal{V}$

For each $b \in \mathbb{R}^{N-n}$. lemma $\Rightarrow$ it is Morse except for $a \in$ some measure zero set

By Fubini, $S = \{ c = (a,b) \in \mathbb{R}^n \times \mathbb{R}^{N-n} \mid f + c\cdot x \text{ is not Morse} \}$

$= \bigcup_{b \in \mathbb{R}^{N-n}} \{ a \in \mathbb{R}^n \mid (f + b\cdot h(x)) + a\cdot x \text{ is not Morse} \}$

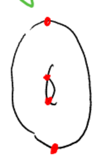
measure zero $\forall b$

still has measure zero.

step 2 There are countably many charts $\Rightarrow$ still measure zero ※

§II. baby Morse theory

1° Morse theory :



The topological structure can be understood from a Morse function.

Topology changes only passing through a critical point (thinking Morse = height)

Cor (Reeb) If $f: M^n \to \mathbb{R}^1$ (compact), Morse with only two critical points (max & min)

Then. M is homeomorphic to S^n.

2° $S^7 \to S^4$ $\mathbb{H} \ni u_0 + i u_1 + j u_2 + k u_3$ $ij = k \cdots$ quaternion algebra

$S^7 \subset \mathbb{H}^2 \cong \mathbb{R}^8$

$= \{ (u, v) \mid |u|^2 + |v|^2 = 1 \}$ $|u|^2 = uu^*$ $u^* = u_0 - iu_1 - ju_2 - ku_3$

$\downarrow \pi$

$(2u^* v, |u|^2 - |v|^2) \in \mathbb{H} \times \mathbb{R} \cong \mathbb{R}^5$

$$4|u^*v|^2 + (|u|^2-|v|^2)^2 = (|u|^2+|v|^2)^2 = 1 \Rightarrow \pi(S^7) \subset S^4$$

$\boxed{\text{check}}$ $\pi: S^7 \to S^4$ is surjective, and $\pi^{-1}(q) \cong S^3$ $\forall q \in S^4$

rmk This is analogous to the Hopf fibration $S^1 \to S^3 \to S^2$

key if $\pi(u,v) = q \in S^4$

Consider $S^3 = \{ g \in \mathbb{H} \cong \mathbb{R}^4 \mid |g| = 1 \}$

$$g \cdot (u,v) \mapsto (gu, gv)$$

$$\pi(gu, gv) = (2u^*g^*gv, |gu|^2 - |gv|^2)$$
$$= (2u^*v, |u|^2-|v|^2) = \pi(u,v)$$

3° Other 7-dim'l manifolds

$$g = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in SU(2) \cong S^3 \qquad |a|^2+|b|^2 = 1$$

Given any two integers: m & n introduce $M_{m,n}$ = 7-dim'l as follows

$$M_{p,q} = \mathbb{R}^4 \times S^3 \cup \mathbb{R}^4 \times S^3 \Big/ \sim$$

two coordinate charts (although S^3 is used)

transition

$$((r,g),h) \mapsto ((\tfrac{1}{r}, g^*), g^m h g^n) = ((\tilde{r},\tilde{g}),\tilde{h})$$

spherical coordinate $\in \mathbb{R}_{\geq 0} \times S^3$

In fact, one can check that $M_{1,0}$ is S^7.

The $M_{1,0}$-case suggests to consider for $m+n=1$

$$f: M_{m,n} \to \mathbb{R}^1$$
$$((r,g),h) \mapsto \frac{\operatorname{tr} h}{(1+r^2)^{\frac{1}{2}}}$$

$$r = \tilde{r}^{-1}, \quad h = (g^*)^m h (g^*)^n$$

$$\Rightarrow \frac{\operatorname{tr} h}{(1+r^2)^{\frac{1}{2}}} = \frac{\operatorname{tr}((g^*)^m \tilde{h} (g^*)^n)}{(1+\frac{1}{\tilde{r}^2})^{\frac{1}{2}}} = \frac{\tilde{r}}{(1+\tilde{r}^2)^{\frac{1}{2}}} \operatorname{tr}((g^*)^n (g^*)^m \tilde{h}) = \frac{\operatorname{tr}((\tilde{r}\tilde{g})\tilde{h})}{(1+\tilde{r}^2)^{\frac{1}{2}}}$$

$\boxed{\text{Check}}$ f is a Morse function with only two critical points on the $((r,g),h)$-chart.

$$\mathbb{R}^4 \times S^3 \ni (0, \pm \mathrm{Id}) \longrightarrow f = \pm 2$$

By Morse / Reeb, $M_{m,n}$ is homeomorphic to S^7 for $m+n=1$

Due to Milnor, when $(m+n)^2 \equiv 1 \pmod 7$, $M_{m,n}$ is NOT diffeomorphism to S^7

rmk $S^3 = \mathbb{R}^2 \times S^1 \cup \mathbb{R}^2 \times S^1 / \sim$

$$\underset{\substack{\| \\ z}}{(re^{i\theta}}, e^{i\alpha}) \sim (\underset{\substack{\| \\ w}}{\tfrac{1}{r}e^{-i\theta}}, e^{i\beta})$$

Hopf fibration: $S^3 \subset \mathbb{C}^2 \to S^2$

$$(\xi, \eta) \longmapsto (2\bar{\xi}\eta, |\xi|^2 - |\eta|^2)$$

$$\left(\frac{1}{\sqrt{1+|z|^2}}, \frac{z}{\sqrt{1+|z|^2}}\right) \longleftarrow \left(\frac{2z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right) \longleftarrow z \in \mathbb{C}$$

$$\left(\frac{e^{i\alpha}}{\sqrt{1+|z|^2}}, \frac{z}{\sqrt{1+|z|^2}}e^{i\alpha}\right) \longleftarrow (z, e^{i\alpha}) \in \mathbb{C} \times S^1$$

$w = \frac{1}{z}$

$$\frac{1}{\sqrt{1+\frac{1}{|w|^2}}}e^{i\alpha} = \frac{|w|}{\sqrt{1+|w|^2}}e^{i\alpha} = \frac{w}{\sqrt{1+|w|^2}}\left(e^{i\alpha}\frac{|w|}{w}\right)$$

$$\frac{z}{\sqrt{1+|z|^2}}e^{i\alpha} = \frac{1}{\sqrt{1+|w|^2}}\left(e^{i\alpha}\frac{|w|}{w}\right)$$

$$\rightsquigarrow (w, e^{i\beta}) \in \mathbb{C} \times S^1 \longmapsto \left(\frac{w}{\sqrt{1+|w|^2}}e^{i\beta}, \frac{1}{\sqrt{1+|w|^2}}e^{i\beta}\right)$$

$\Rightarrow$ transition: $w = \frac{1}{z}$, $\quad e^{i\beta} = e^{i\alpha}\frac{|w|}{w} = e^{i(\alpha - \arg w)} = e^{i(\alpha + \arg z)}$

Morse function: Choose the 1^{st} (real) coordinate $= \dfrac{2\cos\alpha}{\sqrt{1+|z|^2}}$

$$\mathbb{C}^2 \ni (\pm 1, 0) \longleftrightarrow (z=0, e^{i\alpha} = \pm 1)$$