Whitney embedding theorem

<u>thm</u> M : smooth manifold of dim n.
$\Rightarrow$ M admits an embedding into $\mathbb{R}^{2n+1}$

§I. reducing the dimension (injective immersion)

<u>prop 1</u> Suppose that M admits an injective immersion in $\mathbb{R}^{N>2n+1}$
Then, F admits an injective immersion in $\mathbb{R}^{2n+1}$

pf: <u>strategy</u> choose $a \in \mathbb{R}^N$

consider $\pi_a : \mathbb{R}^N \to \mathbb{R}^{N-1}$
$x \mapsto x - \frac{\langle x, a\rangle}{|a|^2} a$ orthogonal projection to $a^\perp$

$$M \xrightarrow{F} \mathbb{R}^N \xrightarrow{\pi_a} \mathbb{R}^{N-1}$$

- injective $\pi_a(F(p)) = \pi_a(F(q)) \Leftrightarrow p = q$
  $\Leftrightarrow F(p) - F(q) \,/\!/\, a$
- immersion $D(\pi_a \circ F)|_p$ is injective $\forall p$

π_a is linear $\rightarrow$ $\parallel$

$$\underset{(N-1)\times N}{\pi_a} \circ \underset{N\times n-1}{DF|_p} \leftarrow \text{known to be injective}$$

$\pi_a \circ DF|_p$ is NOT injective $\Leftrightarrow \exists\, v \in T_pM$ w/ $DF|_p(v) /\!/ a$, $v \neq 0$
$\Leftrightarrow DF|_p(\tilde{v}) = a$

Consider
$$\begin{cases} S : M \times M \times \mathbb{R} \to \mathbb{R}^N & 2n+1 < N \\ (p, q, \lambda) \mapsto \lambda (F(p) - F(q)) & \\ T : TM \to \mathbb{R}^N & 2n < N \\ v \in T_pM \mapsto (DF|_p)(v) & \end{cases}$$
$\Rightarrow$ singular value = image

By Sard, $\exists\, a \in \mathbb{R}^N$ regular value of S and T
$\Leftrightarrow$ NOT in the image of S and T.

Hence $a \neq 0$. and $\begin{cases} F(p) - F(q) \not/\!/ a \\ DF|_p(v) \neq a \quad \forall p \in M,\ v \in T_pM \end{cases}$

The above discussion shows that $\pi_a \circ F : M \to \mathbb{R}^{N-1}$
is still an injective immersion. #

§II compact case

<u>recall</u> [partition of unity subordinate to an open cover]. M can be non-compact
Given any open cover $\{U_\alpha\}_{\alpha \in A}$ of M $\exists \{\theta_j\}_{j=1}^\infty \subset C^\infty(M)$
such that

- $0 \le \theta_j \le 1$
- $\operatorname{supp} \theta_j \subset$ some U_α $\quad \operatorname{supp} \theta_\alpha = \text{closure}\{p \in M \mid \theta_\alpha(p) \neq 0\}$
- $\forall p \in M$, $\exists\, V$: open nbd $\ni$ $\theta_j|_V \equiv 0$
  except for finitely many j's
- $\sum_j \theta_j(p) \equiv 1 \quad \forall p \in M$

finite sum $\forall p$

i) Now, suppose that M is compact.

Choose a finite coordinate cover $\{(\mathcal{U}_\mu, \Phi_\mu)\}_{\mu=1}^{L}$

Then, choose a partition of unity subordinate to $\{\mathcal{U}_\mu\}_{\mu=1}^{L}$: $\{\theta_\mu\}_{\mu=1}^{L}$

Since finite cover, we can do so.

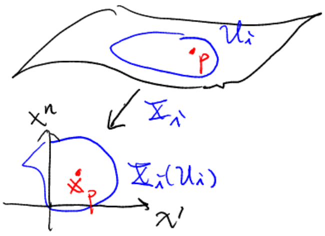
ii) Consider $M \longrightarrow \mathbb{R}^n$

$$p \longmapsto (\theta_\mu \cdot \Phi_\mu)(p) \in C^\infty(M) \qquad \theta_\mu \cdot \Phi_\mu \big|_{M \setminus \mathcal{U}_\mu} = 0$$

Try $M \to \mathbb{R}^{nL}$

$$p \longmapsto \left((\theta_1 \cdot \Phi_1)(p), \cdots, (\theta_L \cdot \Phi_L)(p) \right)$$

immersion? $\forall p \in M$, $\exists \mu \ni \theta_\mu(p) \neq 0$ $\qquad x_p = \Phi(p)$



$D(\theta_\mu \cdot \Phi_\mu) = ?$ in terms of the chart $(\mathcal{U}_\mu, \Phi_\mu)$

$$\theta_\mu \cdot \Phi_\mu : (x^1, \cdots, x^n) \longmapsto (\theta_\mu(x)\, x^1, \cdots, \theta_\mu(x)\, x^n)$$

$\theta_\mu(x) \to (\theta_\mu \circ \Phi_\mu^{-1})$

$$\underbrace{D(\theta_\mu \cdot \Phi_\mu)}_{n \times n}\Big|_{x_p} = \theta_\mu(x)\, \mathbb{I} + \begin{bmatrix} x_p^1 \\ \vdots \\ x_p^n \end{bmatrix} \left[\partial_1 \theta_\mu \cdots \partial_n \theta_\mu \right] \Big|_{x_p}$$

It is not clear $D(\theta_\mu \cdot \Phi_\mu)\big|_{x_p}$ is injective ($\Leftrightarrow$ invertible)

What about $(\theta_\mu \cdot \Phi_\mu, \theta_\mu) : \mathcal{U}_i \to \mathbb{R}^{n+1}$

$$D\begin{pmatrix} \theta_\mu \Phi_\mu \\ \theta_\mu \end{pmatrix}\Big|_{x_p} = \begin{array}{c} n \\ \\ 1 \end{array} \left[\begin{array}{c} \overset{\neq 0}{\theta_\mu(x_p)}\, \mathbb{I} \\ + x_p (\nabla \theta_\mu)^T \big|_{x_p} \\ \hline (\nabla \theta_\mu)^T \big|_{x_p} \end{array} \right] \underset{\text{operation}}{\overset{\text{row}}{\sim}} \left[\begin{array}{c} \theta_\mu(p)\, \mathbb{I} \\ \hline (\nabla \theta_\mu)^T \big|_{x_p} \end{array} \right]$$

NO kernel!

iii) Consider $F : M \longrightarrow \mathbb{R}^{(n+1)L}$

$$p \longmapsto \left((\theta_1 \cdot \Phi_1)(p), \theta_1(p), \cdots, (\theta_L \cdot \Phi_L)(p), \theta_L(p) \right)$$

From the above discussion, it is an immersion

Injectivity? if $F(p) = F(q)$ $\quad \exists \mu \in \{1, \cdots, L\}$ such that $\theta_\mu(p) = \theta_\mu(q) \neq 0$

$\Rightarrow p, q \in \mathcal{U}_\mu$

Also, $(\theta_\mu \cdot \Phi_\mu)(p) = \theta_\mu(q)\, \Phi_\mu(q) \Rightarrow \Phi_\mu(p) = \Phi_\mu(q)$ ※

iv) Summary Any compact, smooth manifold admits an injective embedding into $\mathbb{R}^{N \gg 1}$.

By Prop 1, the dimension can be reduced to $\mathbb{R}^{2n+1}$

Since M is compact, injective immersion = embedding

This finishes the proof of Whitney theorem for compact manifold.

§ III smooth exhaustion function

prop 2 $\exists \rho \in C^\infty(M)$. $\rho \ge 0$ and $\rho^{-1}((-\infty, c])$ is compact $\forall c > 0$

(This $\rho : M \to \mathbb{R}^1$ is proper)

pf: $\forall p$ $\exists$ coordinate chart $(\tilde{U}_p, \Phi_p)$ around p.

we may assume $\Phi_p(\tilde{U}_p) = B(0;2)$

Let $U_p = \Phi_p^{-1}(B(0;1)) \Rightarrow \{(U_p, \Phi_p)\}_{p\in M}$ is a coordinat cover

and $\overline{U_p} \subset M$ is compact

Let $\{\theta_i\}_{i=1}^\infty$ be a partition of unity subordinate to $\{U_p\}_{p\in M}$ $\Leftarrow$ $\operatorname{supp}\theta_i$ is compact $\forall i$

Consider $\rho(p) = \sum_{i=1}^\infty i\,\theta_i(p)$

Fix $n \in \mathbb{N}$. if $\rho(p) \le n$. then $\{\theta_i(p)\}_{i=1}^n$ cannot be all zero

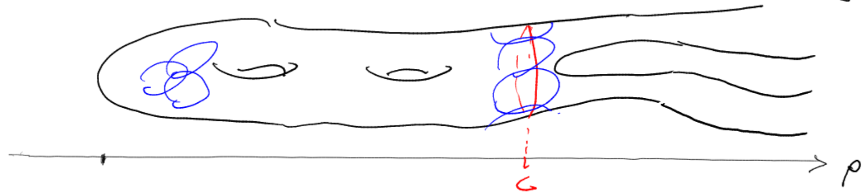
otherwise $\rho(p) \ge (n+1) \sum_{i=n+1}^\infty \theta_i(p) = n+1$

$\Rightarrow \rho^{-1}((-\infty, n]) \subset \bigcup_{i=1}^n \operatorname{supp}(\theta_i)$: compact ※

§ IV non-compact case.

lemma 1 ρ : smooth exhaustion of M^n

then $\forall c > 0$, $\rho^{-1}((-\infty, c))$ admits an injective immersion to $\mathbb{R}^{2n+1}$



pf: (Similar to prop 1) $\forall p \in \rho^{-1}((-\infty, c])$ $\leftarrow$ compact!

Chose (U_p, Φ_p) coordinate around p in M

Since $\rho^{-1}((-\infty,c])$ is compact, $\exists \{P_1, \cdots, P_L\} \to \bigcup_{j=1}^L U_{p_j} \supset \rho^{-1}((-\infty, c])$

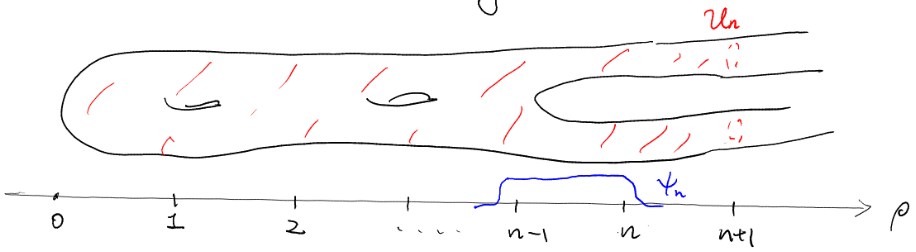
$\bigcup_{j=1}^L U_{p_j}$ is an open subset of M. and thus is a manifold

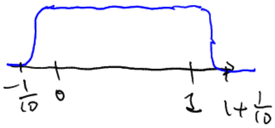
Let $\{\theta_j\}_{j=1}^L$ be a partition of unity subordinate to it.

$\Rightarrow$ restriction of $\big((\theta_1 \cdot \Phi_{p_1}, \theta_1), (\theta_2 \cdot \Phi_{p_2}, \theta_2), \cdots, (\theta_L \cdot \Phi_{p_L}, \theta_L)\big)$

on $\rho^{-1}((-\infty, c))$ is an injective immersion ※

lemma 2 M admits an embedding into $\mathbb{R}^{N \gg 1}$



pf: $\psi =$  $\in \mathcal{C}^\infty(\mathbb{R}^1)$ Let $\psi_n = \psi(\rho(p) - (n-1)) \in \mathcal{C}^\infty(M)$

Let $\mathcal{U}_n = \rho^{-1}((-\infty, n+1))$, Σ_n injective immersion of $\mathcal{U}_n$ to $\mathbb{R}^{2n+1}$ given by lemma 1

Consider $F : M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$

$$p \mapsto \Big(\sum_{n=odd} (\psi_n \cdot \Sigma_n)(p),\ \sum_{n=even} (\psi_n \cdot \Sigma_n)(p),\ \rho(p) \Big)$$

- injectivity: if $F(p) = F(q) \Rightarrow \rho(p) = \rho(q) \in [n-1, n)$ for some $n \in \mathbb{N}$

  Suppose n is odd, note that $\psi_{2m+1}(p) = 0$ except for $n = 2m+1$

  $$\Rightarrow \underbrace{\psi_n(p)}_{=1} \Sigma_n(p) = \underbrace{\psi_n(q)}_{=1} \Sigma_n(q) \Rightarrow \Sigma_n(p) = \Sigma_n(q)$$
  $$\Rightarrow p = q$$

- immersion: $\rho(p) \in [n-1, n)$ for some $n \in \mathbb{N}$

  Suppose n is odd.

  On an open nbd V of p, $\sum_{n=odd} \psi_n \cdot \Sigma_n \big|_V = \Sigma_n \big|_V$

  $\Rightarrow$ differential is injective

- embedding: Since $\rho : M \to \mathbb{R}^1$ is proper, F is proper as well ※

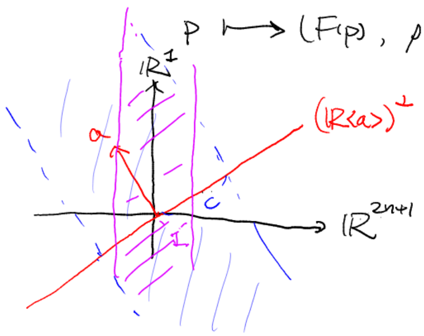
Proof of Whitney embedding theorem

It remains to prove that if there is an injective immersion $F : M \to \mathbb{R}^{2n+1}$, we can construct an embedding $\tilde{F} : M \to \mathbb{R}^{2n+1}$

key $\rho : M \to \mathbb{R}^1$ is proper (from prop 2)

$\Rightarrow \tilde{F} : M \to \mathbb{R}^{2n+1} \times \mathbb{R}$ is proper. $\Rightarrow a \in \mathbb{R}^{2n+1} \setminus \{0\}$, take $\pi_a \circ \tilde{F}$

$p \mapsto (F(p), \rho(p))$



If $|(\pi_a \circ F)(p)| < c \Rightarrow |\rho(p)| < \tilde{c}$?

- Since $\mathbb{R}^{2n+1} \cong$ open ball in $\mathbb{R}^{2n+1}$ (diffeomorphic)

  We may assume $|F(M)| < 1$

- Choose a : regular value in the proof of prop 1 and $a \ne (0, \cdots, 0, 1)$

  We may assume $|a| = 1$, $a = (\underset{\mathbb{R}^{2n+1}}{a_0}, \underset{\mathbb{R}^1}{a_1})$ $0 < |a_1| < 1$

  if $a_1 = 0$, last component after projection is ρ $\Rightarrow$ DONE

- $\forall c > 1$. If $|(\pi_a \circ \tilde{F})(p)| \le c$, then ?

$$|\rho(p)|^2 \le |F(p)|^2 + |\rho(p)|^2 = |\tilde{F}(p)|^2 = |(\pi_a \circ \tilde{F})(p)|^2 + (\underbrace{\langle \tilde{F}(p), a \rangle}_{\langle F(p), a_0 \rangle + a_1 \rho(p)})^2$$

recall $(u+v)^2 \le (1+\frac{1}{\varepsilon^2})u^2 + (1+\varepsilon^2)v^2$

Since $|a_1| < 1$, we can choose $\varepsilon = \varepsilon(a_1) \ni (1+\varepsilon^2)a_1^2 < 1$

$\Rightarrow \quad |\rho(p)|^2 \le C^2 + (1+\frac{1}{\varepsilon^2})|\langle F(p), a_0\rangle|^2 + (1+\varepsilon^2)a_1^2 |\rho(p)|^2$

$\Rightarrow \quad (1-(1+\varepsilon^2)a_1^2)\,|\rho(p)|^2 \le C^2 + (1+\frac{1}{\varepsilon^2})|a_0|^2$

Hence, if $|(\pi_a \circ \tilde{F})(p)| \le C$, $|\rho(p)| \le \tilde{C}$ $\Rightarrow$ $\pi_a \circ \tilde{F}$ is proper

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