STABILITY AND AREA-MINIMIZING PROPERTY OF HIGHER-DIMENSIONAL HELICOIDS

CHUNG-JUN TSAI, MAO-PEI TSUI, JINGBO WAN, AND MU-TAO WANG

Abstract. For each integer $k \geq 1$, we study the $(k+1)$-dimensional helicoid $H_k \subset \mathbb{R}^{2k+1}$ parametrized by

$$(u_1, \ldots, u_k, s) \longmapsto \left(u_1 e^{is}, \ldots, u_k e^{is}, s\right) \in \mathbb{C}^k \times \mathbb{R} \cong \mathbb{R}^{2k+1}.$$

These helicoids form a basic and distinguished family of complete, properly embedded minimal submanifolds diffeomorphic to $\mathbb{R}^{k+1}$, and provide natural higher-dimensional analogues of the classical helicoid in $\mathbb{R}^3$.

We completely determine their stability: H_k is stable for $k \geq 3$ and unstable for $k \leq 2$. The sharp transition at $k = 3$ is particularly striking: while the classical helicoid ($k = 1$) and its first higher-dimensional analogue ($k = 2$) are unstable, the four-dimensional helicoid $H_3 \subset \mathbb{R}^7$ is already stable.

For $k \geq 3$, we also determine their area-minimizing property: H_k is area-minimizing when k is even and not area-minimizing when k is odd. The area-minimizing result is proved by constructing an explicit calibration, while the non-area-minimizing result follows from an explicit competitor. In particular, for every even $k \geq 4$, the $(k+1)$-dimensional helicoid H_k is an entire minimal graph in $\mathbb{R}^{2k+1}$ that is area-minimizing.

1. Introduction

The classical helicoid, first described by Meusnier in 1776, is one of the most fundamental models in the theory of minimal surfaces. The work of Colding and Minicozzi shows, in a precise sense, that embedded minimal disks in a ball in $\mathbb{R}^3$ are modeled either on planes or on helicoids [4]. Meeks and Rosenberg proved that the plane and the helicoid are the only complete, properly embedded, simply connected minimal surfaces in $\mathbb{R}^3$ [13]. Bernstein and Breiner subsequently showed that every complete, non-flat, properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and one end is asymptotic to a helicoid [2]. These results underscore the central role of the helicoid in the local and global geometry of embedded minimal surfaces.

Despite this rigidity, the classical helicoid is unstable: it admits a compactly supported normal variation with negative second variation of area. The pioneering work of Fischer-Colbrie and Schoen [6] (see also do Carmo-Peng [5] and references in [6] for earlier results) on stable minimal surfaces shows that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane. Their arguments imply the instability of the helicoid indirectly through this rigidity theorem.

C.-J. Tsai is supported in part by the National Science and Technology Council grant 112-2628-M-002-004-MY4. M.-P. Tsui is supported in part by the National Science and Technology Council grant 115-2115-M-002-005-MY3. J. Wan is supported by ERC-2023 AdG 101141855 BLaHST. M.-T. Wang is supported in part by the National Science Foundation under Grant DMS-2404945. This work was supported by grants from the Simons Foundation [MPS-TSM-00007411, M.-T. W.]. Part of this work was carried out when M.-T. Wang was visiting the National Center for Theoretical Sciences.

The above contrast motivates the study of higher-dimensional helicoids. For any integer $k \geq 1$, let H_k be the image of the proper embedding

$$\begin{array}{rcl} \mathbb{R}^k \times \mathbb{R} & \to & \mathbb{C}^k \times \mathbb{R} \cong \mathbb{R}^{2k+1}, \\ (u_1, \ldots, u_k, s) & \mapsto & \big(u_1 e^{is}, \ldots, u_k e^{is}, s\big). \end{array} \tag{1}$$

It is not hard to see that H_k is a $(k+1)$-dimensional properly embedded minimal submanifold of $\mathbb{R}^{2k+1}$, and H_1 is the classical helicoid. This family belongs to the broader class of higher-dimensional ruled minimal submanifolds studied in [1].

Our first main theorem gives a complete and unexpectedly sharp answer to the stability question.

Theorem 1.1. *The $(k+1)$-dimensional helicoid H_k is stable if and only if $k \geq 3$. Namely,*

$$H_k \text{ is unstable for } k \leq 2, \qquad H_k \text{ is stable for } k \geq 3.$$

The proof proceeds by first showing that the stability condition is equivalent to the functional inequality (4) for vector-valued functions on $\mathbb{R}^k$ and then establishing this inequality when $k \geq 3$. When $k \leq 2$, the instability is proved by constructing unstable perturbations.

The dichotomy in Theorem 1.1 is perhaps the most surprising feature of the result. One might expect the instability of the classical helicoid to persist through several dimensions, or even throughout the whole family. Instead, stability appears abruptly at $k = 3$: the three-dimensional $H_2 \subset \mathbb{R}^5$ is unstable, whereas the four-dimensional $H_3 \subset \mathbb{R}^7$ is stable.

For $k \geq 3$, we consider the substantially stronger property of area minimization. Stability is infinitesimal: it asserts the nonnegativity of the second variation under compactly supported normal variations. Area minimization is global, requiring comparison with arbitrary compactly supported competitors having the same boundary. Our second main theorem shows that, for $k \geq 3$, the area-minimizing property of H_k is completely determined by the parity of k.

Theorem 1.2. *For $k \geq 3$, the $(k+1)$-dimensional helicoid H_k is area-minimizing when k is even, and not area-minimizing when k is odd.*

The area-minimizing result follows from the construction of an explicit calibration in the sense of Harvey and Lawson [7]. The construction of this calibration is closely related to the calibrations developed by Lawlor [10] and by Kerckhove and Lawlor [8] for the determinantal variety $C(2, k, 1)$; see also Morgan's survey [14] for a broader account of calibration methods. There is a uniform construction of the calibration for all even $k \geq 6$. However, this construction breaks down when $k = 4$ due to the failure of a key lemma (Lemma 4.2). To handle this exceptional case, we exploit the quaternionic structure in dimension four to construct a different calibration.

When $k \geq 3$ is odd, we identify a portion of H_k and construct a competitor with the same boundary but strictly smaller area, thereby showing that H_k is not area-minimizing.

Theorems 1.1 and 1.2 together exhibit a striking dependence on dimension: H_k becomes stable at $k = 3$, while its area-minimization property for $k \geq 3$ is determined entirely by the parity of k.

Finally, we study the broader family of generalized helicoids introduced by Barbosa–Dajczer–Jorge and by Bryant [1, 3]. For positive real parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, we write $\mathcal{H}_{\boldsymbol{\lambda}}$ for the generalized helicoids parametrized by

$$(u_1, \ldots, u_k, s) \mapsto \big(u_1 e^{i\lambda_1 s}, \ldots, u_k e^{i\lambda_k s}, s\big), \tag{2}$$

so that H_k is the case $\lambda_1 = \cdots = \lambda_k = 1$. In our earlier paper [16], we obtained a sufficient condition, stated in (91), for a generalized helicoid to be realized as an entire minimal graph. Here we prove that this condition is also necessary. In particular, we conclude that entireness of H_k is again determined by the parity of k: H_k is entire if and only if k is even. Regarding the stability of generalized helicoids, we show that, in the moduli space of generalized helicoids, the unstable members form an open subset. We also give various criteria, stated directly in terms of the parameters $\lambda_1, \ldots, \lambda_k$, that distinguish stable generalized helicoids from unstable ones.

For the family of $(k+1)$-dimensional helicoids H_k, the preceding results are summarized in Table 1.

k	**Stability**	**Area-Minimizing**	**Entire**
1	No	No	No
2	No	No	Yes
Odd $k \geq 3$	Yes	No	No
Even $k \geq 4$	Yes	Yes	Yes

TABLE 1. Variational and global properties of $H_k \subset \mathbb{R}^{2k+1}$

In particular, we obtain the following.

Corollary 1.3. *For every even $k \geq 4$, the $(k+1)$-dimensional helicoid H_k is an entire minimal graph in $\mathbb{R}^{2k+1}$ that is area-minimizing.*

Section 2 reduces the stability problem for generalized helicoids to a functional inequality for vector-valued functions on $\mathbb{R}^k$. Section 3 proves this functional inequality to establish the stability of H_k for $k \geq 3$. Sections 4 and 5 prove the area-minimizing property of H_k for even $k \geq 6$ and of H_4 by constructing respective calibrations, while Section 6 proves that H_k is not area-minimizing for odd k by constructing explicit competitors. Finally, Section 7.1 identifies a stable region in the moduli space of generalized helicoids, Section 7.2 develops the instability theory, including the instability of H_2 and the openness of the unstable region, while Section 8 studies the entireness problem and determines the entireness of H_k for all k.

Disclosure of AI use: The authors used AI tools, including ChatGPT 5.6 Sol and Claude Fable 5, to assist with literature searches, explore ideas, and perform symbolic calculations in the preparation of this work. All proofs and calculations, however, were independently verified by the authors.

2. Reduction of the stability condition for generalized helicoids

In this section, we study the second variation of generalized helicoids. In particular, we reduce the stability condition to a functional inequality for vector-valued functions on $\mathbb{R}^k$. Identify $\mathbb{R}^{2k+1}$ with $\mathbb{C}^k \times \mathbb{R}$. Denote $\operatorname{diag}(\boldsymbol{\lambda})$ by Λ. The parametrization

$$F_{\boldsymbol{\lambda}}(u, s) = \big(e^{i\Lambda s}u,\, s\big)$$

of generalized helicoid $\mathcal{H}_{\boldsymbol{\lambda}}$ in (2) has derivatives

$$\partial_{u_j} F_{\boldsymbol{\lambda}} = \big(e^{i\lambda_j s}\mathbf{e}_j,\, 0\big), \qquad \partial_s F_{\boldsymbol{\lambda}} = \big(i\Lambda e^{i\Lambda s}u,\, 1\big).$$

Set $h^2 = 1 + |\Lambda u|^2$, where $|\Lambda u|^2 = \sum_{j=1}^{k}(\lambda_j u_j)^2$. An orthonormal tangent frame on $\mathcal{H}_{\boldsymbol{\lambda}}$ is $E_j = \partial_{u_j} F_{\boldsymbol{\lambda}}$ $(1 \le j \le k)$ and $E_{k+1} = h^{-1}\partial_s F_{\boldsymbol{\lambda}}$, giving

$$g = \sum_{j=1}^{k} du_j^2 + h^2 ds^2, \qquad d\mu = h\, du\, ds.$$

Second derivatives satisfy

$$\partial^2_{u_j u_j} F_{\boldsymbol{\lambda}} = 0, \qquad \partial^2_{ss} F_{\boldsymbol{\lambda}} = \left(-\Lambda^2 e^{i\Lambda s} u,\, 0\right) = -\sum_{j=1}^{k} \lambda_j^2 u_j \partial_{u_j} F_{\boldsymbol{\lambda}} \in T\mathcal{H}_{\boldsymbol{\lambda}}.$$

Since $g^{js} = 0$, the mean curvature of $\mathcal{H}_{\boldsymbol{\lambda}}$ vanishes:

$$H = \left(\sum_{j=1}^{k} \partial^2_{u_j u_j} F_{\boldsymbol{\lambda}} + h^{-2}\partial^2_{ss} F_{\boldsymbol{\lambda}}\right)^{\perp} = 0.$$

We now study the stability of $\mathcal{H}_{\boldsymbol{\lambda}}$ as a minimal submanifold. Recall that the second variation of area associated with a normal vector field V is given by

$$Q(V,V) = \int_{\mathcal{H}_{\boldsymbol{\lambda}}} \sum_{a=1}^{k+1} |\nabla^{\perp}_{E_a} V|^2 - \sum_{a,b=1}^{k+1} \langle \mathrm{I\!I}(E_a, E_b), V\rangle^2,$$

where $\{E_a\}_{a=1}^{k+1}$ is any local orthonormal tangent frame and $\mathrm{I\!I}$ denotes the second fundamental form. The minimal submanifold $\mathcal{H}_{\boldsymbol{\lambda}}$ is stable if and only if $Q(V,V) \ge 0$ for every compactly supported normal field V. It turns out that the nonnegativity of Q is equivalent to the nonnegativity of $q_{\boldsymbol{\lambda}}$ in the following definition.

Definition 2.1. For $f = (f^1, \ldots, f^k) \in C_c^\infty(\mathbb{R}^k; \mathbb{R}^k)$, define

$$q_{\boldsymbol{\lambda}}[f] = \int_{\mathbb{R}^k} h \left[\sum_{j=1}^{k} |\nabla f^j|^2 + \left|\nabla\left(\Lambda u \cdot f\right)\right|^2 - \frac{3|\Lambda f|^2}{h^2}\right], \tag{3}$$

where $u_1, \ldots, u_k$ are standard coordinates on $\mathbb{R}^k$, $\Lambda u \cdot f$ is the scalar function $\sum_{j=1}^{k} \lambda_j u_j f^j$, $|\Lambda f|^2 = \sum_{j=1}^{k} (\lambda_j f^j)^2$, $h = \sqrt{1 + |\Lambda u|^2} = \sqrt{1 + \sum_{j=1}^{k}(\lambda_j u_j)^2}$, and ∇ is the standard gradient operator for scalar functions on $\mathbb{R}^k$.

Proposition 2.2. *The generalized helicoid $\mathcal{H}_{\boldsymbol{\lambda}}$ is stable if and only if*

$$q_{\boldsymbol{\lambda}}[f] \ge 0 \quad \forall f \in C_c^\infty(\mathbb{R}^k; \mathbb{R}^k). \tag{4}$$

Moreover, if (4) *fails, then* $\operatorname{ind} \mathcal{H}_{\boldsymbol{\lambda}} = \infty$.

Proof. Step 1: Parametrization of normal vector fields. Given $\tilde f \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}; \mathbb{R}^k)$, define

$$V(u,s) = \left(i e^{i\Lambda s} \tilde f(u,s),\, -(\Lambda u)\cdot \tilde f(u,s)\right) \in \mathbb{C}^k \times \mathbb{R}. \tag{5}$$

Taking ambient inner products with the tangent vectors gives

$$\langle V, \partial_{u_j} F_{\boldsymbol{\lambda}}\rangle = \mathrm{Re}\langle i e^{i\lambda_j s} \tilde f^j, e^{i\lambda_j s}\rangle_{\mathbb{C}} = 0,$$

$$\langle V, \partial_s F_{\boldsymbol{\lambda}}\rangle = \mathrm{Re}\langle i e^{i\Lambda s}\tilde f, i\Lambda e^{i\Lambda s} u\rangle_{\mathbb{C}^k} - (\Lambda u)\cdot \tilde f = 0.$$

Thus $V \in T^{\perp}\mathcal{H}_{\boldsymbol{\lambda}}$ is normal. Since $\dim(T^{\perp}\mathcal{H}_{\boldsymbol{\lambda}}) = k$, the representation (5) uniquely parametrizes all compactly supported normal fields.

Step 2: Computation of the second fundamental form. Differentiating the parametrization yields

$$\partial^2_{u_j u_m} F_{\boldsymbol{\lambda}} = 0, \qquad \partial^2_{ss} F_{\boldsymbol{\lambda}} \in T\mathcal{H}_{\boldsymbol{\lambda}}, \qquad \partial^2_{u_j s} F_{\boldsymbol{\lambda}} = \big(i\lambda_j e^{i\lambda_j s}\mathbf{e}_j,\, 0\big).$$

Taking normal projections gives $\mathrm{I\!I}(E_j, E_m) = 0$ and $\mathrm{I\!I}(E_{k+1}, E_{k+1}) = 0$. For the cross terms,

$$\langle \mathrm{I\!I}(E_j, E_{k+1}), V\rangle = h^{-1}\langle \partial^2_{u_j s} F_{\boldsymbol{\lambda}}, V\rangle = h^{-1}\,\mathrm{Re}\langle i\lambda_j e^{i\lambda_j s}\mathbf{e}_j, i e^{i\lambda_j s}\tilde f^j\rangle = \frac{\lambda_j \tilde f^j}{h}.$$

Therefore,

$$\sum_{a,b=1}^{k+1} \langle \mathrm{I\!I}(E_a, E_b), V\rangle^2 = 2\sum_{j=1}^{k} \langle \mathrm{I\!I}(E_j, E_{k+1}), V\rangle^2 = \frac{2|\Lambda \tilde f|^2}{h^2}. \tag{6}$$

Step 3: Derivatives along u_j. Differentiating V along ∂_{u_j} gives

$$\overline{\nabla}_{\partial_{u_j}} V = \big(i e^{i\Lambda s}\partial_{u_j}\tilde f,\, -\partial_{u_j}\big((\Lambda u)\cdot \tilde f\big)\big).$$

Its tangential components are

$$\langle \overline{\nabla}_{\partial_{u_j}} V, E_m\rangle = 0 \quad (1 \le m \le k),$$

$$\langle \overline{\nabla}_{\partial_{u_j}} V, E_{k+1}\rangle = h^{-1}\Big((\Lambda u)\cdot \partial_{u_j}\tilde f - \partial_{u_j}\big((\Lambda u)\cdot \tilde f\big)\Big) = -h^{-1}\lambda_j \tilde f^j.$$

Subtracting the tangential component from the ambient norm $|\overline{\nabla}_{\partial_{u_j}} V|^2$ gives

$$|\nabla^{\perp}_{E_j} V|^2 = |\overline{\nabla}_{\partial_{u_j}} V|^2 - \langle \overline{\nabla}_{\partial_{u_j}} V, E_{k+1}\rangle^2 = |\partial_{u_j}\tilde f|^2 + \Big|\partial_{u_j}\big((\Lambda u)\cdot \tilde f\big)\Big|^2 - \frac{\lambda_j^2 (\tilde f^j)^2}{h^2}. \tag{7}$$

Step 4: Derivative along s. Differentiating V along ∂_s gives the orthogonal decomposition into normal and tangential parts:

$$\overline{\nabla}_{\partial_s} V = \underbrace{\big(i e^{i\Lambda s}\partial_s \tilde f,\, -(\Lambda u)\cdot \partial_s \tilde f\big)}_{\in T^{\perp}\mathcal{H}_{\boldsymbol{\lambda}}} - \underbrace{\sum_{j=1}^{k} \lambda_j \tilde f^j \partial_{u_j} F_{\boldsymbol{\lambda}}}_{\in T\mathcal{H}_{\boldsymbol{\lambda}}}.$$

Taking the norm of the normal component and scaling by $E_{k+1} = h^{-1}\partial_s$ gives

$$|\nabla^{\perp}_{E_{k+1}} V|^2 = h^{-2}\left|(\overline{\nabla}_{\partial_s} V)^{\perp}\right|^2 = \frac{|\partial_s \tilde f|^2 + \big((\Lambda u)\cdot \partial_s \tilde f\big)^2}{h^2}. \tag{8}$$

Step 5: Second variation and stability. Combining (7) and (8), we obtain

$$|\nabla^{\perp} V|^2 = \sum_{j=1}^{k} |\partial_{u_j}\tilde f|^2 + \Big|\nabla_u\big((\Lambda u)\cdot \tilde f\big)\Big|^2 + \frac{|\partial_s \tilde f|^2 + \big((\Lambda u)\cdot \partial_s \tilde f\big)^2 - |\Lambda \tilde f|^2}{h^2}.$$

Subtracting (6) and integrating with respect to $d\mu = h\, du\, ds$ yields

$$Q(V,V) = \int_{\mathbb{R}} q_{\boldsymbol{\lambda}}[\tilde f(\cdot, s)]\, ds + \int_{\mathbb{R}^{k+1}} h^{-1}\left[|\partial_s \tilde f|^2 + \big((\Lambda u)\cdot \partial_s \tilde f\big)^2\right] du\, ds.$$

If $q_{\boldsymbol{\lambda}}[f] \ge 0$ for all $f \in C_c^{\infty}(\mathbb{R}^k; \mathbb{R}^k)$, then $Q(V,V) \ge 0$, establishing stability.

Conversely, if $q_{\boldsymbol{\lambda}}[f_0] < 0$ for some f_0, choose $0 \neq \chi \in C_c^\infty(-1,1)$ and set

$$\tilde{f}_m(u,s) = \chi\left(\frac{s-3mL}{L}\right) f_0(u)$$

for $m \in \mathbb{Z}$. Then for $V_m = \left(ie^{i\Lambda s}\tilde{f}_m,\, -(\Lambda u)\cdot \tilde{f}_m\right)$

$$Q(V_m,V_m) = L\|\chi\|_2^2 q_{\boldsymbol{\lambda}}[f_0] + L^{-1}\|\chi'\|_2^2 \int_{\mathbb{R}^k} h^{-1}\left[|f_0|^2 + \left((\Lambda u)\cdot f_0\right)^2\right] du.$$

For sufficiently large L, $Q(V_m,V_m) < 0$. Since $\{V_m\}_{m\in\mathbb{Z}}$ have disjoint compact supports in s, they span an infinite-dimensional negative subspace, proving $\operatorname{ind}\mathcal{H}_{\boldsymbol{\lambda}} = \infty$. □

3. Stability of H_k for every $k \geq 3$

We prove the following sharp functional inequality for vector-valued functions on $\mathbb{R}^k$ for every $k \geq 3$.

Proposition 3.1. *Let $k \geq 3$, let $x = (x_1,\ldots,x_k)$ be the standard coordinates on $\mathbb{R}^k$, and set*

$$h(x) = \sqrt{1+|x|^2}, \qquad c_k = \frac{(k-1)^2+8}{4}.$$

Then every $f = (f^1,\ldots,f^k) \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ satisfies

$$\mathcal{Q}[f] := \int_{\mathbb{R}^k} h\left(\sum_{j=1}^k |\nabla f^j|^2 + |\nabla(x\cdot f)|^2 - c_k\frac{|f|^2}{h^2}\right) \geq 0. \tag{9}$$

The proof separates the radial and spherical behavior of f. We begin with the spherical estimates.

We write $r = |x|$ and $x = r\omega$, where $\omega \in S^{k-1}$. On S^{k-1}, let ∇_S, div_S, and D denote the gradient, divergence, and Levi–Civita connection, respectively. If F is tangent to S^{k-1}, let $F^\flat$ be its metric-dual one-form. We use the convention

$$|dF^\flat|^2 = \sum_{i<j}(D_iF_j - D_jF_i)^2$$

in a local orthonormal frame.

Lemma 3.2. *Let $k \geq 3$. First, every smooth function $u : S^{k-1} \to \mathbb{R}$ with zero spherical mean satisfies*

$$\int_{S^{k-1}} |\nabla_S u|^2\, d\omega \geq (k-1)\int_{S^{k-1}} u^2\, d\omega. \tag{10}$$

Second, every smooth tangent vector field F on S^{k-1} satisfies

$$\int_{S^{k-1}} \left((\operatorname{div}_S F)^2 + |dF^\flat|^2\right) d\omega \geq (k-1)\int_{S^{k-1}} |F|^2\, d\omega. \tag{11}$$

Proof. The scalar estimate (10) is the first nonzero eigenvalue identity $\lambda_1(S^{k-1}) = k-1$. Let $\eta = F^\flat$. By the Bochner–Weitzenböck formula for one-forms [12, Chapter 3],

$$\int_{S^{k-1}} \left(|d\eta|^2 + |\delta\eta|^2\right) d\omega = \int_{S^{k-1}} \left(|\nabla\eta|^2 + (k-2)|\eta|^2\right) d\omega. \tag{12}$$

Decompose $\nabla\eta = S + A$ into its symmetric and antisymmetric parts. Our normalization gives $|A|^2 = \frac{1}{2}|d\eta|^2$, while $(\operatorname{tr} S)^2 \le (k-1)|S|^2$. Since $\delta\eta = -\operatorname{div}_S F$, it follows that

$$|\nabla\eta|^2 \ge \frac{1}{k-1}|\delta\eta|^2 + \frac{1}{2}|d\eta|^2.$$

Substitution in (12), followed by rearrangement, gives

$$\int_{S^{k-1}} \left(|\delta\eta|^2 + \frac{k-1}{2(k-2)}|d\eta|^2\right) d\omega \ge (k-1)\int_{S^{k-1}} |\eta|^2\, d\omega. \tag{13}$$

For $k \ge 3$, $(k-1)/(2(k-2)) \le 1$, so (13) implies (11). □

Proof of Proposition 3.1. For $r > 0$, decompose

$$f(r\omega) = a(r,\omega)\omega + F(r,\omega), \qquad F(r,\omega) \in T_\omega S^{k-1}.$$

Then $x \cdot f = ra$ and $|f|^2 = a^2 + |F|^2$. The functions $a = \omega \cdot f(r\omega)$ and $F = f(r\omega) - a\omega$ are smooth on $[0,\infty) \times S^{k-1}$ and vanish for sufficiently large r. Hence all radial boundary terms vanish: at infinity by compact support and at the origin because they contain a factor r^k.

A calculation in the orthonormal frame $\{\partial_r, r^{-1}e_1, \dots, r^{-1}e_{k-1}\}$ gives

$$|\nabla f|^2 = (\partial_r a)^2 + |\partial_r F|^2 + \frac{1}{r^2}\left(|\nabla_S a - F|^2 + |DF|^2 + 2a\operatorname{div}_S F + (k-1)a^2\right),$$

$$|\nabla(x\cdot f)|^2 = \big(\partial_r(ra)\big)^2 + |\nabla_S a|^2.$$

Using integration by parts on the sphere and the Bochner–Weitzenböck formula,

$$\int_{S^{k-1}} |DF|^2\, d\omega = \int_{S^{k-1}} \left((\operatorname{div}_S F)^2 + |dF^\flat|^2 - (k-2)|F|^2\right) d\omega,$$

$\mathcal{Q}[f]$ in (9) equals

$$\begin{aligned}\mathcal{Q}[f] = \int_0^\infty\!\!\int_{S^{k-1}} \Big\{ & hr^{k-1}\big((\partial_r a)^2 + (\partial_r(ra))^2 + |\partial_r F|^2\big) \\ & + h^3 r^{k-3}|\nabla_S a|^2 + \Big((k-1)hr^{k-3} - c_k\frac{r^{k-1}}{h}\Big)a^2 \\ & + \Big((3-k)hr^{k-3} - c_k\frac{r^{k-1}}{h}\Big)|F|^2 + 4hr^{k-3}a\operatorname{div}_S F \\ & + hr^{k-3}\big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big)\Big\}\, d\omega\, dr.\end{aligned} \tag{14}$$

Expanding $\partial_r(ra) = a + r\partial_r a$ and integrating its mixed term by parts gives

$$\begin{aligned}&\int_0^\infty hr^{k-1}\big((\partial_r a)^2 + (\partial_r(ra))^2\big)\, dr \\ &\qquad = \int_0^\infty \Big[h^3 r^{k-1}(\partial_r a)^2 - \Big((k-1)hr^{k-1} + \frac{r^{k+1}}{h}\Big)a^2\Big]\, dr.\end{aligned} \tag{15}$$

The remaining radial derivatives admit the exact square completions

$$\int_0^\infty h^3 r^{k-1}(\partial_r a)^2\,dr = \int_0^\infty \Big[h^3 r^{k-1}\Big(\partial_r a + \frac{(k+1)r}{2h^2}a\Big)^2 + \Big(\frac{k+1}{2}(hr^k)' - \frac{(k+1)^2}{4}\frac{r^{k+1}}{h}\Big)a^2\Big]\,dr, \tag{16}$$

$$\int_0^\infty h r^{k-1}|\partial_r F|^2\,dr = \int_0^\infty \Big[h r^{k-1}\Big|\partial_r F + \frac{(k-1)r}{2h^2}F\Big|^2 + \Big(\frac{k-1}{2}\Big(\frac{r^k}{h}\Big)' - \frac{(k-1)^2}{4}\frac{r^{k+1}}{h^3}\Big)|F|^2\Big]\,dr. \tag{17}$$

Substituting (15) into (14), then applying (16) and (17), discarding the two nonnegative squares, and using $r^2 = h^2 - 1$, we obtain

$$\mathcal{Q}[f] \ge \int_0^\infty \int_{S^{k-1}} r^{k-3}\Big\{h^3|\nabla_S a|^2 + \alpha a^2 + 4ha\,\mathrm{div}_S F + h\big((\mathrm{div}_S F)^2 + |dF^\flat|^2\big) - C|F|^2\Big\}\,d\omega\,dr, \tag{18}$$

where

$$\alpha = \alpha(r) = \frac{4(k-1) + (k^2+4k-9)r^2 + (k-1)^2 r^4}{4h}, \tag{19}$$

$$C = C(r) = (k-1)h - \frac{k^2+7}{4h} + \frac{k^2-1}{4h^3}. \tag{20}$$

For $k \ge 3$, all coefficients in the numerator of (19) are positive, so $\alpha > 0$.

For each fixed r, write $a = \bar a + \mathring{a}$, where $\bar a$ is the spherical mean. Since $\int_{S^{k-1}} \mathrm{div}_S F\,d\omega = 0$, only $\mathring{a}$ occurs in the mixed term. Since $|\nabla_S a| = |\nabla_S \mathring{a}|$, Lemma 3.2 gives

$$\int_{S^{k-1}} h^3|\nabla_S a|^2\,d\omega \ge (k-1)h^3 \int_{S^{k-1}} \mathring{a}^2\,d\omega. \tag{21}$$

For the term $\int_{S^{k-1}} \big(h\big((\mathrm{div}_S F)^2 + |dF^\flat|^2\big) - C|F|^2\big)\,d\omega$ on the right-hand side of (18), direct calculation gives

$$4h^3 C = 4(k-1)r^4 - (k-3)(k-5)r^2 + 4(k-3). \tag{22}$$

The minimum of the right-hand side of (22) over $r^2 \ge 0$ is

$$\begin{cases} 4(k-3), & 3 \le k \le 5, \\ \dfrac{k-3}{16(k-1)}\left[64(k-1) - (k-3)(k-5)^2\right], & k \ge 6. \end{cases}$$

Moreover, $(k-3)(k-5)^2 - 64(k-1) = k^3 - 13k^2 - 9k - 11$, whose derivative is $(3k+1)(k-9)$. Direct evaluation gives a negative value for $6 \le k \le 13$, with the value -128 at $k = 13$, and the value 59 at $k = 14$. Therefore

$$C(r) \ge 0 \quad \text{for every } r \text{ when } 3 \le k \le 13, \text{ and } \{r : C(r) < 0\} \ne \varnothing \quad \text{only when } k \ge 14. \tag{23}$$

If $C(r) \geq 0$, multiplying (11) by $\frac{C}{k-1} \geq 0$ and adding $h \int_{S^{k-1}} \big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big)\, d\omega$ gives

$$\begin{aligned}\int_{S^{k-1}} \Big[h\big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big) - C|F|^2\Big]\, d\omega \\ \geq \Big(h - \frac{C}{k-1}\Big) \int_{S^{k-1}} \big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big)\, d\omega \\ = \frac{8 + (k^2+7)r^2}{4(k-1)h^3} \int_{S^{k-1}} \big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big)\, d\omega,\end{aligned}$$

the last equality by substituting $h^2 = 1 + r^2$ in $h - \frac{C}{k-1}$. If $C(r) < 0$, the term $-C|F|^2$ is nonnegative and may be discarded. Hence, at every r,

$$\begin{aligned}\int_{S^{k-1}} \Big[h\big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big) - C|F|^2\Big]\, d\omega \\ \geq \gamma \int_{S^{k-1}} \big((\operatorname{div}_S F)^2 + |dF^\flat|^2\big)\, d\omega,\end{aligned} \tag{24}$$

where

$$\gamma = \begin{cases} \dfrac{8 + (k^2+7)r^2}{4(k-1)h^3}, & C \geq 0, \\ h, & C < 0, \end{cases} \tag{25}$$

and $\gamma > 0$ in both cases.

Set $\beta = (k-1)h^3 + \alpha$. Combining (18), (21), and (24), we find

$$\begin{aligned}\mathcal{Q}[f] \geq \int_0^\infty r^{k-3} \Bigg\{ \alpha \int_{S^{k-1}} \bar{a}^2\, d\omega \\ + \int_{S^{k-1}} \big(\beta \mathring{a}^2 + \gamma (\operatorname{div}_S F)^2 + 4h\mathring{a} \operatorname{div}_S F + \gamma |dF^\flat|^2\big)\, d\omega \Bigg\}\, dr.\end{aligned} \tag{26}$$

The only remaining mixed term is handled by

$$\beta \mathring{a}^2 + \gamma (\operatorname{div}_S F)^2 + 4h\mathring{a} \operatorname{div}_S F = \gamma \Big(\operatorname{div}_S F + \frac{2h}{\gamma}\mathring{a}\Big)^2 + \frac{\beta\gamma - 4h^2}{\gamma}\mathring{a}^2. \tag{27}$$

If $C \geq 0$, substituting the expression in (25) for $C \geq 0$ into (27) gives

$$\begin{aligned}\beta\gamma - 4h^2 = \frac{r^2}{16(k-1)h^4}\Big[8k(k^2-5) + (k^4 + 12k^3 - 2k^2 - 92k + 49)r^2 \\ + (k-1)(k^3 + 3k^2 + 7k - 43)r^4\Big].\end{aligned}$$

All three coefficients in the bracket are positive for every integer $k \geq 3$: at $k = 3$ they equal 96, 160, and 64, and each is increasing in k. Thus $\beta\gamma - 4h^2 \geq 0$ whenever $C \geq 0$. If $C < 0$, then $k \geq 14$ by (23), and, using $\alpha > 0$ and $h \geq 1$,

$$\beta\gamma = \beta h \geq (k-1)h^4 \geq 13h^4 \geq 4h^2.$$

Thus $\beta\gamma - 4h^2 \geq 0$ in both cases, and every term in (26) is nonnegative. This proves (9). □

Remark 3.3. The constant c_k in (9) can be shown to be optimal. A key ingredient is that $F_1 = \nabla_S \omega_1 = e_1 - \langle \omega, e_1 \rangle\, \omega$ attains equality in (11).

To apply Proposition 3.1 to $q_{\boldsymbol{\lambda}}$ in (3), observe that

$$c_k - 3 = \frac{(k-3)(k+1)}{4} \geq 0 \qquad \text{for } k \geq 3. \tag{28}$$

Theorem 3.4. *H_k is stable for every $k \geq 3$.*

Proof. Take $\boldsymbol{\lambda} = (1, \ldots, 1)$ in (3). Then Λ is the identity and $|\Lambda f|^2 = |f|^2$ for any $f \in C_c^\infty(\mathbb{R}^k, \mathbb{R}^k)$, so

$$q_{\boldsymbol{\lambda}}[f] = \int_{\mathbb{R}^k} h \left(\sum_{j=1}^k |\nabla f^j|^2 + |\nabla(u \cdot f)|^2 - \frac{3|f|^2}{h^2} \right) du.$$

Proposition 3.1 and (28) give $q_{\boldsymbol{\lambda}}[f] \geq (c_k - 3) \int_{\mathbb{R}^k} \frac{|f|^2}{h}\, du \geq 0$. Stability now follows from Proposition 2.2. $\square$

This proves the stable half of Theorem 1.1. The instability of H_2 is proved in Section 7.2; the instability of the classical helicoid H_1 follows from [5, 6].

4. Calibration of H_k for even $k \geq 6$

Throughout this section, $k \geq 6$ is even. Let

$$H_k = \left\{ (\cos s\, u, \sin s\, u, s) : u \in \mathbb{R}^k,\ s \in \mathbb{R} \right\} \subset \mathbb{R}^{2k+1}$$

be the $(k+1)$-dimensional helicoid. The projection of H_k onto $\mathbb{R}^{2k}$ is precisely $C(2, k, 1)$, the cone of $2 \times k$ matrices of rank at most 1. Kerckhove and Lawlor [8] studied the area-minimizing property of these minimal cones in $\mathbb{R}^{2k}$ and showed that $C(2, k, 1)$ is area-minimizing if $k \geq 4$; see also [10]. The construction of our calibration is inspired by their work.

4.1. **Coordinates adapted to the helicoid and the induced geometry.**

Write $\mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k$, and denote a point by (x, y), where $x, y \in \mathbb{R}^k$.

Consider the subset $Z_0 \subset \mathbb{R}^{2k}$:

$$Z_0 = \{(x, y) \in \mathbb{R}^{2k} : |x| = |y|,\ \langle x, y \rangle = 0\}, \tag{29}$$

and define $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ by

$$\sigma_1^2 = \frac{|x|^2 + |y|^2}{2} + \frac{1}{2}\sqrt{\left(|x|^2 - |y|^2\right)^2 + 4\langle x, y\rangle^2}$$

and

$$\sigma_2^2 = \frac{|x|^2 + |y|^2}{2} - \frac{1}{2}\sqrt{\left(|x|^2 - |y|^2\right)^2 + 4\langle x, y\rangle^2}.$$

Then σ_1 and σ_2 are the singular values of the $2 \times k$ matrix $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \end{pmatrix}$. Moreover, $Z_0 = \{(x, y) \in \mathbb{R}^{2k} : \sigma_1 = \sigma_2\}$ and therefore $\sigma_1 > \sigma_2$ on $\mathbb{R}^{2k} \setminus Z_0$.

For any $(x, y) \in \mathbb{R}^{2k} \setminus Z_0$, by singular value decomposition there exist $s \in \mathbb{R}$, and orthonormal vectors $w, v \in \mathbb{R}^k$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix}. \tag{30}$$

In particular, w is associated with the larger singular value σ_1.

We also introduce $\rho > 0$ and $0 \le t < \frac{\pi}{4}$ such that

$$\sigma_1 = \rho\cos t, \qquad \sigma_2 = \rho\sin t \tag{31}$$

in $\mathbb{R}^{2k} \setminus Z_0$. Together, (ρ, t, s, w, v) form a coordinate system on $\mathbb{R}^{2k} \setminus Z_0$ with

$$\rho > 0, \qquad 0 \le t < \frac{\pi}{4}, \qquad s \in \mathbb{R}, \qquad w, v \in \mathbb{R}^k, \qquad |w| = |v| = 1, \langle w, v\rangle = 0.$$

Since we choose $\begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \in SO(2)$, the only remaining ambiguity, apart from the standard periodicity $s \sim s + 2\pi$, is $(s, w, v) \sim (s + \pi, -w, -v)$.

For $(x, y) \in \mathbb{R}^{2k} \setminus Z_0$, (31) gives

$$\rho = \sqrt{|x|^2 + |y|^2}, \qquad \sin(2t) = \frac{2\sqrt{|x|^2|y|^2 - \langle x, y\rangle^2}}{|x|^2 + |y|^2}. \tag{32}$$

Adding $z \in \mathbb{R}$ gives adapted coordinates (ρ, t, s, w, v, z) on $\mathbb{R}^{2k+1} \setminus Z$, where $Z = Z_0 \times \mathbb{R}$, in which

$$H_k = \{t = 0,\ z = s\}. \tag{33}$$

Write $d\tau = \langle dw, v\rangle$. Differentiating (30) and computing the Euclidean metric on $\mathbb{R}^{2k} \setminus Z_0$, we obtain

$$\begin{aligned} |dx|^2 + |dy|^2 = {} & |\cos t\, w\, d\rho - \rho\sin t\, w\, dt - \rho\sin t\, v\, ds + \rho\cos t\, dw|^2 \\ & + |\sin t\, v\, d\rho + \rho\cos t\, v\, dt + \rho\cos t\, w\, ds + \rho\sin t\, dv|^2 . \end{aligned}$$

The relations $|w| = |v| = 1$ and $\langle w, v\rangle = 0$ give

$$dw = v\, d\tau + (dw - v\, d\tau), \qquad dv = -w\, d\tau + (dv + w\, d\tau).$$

Hence the Euclidean metric on $\mathbb{R}^{2k+1} \setminus Z$ is

$$\begin{aligned} g = {} & d\rho^2 + \rho^2 dt^2 + \rho^2 \left(ds^2 - 2\sin(2t)\, ds\, d\tau + d\tau^2\right) \\ & + \rho^2\cos^2 t\, |dw - v\, d\tau|^2 + \rho^2\sin^2 t\, |dv + w\, d\tau|^2 + dz^2. \end{aligned} \tag{34}$$

Restricting (34) to H_k as described in (33) gives

$$g_{H_k} = d\rho^2 + (1 + \rho^2)ds^2 + \rho^2|dw|^2. \tag{35}$$

Let Θ denote the standard volume form of the metric $|dw|^2$ on S^{k-1},

$$\Theta = \sum_{i=1}^{k} (-1)^{i-1} w_i\, dw_1 \wedge \cdots \wedge \widehat{dw_i} \wedge \cdots \wedge dw_k. \tag{36}$$

When k is even, Θ is invariant under $(s, w, v) \sim (s + \pi, -w, -v)$ and hence descends to $\mathbb{R}^{2k+1} \setminus Z$.

By (35), we orient the $(k + 1)$-dimensional helicoid H_k by

$$d\mu_{H_k} = \rho^{k-1}\sqrt{1 + \rho^2}\, \Theta \wedge d\rho \wedge ds. \tag{37}$$

Choose a local orthonormal frame $e_1, \ldots, e_{k-2}$ of $\{w, v\}^\perp$ such that $(v, e_1, \ldots, e_{k-2})$ is positively oriented in $T_w S^{k-1}$, and set

$$\theta^a = \langle dw - v\, d\tau, e_a\rangle = \langle dw, e_a\rangle, \qquad 1 \le a \le k - 2.$$

Then

$$\Theta = d\tau \wedge \theta^1 \wedge \cdots \wedge \theta^{k-2} \tag{38}$$

and $|dw - v\,d\tau|^2 = \sum_{a=1}^{k-2}(\theta^a)^2$.

In the rest of this subsection, we consider Θ and $\Theta \wedge ds$ as a $(k-1)$-form and a k-form on $\mathbb{R}^{2k+1} \setminus Z$ and compute their comasses with respect to g in (34).

The relevant part of the inverse of (34) is

$$\langle ds, ds\rangle = \langle d\tau, d\tau\rangle = \frac{1}{\rho^2\cos^2(2t)}, \qquad \langle ds, d\tau\rangle = \frac{\sin(2t)}{\rho^2\cos^2(2t)}.$$

Moreover, $\theta^1, \ldots, \theta^{k-2}$ are mutually orthogonal and orthogonal to ds and $d\tau$, with

$$\|\theta^a\|^2 = \frac{1}{\rho^2\cos^2 t}.$$

By (38), Θ is decomposable, and its comass equals its pointwise norm. Therefore,

$$\|\Theta\|_*^2 = \frac{1}{\rho^2\cos^2(2t)}\left(\frac{1}{\rho^2\cos^2 t}\right)^{k-2} = \frac{1}{\rho^{2k-2}J(t)^2}$$

where

$$J(t) = (\cos t)^{k-2}\cos(2t). \tag{39}$$

Similarly,

$$\|d\tau \wedge ds\|^2 = \det\begin{pmatrix}\langle d\tau, d\tau\rangle & \langle d\tau, ds\rangle \\ \langle ds, d\tau\rangle & \langle ds, ds\rangle\end{pmatrix} = \frac{1}{\rho^4\cos^2(2t)}.$$

Consequently, using (39),

$$\|\Theta\|_* = \frac{1}{\rho^{k-1}J(t)}, \qquad \|\Theta\wedge ds\|_* = \frac{1}{\rho^k J(t)}. \tag{40}$$

4.2. **The calibration.** We construct the calibration ξ for H_k on $\mathbb{R}^{2k+1}\setminus Z$ in this subsection.

Proposition 4.1. *Let*

$$\varphi(t) = \cos(2t), \qquad 0 \le t < \frac{\pi}{4},$$

and set $r = \rho\varphi(t)$. On $\mathbb{R}^{2k+1}\setminus Z$, with coordinates (ρ, t, s, w, v, z) define

$$\xi = \Theta \wedge dr \wedge \left(\frac{r^{k+1}}{\sqrt{1+r^2}}\,ds + \frac{r^{k-1}}{\sqrt{1+r^2}}\,dz\right). \tag{41}$$

Then ξ has the following properties on $\mathbb{R}^{2k+1}\setminus Z$:

(1) *Its comass satisfies*

$$\|\xi\|_* \le 1. \tag{42}$$

(2) *The form ξ satisfies*

$$\xi \in C^\infty(\mathbb{R}^{2k+1}\setminus Z) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{2k+1}), \qquad d\xi = 0 \quad \text{on } \mathbb{R}^{2k+1}\setminus Z. \tag{43}$$

(3) *The restriction of ξ to $H_k \setminus Z$ is the induced volume form:*

$$\xi|_{H_k\setminus Z} = d\mu_{H_k}. \tag{44}$$

Proof. We first estimate the comass of ξ. Differentiating $r = \rho\varphi(t)$ gives

$$dr = \varphi\, d\rho + \rho\varphi'\, dt.$$

By (34), $d\rho \perp dt$, $|d\rho| = 1$, and $|dt| = \rho^{-1}$; moreover, dr is orthogonal to all the one-form factors in Θ, ds, and dz. Hence

$$|dr|^2 = \varphi^2 + (\varphi')^2.$$

The form ξ is decomposable, so its comass equals its pointwise norm. Since dz is a unit one-form orthogonal to ds and to every factor in Θ, (40) gives

$$\begin{aligned}\|\xi\|_*^2 &= \left(\varphi^2 + (\varphi')^2\right)\left(\frac{r^{2k+2}}{1+r^2}\|\Theta\wedge ds\|_*^2 + \frac{r^{2k-2}}{1+r^2}\|\Theta\wedge dz\|_*^2\right)\\ &= \frac{\varphi^2 + (\varphi')^2}{\rho^{2k}J(t)^2}\left(\frac{r^{2k+2}}{1+r^2} + \rho^2\frac{r^{2k-2}}{1+r^2}\right).\end{aligned}$$

Substituting $r = \rho\varphi$ we obtain

$$\|\xi\|_*^2 = \frac{\varphi^{2k-2}\left(\varphi^2 + (\varphi')^2\right)}{J(t)^2}\frac{1+\rho^2\varphi^4}{1+\rho^2\varphi^2}. \tag{45}$$

Since $0 \le \varphi \le 1$, the last factor in (45) is at most one. The comass bound (42) now follows from Lemma 4.2 at the end of this subsection.

To prove (2), let $A, B : [0,\infty) \to \mathbb{R}$ satisfy

$$A'(r) = \frac{r^{k+1}}{\sqrt{1+r^2}}, \qquad B'(r) = \frac{r^{k-1}}{\sqrt{1+r^2}} \tag{46}$$

with $A(0) = B(0) = 0$. On $\mathbb{R}^{2k+1}\setminus Z$, define

$$\Psi = (-1)^{k-1}\left(A(r)\Theta\wedge ds + B(r)\Theta\wedge dz\right). \tag{47}$$

Since $d\Theta = 0$, $d\Psi = \xi$ and thus ξ is closed on $\mathbb{R}^{2k+1}\setminus Z$.

We then control Ψ and ξ near Z. By (31) and (32),

$$\cos(2t) = \frac{\sqrt{(|x|^2-|y|^2)^2 + 4\langle x,y\rangle^2}}{|x|^2+|y|^2},$$

and hence

$$r = \left(\frac{(|x|^2-|y|^2)^2 + 4\langle x,y\rangle^2}{|x|^2+|y|^2}\right)^{1/2}.$$

The function r is smooth on $\mathbb{R}^{2k+1}\setminus Z$. Since Θ, ds, and dz are also smooth there, (46) and (47) give $\Psi, \xi \in C^\infty(\mathbb{R}^{2k+1}\setminus Z)$ and $d\Psi = \xi$ on $\mathbb{R}^{2k+1}\setminus Z$. Directly from (46),

$$0 \le A(r) \le \frac{r^{k+2}}{k+2}, \qquad 0 \le B(r) \le \frac{r^k}{k}.$$

Using (40) and $r = \rho\varphi$ gives

$$\|\Psi\|_* \le A(r)\|\Theta\wedge ds\|_* + B(r)\|\Theta\wedge dz\|_* \le \frac{\rho^2\varphi^{k+2}}{(k+2)J(t)} + \frac{\rho\varphi^k}{kJ(t)}.$$

Moreover,

$$\frac{\varphi^k}{J(t)} = \frac{(\cos(2t))^{k-1}}{(\cos t)^{k-2}}, \qquad \frac{\varphi^{k+2}}{J(t)} = \frac{(\cos(2t))^{k+1}}{(\cos t)^{k-2}}.$$

Both ratios are bounded on $[0, \pi/4]$. Defining $\Psi = \xi = 0$ on Z, which does not change their L^∞_{loc} classes, we obtain (43), that is,

$$\Psi, \xi \in C^\infty(\mathbb{R}^{2k+1} \setminus Z) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^{2k+1}), \qquad d\Psi = \xi \quad \text{on } \mathbb{R}^{2k+1} \setminus Z.$$

The local boundedness of ξ follows from (42).

Finally, on $H_k \setminus Z$, one has $t = 0$, $z = s$, $r = \rho$, $dr = d\rho$, and $dz = ds$. Hence, (41) and (37) give

$$\xi|_{H_k \setminus Z} = \rho^{k-1} \sqrt{1+\rho^2}\, \Theta \wedge d\rho \wedge ds = d\mu_{H_k}.$$

□

Lemma 4.2. *For $k \geq 6$, the function $\varphi(t) = \cos(2t)$ satisfies $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\varphi(\pi/4) = 0$, and*

$$\varphi^{2k-2}\left(\varphi^2 + (\varphi')^2\right) \leq J(t)^2 \qquad \text{on } \left[0, \frac{\pi}{4}\right). \tag{48}$$

Proof. Since $\varphi' = -2\sin(2t)$, $\varphi^2 + (\varphi')^2 = 1 + 3\sin^2(2t)$. Set $\mathsf{t} = \tan^2 t \in [0,1)$. Using

$$\cos(2t) = \frac{1-\mathsf{t}}{1+\mathsf{t}}, \qquad \sin^2(2t) = \frac{4\mathsf{t}}{(1+\mathsf{t})^2}, \qquad \cos^2 t = \frac{1}{1+\mathsf{t}},$$

inequality (48) is equivalent to

$$f(\mathsf{t}) := \frac{(1-\mathsf{t})^{2k-4}(1+14\mathsf{t}+\mathsf{t}^2)}{(1+\mathsf{t})^k} \leq 1.$$

Its logarithmic derivative is

$$\begin{aligned} \frac{d}{d\mathsf{t}} \log f(\mathsf{t}) &= -\frac{2k-4}{1-\mathsf{t}} + \frac{14+2\mathsf{t}}{1+14\mathsf{t}+\mathsf{t}^2} - \frac{k}{1+\mathsf{t}} \\ &= -(k-6)\frac{3+\mathsf{t}}{1-\mathsf{t}^2} + (14+2\mathsf{t})\left(\frac{1}{1+14\mathsf{t}+\mathsf{t}^2} - \frac{1}{1-\mathsf{t}^2}\right) \leq 0. \end{aligned}$$

Thus f is non-increasing on $[0,1)$, and since $f(0) = 1$, the result follows. □

4.3. **Distributional closedness and area minimization.** Let $d(\cdot) = \mathrm{dist}(\cdot, Z)$. The extension across Z uses the following codimension-two tubular estimate.

Lemma 4.3. *For every $R > 0$, there exists $C_{k,R} > 0$ such that*

$$\mathrm{Vol}\big(\{d \leq \epsilon\} \cap B_R^{2k+1}(0)\big) \leq C_{k,R}\epsilon^2, \qquad \forall \epsilon > 0.$$

Proof. Write $Z = Z_0 \times \mathbb{R}$, where, as recall from (29),

$$Z_0 = \{(x,y) \in \mathbb{R}^{2k} : |x| = |y|,\ x \cdot y = 0\},$$

so that $d((x,y,z), Z) = \mathrm{dist}((x,y), Z_0)$. Set $K = Z_0 \cap S^{2k-1}$. On S^{2k-1},

$$K = F^{-1}(0,0), \text{ where } F(x,y) = (|x|^2 - |y|^2,\ 2x \cdot y).$$

At every point of K, the spherical gradients of the components of F are $(2x, -2y)$ and $(2y, 2x)$, which are orthogonal and have norm 2. Thus, $K \subset S^{2k-1}$ is a smooth compact submanifold of codimension 2, and consequently

$$\mathrm{Vol}_{2k-1}\left(\{\theta \in S^{2k-1} : \mathrm{dist}(\theta, K) \leq \delta\}\right) \leq \tilde{c}_k\, \delta^2, \qquad \forall \delta > 0.$$

Let $p = r\theta$ with $r > \epsilon$, $\theta \in S^{2k-1}$ and suppose $\operatorname{dist}(p, Z_0) \le \epsilon$. Choosing a closest point $s\eta \in Z_0$ with $s > 0$ and $\eta \in K$, we have $|r\theta - s\eta| \le \epsilon$ and $|r - s| \le \epsilon$, which yields

$$\operatorname{dist}(\theta, K) \le |\theta - \eta| \le \frac{|r\theta - s\eta| + |r - s|}{r} \le \frac{2\epsilon}{r}.$$

Integrating in polar coordinates for $0 < \epsilon \le R$ gives

$$\begin{aligned}
\operatorname{Vol}\big(\{d \le \epsilon\} \cap B_R^{2k+1}\big) &\le 2R \operatorname{Vol}_{2k}\big(\{\operatorname{dist}(\cdot, Z_0) \le \epsilon\} \cap B_R^{2k}\big) \\
&= 2R\left[\operatorname{Vol}_{2k}\big(\{\operatorname{dist}(\cdot, Z_0) \le \epsilon\} \cap B_\epsilon^{2k}\big)\right. \\
&\qquad\qquad \left.+ \operatorname{Vol}_{2k}\big(\{\operatorname{dist}(\cdot, Z_0) \le \epsilon\} \cap (B_R^{2k} \setminus B_\epsilon^{2k})\big)\right] \\
&\le 2R\left[\operatorname{Vol}_{2k}(B_\epsilon^{2k})\right. \\
&\qquad\qquad \left.+ \int_\epsilon^R r^{2k-1} \operatorname{Vol}_{2k-1}\left(\left\{\theta \in S^{2k-1} : \operatorname{dist}(\theta, K) \le \frac{2\epsilon}{r}\right\}\right) dr\right] \\
&\le 2R\left[\omega_{2k}\epsilon^{2k} + \int_\epsilon^R r^{2k-1} \cdot c_k \left(\frac{2\epsilon}{r}\right)^2 dr\right] \\
&= 2R\left[\omega_{2k}\epsilon^{2k-2} + \frac{2c_k}{k-1}\left(R^{2k-2} - \epsilon^{2k-2}\right)\right]\epsilon^2 \\
&\le C_{k,R}\epsilon^2.
\end{aligned}$$

For $\epsilon > R$, the estimate follows immediately by bounding the volume by $\operatorname{Vol}(B_R^{2k+1})$. $\square$

We now extend the relation $d\Psi = \xi$ across Z. Fix a test k-form $\eta \in C_c^\infty(\mathbb{R}^{2k+1}; \Omega^k)$ with $\operatorname{supp}\eta \subset B_R(0)$. Let $\chi_\epsilon \in C^\infty(\mathbb{R}^{2k+1})$ be a cutoff function with $\chi_\epsilon = 0$ on $\{d \le \epsilon\}$, $\chi_\epsilon = 1$ on $\{d \ge 2\epsilon\}$, and $|\nabla\chi_\epsilon| \le \frac{2}{\epsilon}$. By (43), Ψ and ξ are smooth on $\operatorname{supp}\chi_\epsilon$ and $d\Psi = \xi$ there. Hence

$$\int_{\mathbb{R}^{2k+1}} \chi_\epsilon \xi \wedge \eta = (-1)^{k+1} \int_{\mathbb{R}^{2k+1}} \chi_\epsilon \Psi \wedge d\eta + (-1)^{k+1} \int_{\mathbb{R}^{2k+1}} d\chi_\epsilon \wedge \Psi \wedge \eta.$$

Using Lemma 4.3 and $\Psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2k+1}; \Omega^k)$, the error term satisfies

$$\left|\int_{\mathbb{R}^{2k+1}} d\chi_\epsilon \wedge \Psi \wedge \eta\right| \le \|\Psi\|_{L^\infty(B_R)} \|\eta\|_{L^\infty} \cdot \frac{2}{\epsilon} \operatorname{Vol}(\{\epsilon \le d \le 2\epsilon\} \cap B_R) \le C'_R \epsilon \to 0$$

as $\epsilon \to 0$. Since $\Psi, \xi \in L^\infty_{\mathrm{loc}}$, dominated convergence yields

$$\int_{\mathbb{R}^{2k+1}} \xi \wedge \eta = (-1)^{k+1} \int_{\mathbb{R}^{2k+1}} \Psi \wedge d\eta.$$

Thus $d\Psi = \xi$ distributionally. Consequently, for any $\zeta \in C_c^\infty(\mathbb{R}^{2k+1}; \Omega^{k-1})$,

$$\int_{\mathbb{R}^{2k+1}} \xi \wedge d\zeta = (-1)^{k+1} \int_{\mathbb{R}^{2k+1}} \Psi \wedge d^2\zeta = 0,$$

so

$$d\xi = 0 \tag{49}$$

distributionally on $\mathbb{R}^{2k+1}$.

Proof of Theorem 1.2. The form ξ is distributionally closed by (49), has comass at most one by (42), and restricts to the oriented volume form of $H_k \setminus Z$ by (44). Since $H_k \cap Z$ has zero $(k+1)$-dimensional measure, ξ calibrates H_k.

Let M' be a smooth oriented competitor homologous to H_k that agrees with H_k outside a compact set. Choose compact portions $P \subset H_k$ and $P' \subset M'$ containing the region where they differ, and a compact oriented $(k+2)$-chain S such that $P - P' = \partial S$. Fix a nonnegative function $\vartheta \in C_c^\infty(B_1^{2k+1}(0))$ satisfying $\int_{\mathbb{R}^{2k+1}} \vartheta = 1$. For $\epsilon > 0$, define

$$\vartheta_\epsilon(p) = \epsilon^{-(2k+1)} \vartheta\left(\frac{p}{\epsilon}\right), \qquad \xi_\epsilon(p) = (\vartheta_\epsilon * \xi)(p) = \int_{\mathbb{R}^{2k+1}} \vartheta_\epsilon(q)\xi(p-q)\,dq.$$

Then $\operatorname{supp} \vartheta_\epsilon \subset B_\epsilon^{2k+1}(0)$. Since $d\xi = 0$ distributionally, $d\xi_\epsilon = d(\vartheta_\epsilon * \xi) = \vartheta_\epsilon * d\xi = 0$. Moreover, the comass bound is preserved under convolution

$$\|\xi_\epsilon(p)\|_* \leq \int_{\mathbb{R}^{2k+1}} \vartheta_\epsilon(q)\|\xi(p-q)\|_*\,dq \leq \int_{\mathbb{R}^{2k+1}} \vartheta_\epsilon(q)\,dq = 1.$$

Since ξ_ϵ is smooth and closed, Stokes' theorem gives

$$0 = \int_S d\xi_\epsilon = \int_{\partial S} \xi_\epsilon = \int_P \xi_\epsilon - \int_{P'} \xi_\epsilon.$$

On $H_k \backslash Z$, the forms ξ_ϵ converge locally uniformly to ξ. Therefore, by dominated convergence and (44),

$$\mathrm{Vol}(P) = \lim_{\epsilon\to 0} \int_P \xi_\epsilon = \lim_{\epsilon \to 0} \int_{P'} \xi_\epsilon \leq \mathrm{Vol}(P').$$

Therefore H_k is area-minimizing. $\square$

5. Calibration for H_4

We identify $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$ and write a point of $\mathbb{R}^9$ as (x, y, z), where $x, y \in \mathbb{H}$. Writing $x = x_0 + x_1 i + x_2 j + x_3 k$ and $y = y_0 + y_1 i + y_2 j + y_3 k$, the three Kähler forms induced by simultaneous left multiplication by i, j, k on the two quaternionic factors are

$$\begin{aligned}
\omega_1 &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3 + dy_0 \wedge dy_1 + dy_2 \wedge dy_3,\\
\omega_2 &= dx_0 \wedge dx_2 - dx_1 \wedge dx_3 + dy_0 \wedge dy_2 - dy_1 \wedge dy_3,\\
\omega_3 &= dx_0 \wedge dx_3 + dx_1 \wedge dx_2 + dy_0 \wedge dy_3 + dy_1 \wedge dy_2.
\end{aligned}$$

The 4-form introduced by Kraines [9] is

$$\kappa = \frac{1}{6}(\omega_1^2 + \omega_2^2 + \omega_3^2), \tag{50}$$

normalized to have comass one. Set $\rho = \sqrt{|x|^2 + |y|^2}$ and $h = \sqrt{1+\rho^2}$. On the same $\mathbb{R}^8$, define the 2-form

$$\omega = \sum_{a=0}^{3} dx_a \wedge dy_a.$$

Equivalently, $\omega(U, V) = \langle(-U_2, U_1), V\rangle$ for $U = (U_1, U_2)$ and $V \in \mathbb{H} \oplus \mathbb{H}$. It is the Kähler form of the orthogonal complex structure $(x, y) \mapsto (-y, x)$. Since $\rho\partial_\rho = (x, y)$, one has $\iota_{\rho\partial_\rho}\omega = (-y, x)^\flat$, so this contraction is precisely the horizontal part of the screw direction. All these forms are defined on $\mathbb{R}^8$ and are pulled back to $\mathbb{R}^9 = \mathbb{R}^8 \times \mathbb{R}_z$ without further notation.

Recall from (1) that

$$H_4 = \{(\cos s\, u, \sin s\, u, s) : u \in \mathbb{H},\ s \in \mathbb{R}\}.$$

The restriction of ρ to H_4 is $|u|$. Writing $u = u_0 + u_1 i + u_2 j + u_3 k$, the induced metric and volume form are

$$\begin{aligned} g_{H_4} &= \sum_{a=0}^{3} (du^a)^2 + (1 + |u|^2) ds^2, \\ d\mu_{H_4} &= \sqrt{1 + |u|^2}\, du^0 \wedge du^1 \wedge du^2 \wedge du^3 \wedge ds. \end{aligned} \tag{51}$$

At $(x, y, z) = (\cos z\, u, \sin z\, u, z) \in H_4$,

$$T_{(x,y,z)} H_4 = \{(\cos z\, v, \sin z\, v, 0) : v \in \mathbb{H}\} \oplus \operatorname{span}\{(-y, x) + \partial_z\}. \tag{52}$$

We orient H_4 by the parametrization $(u, s) \mapsto (\cos s\, u, \sin s\, u, s)$. The first summand of (52) is the quaternionic ruling at height z. Denote its oriented unit simple 4-vector by P_z, where

$$P_z = (\cos z, \sin z, 0) \wedge (\cos z\, i, \sin z\, i, 0) \wedge (\cos z\, j, \sin z\, j, 0) \wedge (\cos z\, k, \sin z\, k, 0). \tag{53}$$

Direct substitution of (53) into (50) gives $\kappa(P_z) = 1$. By (51), the splitting in (52) is orthogonal, $|P_z| = 1$, and

$$d\mu_{H_4}(P_z, (-y, x) + \partial_z) = |P_z \wedge ((-y, x) + \partial_z)| = \sqrt{1 + \rho^2} = h. \tag{54}$$

Set

$$A(\rho) = \frac{3h^2 + 9h + 8}{15(h + 1)^3}, \qquad B(\rho) = \frac{h + 2}{3(h + 1)^2} \tag{55}$$

and define

$$\Psi = \iota_{\rho\partial_\rho} \kappa \wedge \left(A(\rho)\, \iota_{\rho\partial_\rho} \omega + B(\rho)\, dz \right), \qquad \xi = d\Psi. \tag{56}$$

In the following proposition, ξ will be shown to be a calibration form on $\mathbb{R}^9$ that calibrates H_4.

Remark 5.1. The motivation for the ansatz (56) is as follows. Since

$$d\left(\frac{1}{4} \iota_{\rho\partial_\rho} \kappa \right) = \kappa, \qquad \kappa(P_z) = 1,$$

the form $\frac{1}{4} \iota_{\rho\partial_\rho} \kappa$ provides a natural 3-form whose exterior derivative agrees with the volume form along each quaternionic ruling. The vertical and horizontal parts of the remaining tangent direction $(-y, x) + \partial_z$ are detected by dz and $\iota_{\rho\partial_\rho} \omega = (-y, x)^\flat$, respectively. This leads to the two terms in (56). Comparing with the form Ψ defined in (47) for even $k \geq 6$, the two constructions correspond under

$$\Theta \wedge ds \longleftrightarrow \iota_{\rho\partial_\rho} \kappa \wedge \iota_{\rho\partial_\rho} \omega, \qquad \Theta \wedge dz \longleftrightarrow \iota_{\rho\partial_\rho} \kappa \wedge dz.$$

Proposition 5.2. *The functions* (55) *satisfy*

$$\rho A_\rho + 6A = \frac{1}{h}, \quad \rho B_\rho + 4B = \frac{1}{h}, \tag{57}$$

and

$$16B^2 + 24\rho^2 A^2 \leq 1. \tag{58}$$

With (57) *and* (58), ξ *is a smooth calibration form on* $\mathbb{R}^9$, *and calibrates* H_4. *Consequently,* H_4 *is area-minimizing.*

Proof. The verifications of (57) and (58) are direct computations.

Step 1: smoothness. The function $h = (1 + |x|^2 + |y|^2)^{1/2}$ is smooth and positive on $\mathbb{R}^8$, so A and B in (55) are smooth; and $\rho\partial_\rho = (x, y)$ is the Euler field of $\mathbb{R}^8$, so the two contractions in (56) have polynomial coefficients. Hence Ψ, and therefore $\xi = d\Psi$, is smooth on $\mathbb{R}^9$.

Step 2: restriction to H_4. A straightforward computation using Cartan's formula shows that

$$d\iota_{\rho\partial_\rho}\kappa = 4\kappa, \qquad d\iota_{\rho\partial_\rho}\omega = 2\omega.$$

Taking the exterior derivative on (56) gives that

$$\begin{aligned} \xi &= A_\rho \, d\rho \wedge \iota_{\rho\partial_\rho}\kappa \wedge \iota_{\rho\partial_\rho}\omega + 4A\,\kappa \wedge \iota_{\rho\partial_\rho}\omega - 2A\,\iota_{\rho\partial_\rho}\kappa \wedge \omega \\ &\quad + B_\rho \, d\rho \wedge \iota_{\rho\partial_\rho}\kappa \wedge dz + 4B\,\kappa \wedge dz. \end{aligned} \tag{59}$$

Contracting (59) against the frame (52) is a straightforward computation:

$$\xi(P_z, (-y, x) + \partial_z) = \rho^2(\rho A_\rho + 6A) + (\rho B_\rho + 4B).$$

A useful observation for the computation is that at any point of H_4, $\rho\partial_\rho$ belongs to the ruling P_z. Due to (57), the preceding expression equals h. Together with (54), this gives $\xi|_{H_4} = d\mu_{H_4}$. We retain the two identities in (57) separately because both are used in the comass estimate.

Step 3: the comass estimate. Put $e = (x, y)/\rho$ and write κ by using the decomposition $\Lambda^4\mathbb{R}^8 = \Lambda^4 e^\perp \oplus e^\flat \wedge \Lambda^3 e^\perp$:

$$\kappa = e^\flat \wedge \alpha + \beta, \qquad \alpha = \iota_e\kappa, \qquad \iota_e\beta = 0.$$

Since the form κ in (50) is self-dual,

$$*_8(e^\flat \wedge \alpha) = \beta, \qquad *_8\beta = e^\flat \wedge \alpha. \tag{60}$$

Combining these with the identity $*_8(\,\cdot \wedge v^\flat) = \iota_v\,(*_8\,\cdot\,)$ gives, for every unit vector v orthogonal to e,

$$*_8(e^\flat \wedge \alpha \wedge v^\flat) = \iota_v\beta, \qquad *_8(\beta \wedge v^\flat) = \iota_v\left(e^\flat \wedge \alpha\right) = -e^\flat \wedge \iota_v\alpha. \tag{61}$$

Let $\mathcal{J}(x, y) = (-y, x)$ be the orthogonal complex structure whose Kähler form is ω; $\mathcal{J}$ commutes with the quaternionic multiplication on $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$. Consider

$$\mathcal{J}e = \frac{(-y, x)}{\rho}.$$

Note that $\mathcal{J}e$ is a unit vector orthogonal to e. Decomposing $\mathcal{J}e$ with respect to $(\mathbb{H}e)^\perp \oplus \mathbb{H}e$ therefore gives, for a suitable I in the S^2-family of quaternionic complex structures, a unit vector $q \in (\mathbb{H}e)^\perp$ and an angle $0 \le t \le \frac{\pi}{4}$ such that

$$\mathcal{J}e = \cos(2t)\,q - \sin(2t)\,Ie, \qquad \sin(2t) = \frac{2\,|\mathrm{Im}(x\bar{y})|}{\rho^2}. \tag{62}$$

Since $|\,\mathrm{Im}(x\bar{y})|^2 = |x|^2|y|^2 - \langle x, y\rangle^2$, this is the same angular variable t as in (32). The set where $\rho > 0$ and $t < \frac{\pi}{4}$ is the complement in $\mathbb{R}^9$ of the zero set of the z-independent nonzero polynomial $(|x|^2 + |y|^2)^2 - 4|\,\mathrm{Im}(x\bar{y})|^2$, hence open and dense. Since $\|\xi\|_*$ is a continuous function of the point, it suffices to prove $\|\xi\|_* \le 1$ on this set. Fix such a point and write $L = \mathbb{H}e$, so that $L^\perp = \mathbb{H}q$; let ω_I be the Kähler form of I and set

$$\varsigma = \frac{2}{3}\,\omega_I|_{\langle e, Ie, q, Iq\rangle^\perp}.$$

(a) An adapted frame. Complete I to an oriented orthonormal triple $I_1 = I$, I_2, I_3 in the S^2-family, so that $I_1 I_2 = I_3$, and for the rest of Step 3 let $\omega_1, \omega_2, \omega_3$ denote the Kähler forms of this triple; the expression $\kappa = \frac{1}{6}(\omega_1^2 + \omega_2^2 + \omega_3^2)$ is invariant under such rotations of the triple. Since $q \perp L$ and the I_i are isometries preserving L and $L^\perp$,

$$e_0 = e, \quad e_a = I_a e, \quad f_0 = q, \quad f_a = I_a q \qquad (a = 1, 2, 3)$$

is an orthonormal frame of $\mathbb{R}^8$ in which each ω_i takes its standard coordinate form. Each I_i preserves L, so $\omega_i = \omega_i^L + \omega_i^{L^\perp}$, and squaring gives

$$\kappa = \mathrm{vol}_L + \mathrm{vol}_{L^\perp} + \frac{1}{3} \sum_{i=1}^{3} \omega_i^L \wedge \omega_i^{L^\perp}. \tag{63}$$

Write $\omega_i^L = e^{0i} + \varepsilon_i$ and $\omega_i^{L^\perp} = f^{0i} + \tau_i$, where $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (e^{23}, e^{31}, e^{12})$ and τ_i is the same expression in the f's. Then (63) gives

$$\alpha = e^{123} + \frac{1}{3} \sum_i e^i \wedge \omega_i^{L^\perp}, \qquad \beta = \mathrm{vol}_{L^\perp} + \frac{1}{3} \sum_i \varepsilon_i \wedge \omega_i^{L^\perp},$$

and

$$\iota_q \beta = f^{123} + \frac{1}{3} \sum_i \varepsilon_i \wedge f^i, \qquad \beta_q := \beta - q^\flat \wedge \iota_q \beta = \frac{1}{3} \sum_i \varepsilon_i \wedge \tau_i.$$

It follows that $\omega_1 = e^{01} + \varepsilon_1 + f^{01} + \tau_1$, $\langle e, Ie, q, Iq \rangle^\perp = \langle e_2, e_3, f_2, f_3 \rangle$, and

$$\varsigma = \frac{2}{3} (\varepsilon_1 + \tau_1), \qquad \text{whence} \quad \|\varsigma\|_* = \frac{2}{3}.$$

(b) A useful identity. The following identity will be useful for the estimate of the comass:

$$e^\flat \wedge (\omega + 3\, \iota_{\mathcal{J}e} \alpha) = -3 \sin(2t)\, e^\flat \wedge \varsigma, \tag{64}$$

where $e^\flat = e^0$ in the frame introduced in (a). To prove this identity, first observe that (62) implies

$$\mathcal{J} e_1 = \cos(2t)\, f_1 + \sin(2t)\, e, \quad \mathcal{J} e_2 = \cos(2t)\, f_2 + \sin(2t)\, e_3, \quad \mathcal{J} e_3 = \cos(2t)\, f_3 - \sin(2t)\, e_2.$$

Since $t < \frac{\pi}{4}$, it is not hard to find that $\mathcal{J} q = -\cos(2t)\, e + \sin(2t)\, f_1$, and hence,

$$\mathcal{J} f_1 = -\cos(2t)\, e_1 - \sin(2t)\, q, \quad \mathcal{J} f_2 = -\cos(2t)\, e_2 - \sin(2t)\, f_3, \quad \mathcal{J} f_3 = -\cos(2t)\, e_3 + \sin(2t)\, f_2.$$

These relations allow us to write down ω in terms of the basis $\{e_0, \ldots, e_3, f_0, \cdots, f_3\}$:

$$\omega = \cos(2t) \sum_{a=0}^{3} e^a \wedge f^a + \sin(2t) \left(-e^0 \wedge e^1 + f^0 \wedge f^1 + e^2 \wedge e^3 - f^2 \wedge f^3 \right).$$

On the other hand, contracting the formulas of (a) gives $\iota_q \alpha = -\frac{1}{3} \sum_{i=1}^{3} e^i \wedge f^i$ and $\iota_{Ie} \alpha = e^{23} + \frac{1}{3} (f^{01} + f^{23})$, so by (62),

$$\omega + 3\, \iota_{\mathcal{J}e} \alpha = \cos(2t)\, e^0 \wedge f^0 - \sin(2t)\, e^0 \wedge e^1 - 2 \sin(2t) \left(e^{23} + f^{23} \right);$$

wedging the formula with $e^\flat = e^0$ proves (64).

(c) The Hodge star of ξ. Since $d\rho = e^\flat$, $\iota_{\rho \partial_\rho} \kappa = \rho \alpha$, $\iota_{\rho \partial_\rho} \omega = \rho (\mathcal{J} e)^\flat$ and $\kappa \wedge \omega = - *_8\, \omega$, (59) can be rewritten as

$$\xi = \rho^2 A_\rho\, e^\flat \wedge \alpha \wedge (\mathcal{J} e)^\flat + 6 \rho A\, \kappa \wedge (\mathcal{J} e)^\flat + 2 \rho A\, \iota_e\, (*_8 \omega) + \rho B_\rho\, e^\flat \wedge \alpha \wedge dz + 4 B\, \kappa \wedge dz.$$

Equip $\mathbb{R}^8 \oplus \mathbb{R}\partial_z$ with the product orientation. By (60), (61), $\iota_e *_8 \omega = *_8(\omega \wedge e^\flat)$, and writing $\beta = (\mathcal{J}e)^\flat \wedge \iota_{\mathcal{J}e}\beta + \beta_{\mathcal{J}e}$ with $\iota_{\mathcal{J}e}\beta_{\mathcal{J}e} = 0$, the Hodge star of ξ is

$$\begin{aligned} *_9\xi &= (\rho^2 A_\rho + 6\rho A)\, \iota_{\mathcal{J}e}\beta \wedge dz - 6\rho A\, e^\flat \wedge \iota_{\mathcal{J}e}\alpha \wedge dz - 2\rho A\, \omega \wedge e^\flat \wedge dz \\ &\quad + \rho B_\rho\, \beta + 4B\, \kappa \\ &= 4B\, e^\flat \wedge \alpha + (4B + \rho B_\rho)\, \beta - \rho\, (\rho A_\rho + 6A)\, dz \wedge \iota_{\mathcal{J}e}\beta + 2\rho A\, dz \wedge e^\flat \wedge (\omega + 3\, \iota_{\mathcal{J}e}\alpha)\,. \end{aligned} \tag{65}$$

Applying (57) and (64) to (65), we obtain

$$*_9\xi = 4B\, e^\flat \wedge \alpha + \frac{1}{h}\beta - dz \wedge \Big(\frac{\rho}{h}\iota_{\mathcal{J}e}\beta + 6\rho A \sin(2t)\, e^\flat \wedge \varsigma\Big)\,. \tag{66}$$

(d) The reduction. Consider the horizontal–vertical decomposition, $\mathbb{R}^8 \oplus \mathbb{R}\partial_z$. For any oriented unit simple *four*-plane Π in $\mathbb{R}^8 \oplus \mathbb{R}\partial_z$, $\Pi \cap (\mathbb{R}^8 \oplus \{0\})$ has dimension at least three. Choose an oriented orthonormal three-vector Π_h in this intersection. The remaining unit vector of Π can then be written $\cos\theta\, v + \sin\theta\, \partial_z$, where v is horizontal, unit, and orthogonal to Π_h.

For any fixed Π_h, it follows from (66) that

$$\begin{aligned} &\max_{v,\theta}\big[(*_9\xi)\,(\Pi_h \wedge (\cos\theta\, v + \sin\theta\, \partial_z))\big]^2 \\ &= \left|\iota_{\Pi_h}\left(4B\, e^\flat \wedge \alpha + \frac{1}{h}\beta\right)\right|^2 + \left[\frac{\rho}{h}\,(\iota_{\mathcal{J}e}\beta)\,(\Pi_h) + 6\rho A \sin(2t)(e^\flat \wedge \varsigma)(\Pi_h)\right]^2. \end{aligned}$$

By expanding $\mathcal{J}e$ by (62) and using the basic inequality $(c_0\cos(2t) + c_1\sin(2t))^2 \le c_0^2 + c_1^2$, it suffices to show that

$$\left|\iota_{\Pi_h}\left(4B\, e^\flat \wedge \alpha + \frac{1}{h}\beta\right)\right|^2 + \frac{\rho^2}{h^2}(\iota_q\beta)(\Pi_h)^2 + \left[-\frac{\rho}{h}(\iota_{Ie}\beta)(\Pi_h) + 6\rho A(e^\flat \wedge \varsigma)(\Pi_h)\right]^2 \le 1 \tag{67}$$

for every unit simple horizontal three-vector Π_h.

To proceed, note that $\Pi_h \cap e^\perp$ has dimension at least two. Thus, we can choose[1] $u, v, w \in e^\perp$ and $\Pi_h = (\cos\varphi\, e + \sin\varphi\, u) \wedge v \wedge w$ for some angle φ. With this understood, we fix any orthonormal triple $u, v, w \in e^\perp$, and analyze the maximum of the left-hand side of (67) over φ, or more precisely, over $\Pi_h(\varphi) = (\cos\varphi\, e + \sin\varphi\, u) \wedge v \wedge w$.

(e) Some inequalities. Let

$$\mathcal{A} = \alpha(v, w, \cdot) \quad \text{and} \quad \mathcal{B} = \beta(u, v, w, \cdot).$$

Since $\iota_e\alpha = 0$ and $\iota_e\beta = 0$, both annihilate e.

Since κ has comass 1, $\left|\iota_{\Pi_h(\varphi)}\kappa\right|^2 \le 1$ for any φ. With[2]

$$\iota_{\Pi_h(\varphi)}\kappa = \cos\varphi\, \mathcal{A} + \sin\varphi\, \big(-\alpha(u, v, w)\, e^\flat + \mathcal{B}\big)\,,$$

we find that

$$\begin{aligned} |\mathcal{A}| \le 1, \qquad \alpha(u, v, w)^2 + |\mathcal{B}|^2 \le 1, \\ \langle \mathcal{A}, \mathcal{B}\rangle^2 \le \big(1 - |\mathcal{A}|^2\big)\big(1 - \alpha(u, v, w)^2 - |\mathcal{B}|^2\big)\,. \end{aligned} \tag{68}$$

We need two further inequalities:

$$|\varsigma(v, w)| \le \frac{2}{3}, \qquad \mathcal{B}(q)^2 + \alpha(u, v, w)^2 + 3\mathcal{B}(Ie)^2 \le 1. \tag{69}$$

[1]The notation v is used in three different places of this proof; the v's in different parts are unrelated.
[2]The convention is $\iota_{X_1 \wedge X_2 \wedge X_3}\Phi = \Phi(X_1, X_2, X_3, \cdot)$.

The first one follows from the definition of ς, and the second one is precisely (70), proved in Lemma 5.3 (applied to $u \wedge v \wedge w$).

(f) The estimate. Direct contraction gives

$$\begin{aligned}
\iota_{\Pi_h(\varphi)}\left(4B\, e^\flat \wedge \alpha + \tfrac{1}{h}\beta\right) &= 4B\cos\varphi\,\mathcal{A} + \sin\varphi\left(-4B\,\alpha(u,v,w)\,e^\flat + \tfrac{1}{h}\mathcal{B}\right),\\
\tfrac{\rho}{h}\,(\iota_q\beta)(\Pi_h(\varphi)) &= -\tfrac{\rho\sin\varphi}{h}\,\mathcal{B}(q),\\
-\tfrac{\rho}{h}\,(\iota_{Ie}\beta)(\Pi_h(\varphi)) + 6\rho A\,(e^\flat\wedge\varsigma)(\Pi_h(\varphi)) &= \tfrac{\rho\sin\varphi}{h}\,\mathcal{B}(Ie) + 6\rho A\cos\varphi\,\varsigma(v,w).
\end{aligned}$$

Consider the following map

$$(\cos\varphi,\sin\varphi)\mapsto\left(\iota_{\Pi_h(\varphi)}\left(4B\,e^\flat\wedge\alpha+\tfrac{1}{h}\beta\right),\ \tfrac{\rho}{h}(\iota_q\beta)(\Pi_h(\varphi)),\ -\tfrac{\rho}{h}(\iota_{Ie}\beta)(\Pi_h(\varphi))+6\rho A\,(e^\flat\wedge\varsigma)(\Pi_h(\varphi))\right),$$

which takes values in $\Lambda^1\mathbb{R}^8\oplus\mathbb{R}^2$. Clearly, (67) holds if and only if the squared norm of its value is at most 1. By regarding it as a quadratic form on $\mathbb{R}^2$, it has coefficients

$$\begin{aligned}
\mathsf{E} &= 16B^2|\mathcal{A}|^2 + 36\rho^2A^2\,\varsigma(v,w)^2,\\
\mathsf{G} &= 16B^2\alpha(u,v,w)^2 + \frac{1}{h^2}|\mathcal{B}|^2 + \frac{\rho^2}{h^2}\left(\mathcal{B}(q)^2+\mathcal{B}(Ie)^2\right),\\
\mathsf{F} &= \frac{4B}{h}\langle\mathcal{A},\mathcal{B}\rangle + \frac{6\rho^2A}{h}\,\mathcal{B}(Ie)\,\varsigma(v,w).
\end{aligned}$$

Therefore, it remains to prove the following:

$$\mathsf{E}\le 1,\quad \mathsf{G}\le 1,\quad\text{and }\ \mathsf{F}^2\le(1-\mathsf{E})(1-\mathsf{G}).$$

For $\mathsf{E}\le 1$: according to (58), (68) and (69),

$$\begin{aligned}
1-\mathsf{E} &= 16B^2\left(1-|\mathcal{A}|^2\right) + \left(1-16B^2-24\rho^2A^2\right) + 36\rho^2A^2\left(\frac{2}{3}-\varsigma(v,w)^2\right)\\
&\ge 16B^2\left(1-|\mathcal{A}|^2\right) + 36\rho^2A^2\left(\frac{2}{3}-\varsigma(v,w)^2\right)\ge 0.
\end{aligned}$$

For $\mathsf{G}\le 1$: using $1=h^{-2}+\rho^2h^{-2}$, (58) (which gives $16B^2\le 1$) and (69),

$$\begin{aligned}
1-\mathsf{G} &= \frac{1}{h^2}\left(1-\alpha(u,v,w)^2-|\mathcal{B}|^2\right) + \frac{\rho^2}{h^2}\left(1-\mathcal{B}(q)^2-\mathcal{B}(Ie)^2\right)\\
&\quad-\left(16B^2-\frac{1}{h^2}\right)\alpha(u,v,w)^2\\
&\ge \frac{1}{h^2}\left(1-\alpha(u,v,w)^2-|\mathcal{B}|^2\right) + \frac{\rho^2}{h^2}\left(1-\mathcal{B}(q)^2-\mathcal{B}(Ie)^2-\alpha(u,v,w)^2\right)\\
&\ge \frac{1}{h^2}\left(1-\alpha(u,v,w)^2-|\mathcal{B}|^2\right) + \frac{2\rho^2}{h^2}\mathcal{B}(Ie)^2\ge 0.
\end{aligned}$$

Note that $|\varsigma(v,w)| \le \frac{2}{3}$ leads to $\varsigma(v,w)^2 \le 2\left(\frac{2}{3} - \varsigma(v,w)^2\right)$. By the Cauchy–Schwarz inequality and (68),

$$\begin{aligned}
|\mathsf{F}| &\le \frac{4|B|}{h}\sqrt{(1-|\mathcal{A}|^2)(1-\alpha(u,v,w)^2-|\mathcal{B}|^2)} + \frac{6\rho^2|A|}{h}\,|\mathcal{B}(Ie)\,\varsigma(v,w)| \\
&\le \sqrt{16B^2(1-|\mathcal{A}|^2)}\sqrt{\frac{1}{h^2}(1-\alpha(u,v,w)^2-|\mathcal{B}|^2)} \\
&\quad + \sqrt{36\rho^2A^2\left(\frac{2}{3}-\varsigma(v,w)^2\right)}\sqrt{\frac{2\rho^2}{h^2}\mathcal{B}(Ie)^2} \\
&\le \left[16B^2(1-|\mathcal{A}|^2) + 36\rho^2A^2\left(\frac{2}{3}-\varsigma(v,w)^2\right)\right]^{1/2} \\
&\quad \times \left[\frac{1}{h^2}(1-\alpha(u,v,w)^2-|\mathcal{B}|^2) + \frac{2\rho^2}{h^2}\mathcal{B}(Ie)^2\right]^{1/2} \\
&\le \sqrt{1-\mathsf{E}}\,\sqrt{1-\mathsf{G}}.
\end{aligned}$$

Hence, $\xi = d\Psi$ is smooth and closed, $\|\xi\|_* \le 1$, and $\xi|_{H_4} = d\mu_{H_4}$, so ξ calibrates H_4. Consequently, H_4 is area-minimizing. $\square$

The lemma below establishes an inequality claimed in Step 3 (e).

Lemma 5.3. *Let $e \in \mathbb{R}^8$ be a unit vector and write the 4-form κ in (50) as*

$$\kappa = e^\flat \wedge \alpha + \beta, \qquad \alpha = \iota_e\kappa, \qquad \iota_e\beta = 0.$$

Let I be any quaternionic complex structure and let q be any unit vector orthogonal to the quaternionic line $\mathbb{H}e$. Then, for every unit simple three-vector $\eta \in \Lambda^3 e^\perp$,

$$(\iota_q\beta)(\eta)^2 + \alpha(\eta)^2 + 3(\iota_{Ie}\beta)(\eta)^2 \le 1. \tag{70}$$

Proof. Complete I to a hyperkähler triple I, J, K, with Kähler forms $\omega_I, \omega_J, \omega_K$, and set

$$\Phi_1 = \omega_I^2 - 3\kappa, \qquad \Phi_2 = \omega_J^2 - 3\kappa, \qquad \Phi_3 = \omega_K^2 - 3\kappa.$$

These are Cayley calibrations, and $\omega_I^2 + \omega_J^2 + \omega_K^2 = 6\kappa$ gives $\Phi_1 + \Phi_2 + \Phi_3 = -3\kappa$. Writing $\gamma_i = \iota_\eta\Phi_i$ and contracting this identity with η,

$$\frac{1}{3}\sum_{i=1}^{3}\gamma_i(X) = -\kappa(\eta, X) = \kappa(X,\eta) \qquad \text{for every } X \in \mathbb{R}^8.$$

Since $\eta \in \Lambda^3 e^\perp$, the decomposition $\kappa = e^\flat \wedge \alpha + \beta$ gives $\kappa(e,\eta) = \alpha(\eta)$, $\kappa(q,\eta) = (\iota_q\beta)(\eta)$ and $\kappa(Ie,\eta) = (\iota_{Ie}\beta)(\eta)$; taking $X = e$ and $X = q$ above therefore yields

$$\frac{1}{3}\sum_{i=1}^{3}\gamma_i(e) = \alpha(\eta), \qquad \frac{1}{3}\sum_{i=1}^{3}\gamma_i(q) = (\iota_q\beta)(\eta).$$

For the third quantity only Φ_1 is needed: since $\iota_{Ie}\omega_I = -e^\flat$, one has $\iota_{Ie}\omega_I^2 = 2\,(\iota_{Ie}\omega_I)\wedge\omega_I = -2\,e^\flat \wedge \omega_I$, which annihilates $\Lambda^3 e^\perp$, so $\omega_I^2(\eta, Ie) = 0$ and hence

$$\gamma_1(Ie) = \Phi_1(\eta, Ie) = -3\kappa(\eta, Ie) = 3(\iota_{Ie}\beta)(\eta).$$

Each Φ_i has comass one and η is a unit simple three-vector, so $|\gamma_i| \leq 1$. By Cauchy–Schwarz, $\alpha(\eta)^2 \leq \frac{1}{3}\sum_i \gamma_i(e)^2$ and $(\iota_q\beta)(\eta)^2 \leq \frac{1}{3}\sum_i \gamma_i(q)^2$, while e, q and Ie are orthonormal, so $\gamma_i(e)^2 + \gamma_i(q)^2 + \gamma_i(Ie)^2 \leq |\gamma_i|^2$ for each i. Therefore,

$$(\iota_q\beta)(\eta)^2 + \alpha(\eta)^2 + 3(\iota_{Ie}\beta)(\eta)^2 \leq \frac{1}{3}\sum_{i=1}^{3}\big(\gamma_i(q)^2 + \gamma_i(e)^2\big) + \frac{1}{3}\gamma_1(Ie)^2 \leq \frac{1}{3}\sum_{i=1}^{3}|\gamma_i|^2 \leq 1.$$

This finishes the proof of this lemma. □

6. Construction of competitors for H_k with odd k

We write (x, y, z) for the coordinates on $\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}$. By (1), H_k is parametrized by $F_k(u, s) = (\cos s\, u, \sin s\, u, s)$, where $u \in \mathbb{R}^k$ and $s \in \mathbb{R}$.

Theorem 6.1. *If k is odd, then H_k is not area-minimizing.*

6.1. **The oriented replacement.** Orient H_k by the coordinates $(u_1, \ldots, u_k, s)$. In polar coordinates $u = \rho w$, the induced metric and volume form are

$$g_{H_k} = d\rho^2 + (1+\rho^2)ds^2 + \rho^2|dw|^2, \qquad d\mu_{H_k} = \rho^{k-1}\sqrt{1+\rho^2}\, d\rho\, d\mu_{S^{k-1}}\, ds. \tag{71}$$

Fix $R > 0$ and write $B_R = \{u \in \mathbb{R}^k : |u| \leq R\}$. Consider the following portion of H_k:

$$P_R = \{F_k(u, s) : u \in B_R,\ 0 \leq s \leq 2\pi\}. \tag{72}$$

The last coordinate makes F_k injective on $B_R \times [0, 2\pi]$. By (71),

$$\mathrm{Vol}(P_R) = 2\pi\,\mathrm{Vol}(S^{k-1})\int_0^R \rho^{k-1}\sqrt{1+\rho^2}\, d\rho. \tag{73}$$

Here Vol denotes Hausdorff measure in the dimension of the set under consideration, and $\mathrm{Vol}(S^0) = 2$.

Let $Q_R = B_R \times [0,\pi]^2 \subset \mathbb{R}^{k+2}$, with coordinates (u, s, t), and define

$$G : Q_R \longrightarrow \mathbb{R}^{2k+1}, \qquad G(u, s, t) = (\cos s\, u, \sin s\, u, s + t).$$

The $(k+2)$-dimensional Jacobian $J_{k+2}G$ measures the local change of $(k+2)$-dimensional volume from Q_R to its image $G(Q_R) \subset \mathbb{R}^{2k+1}$. Away from $u = 0$, the map G is injective except for the codimension-two identification $(u, 0, \pi) \sim (-u, \pi, 0)$. At $u = 0$, its rank drops to $k + 1$. Indeed,

$$G_{u_j} = (\cos s\, e_j, \sin s\, e_j, 0), \quad G_s = (-\sin s\, u, \cos s\, u, 1), \quad G_t = (0, 0, 1).$$

The vectors $G_{u_1}, \ldots, G_{u_k}, G_s - G_t, G_t$ are mutually orthogonal, with lengths $1, \ldots, 1, |u|, 1$. Hence

$$J_{k+2}G = |G_{u_1} \wedge \cdots \wedge G_{u_k} \wedge (G_s - G_t) \wedge G_t| = |u|.$$

The set $u = 0$ maps under G to the vertical axis and contributes no $(k+2)$-dimensional mass. The images of the faces $t = 0, \pi$ under G together form P_R:

$$G(u, s, 0) = F_k(u, s), \qquad G(u, s, \pi) = F_k(-u, s + \pi).$$

Under G, the faces $s = 0$ and $s = \pi$ together map onto the cylinder

$$C_R = \{(u, 0, z) : u \in B_R,\ 0 \leq z \leq 2\pi\},$$

and the face $|u| = R$ maps onto

$$L_R = \{(R\cos s\, w, R\sin s\, w, z) : 0 \leq s \leq \pi,\ s \leq z \leq s + \pi,\ w \in S^{k-1}\}.$$

To determine the orientations of the two helicoid faces, write (v, σ) for the coordinates on the target helicoid,

$$F_k(v, \sigma) = (\cos\sigma\, v, \sin\sigma\, v, \sigma).$$

Orient Q_R by

$$\Omega = dt \wedge du_1 \wedge \cdots \wedge du_k \wedge ds, \quad \text{and denote} \quad \omega = du_1 \wedge \cdots \wedge du_k \wedge ds.$$

Using the outward-normal-first convention on Q_R's boundary, the boundary orientations are $-\omega$ on $t = 0$ and $+\omega$ on $t = \pi$. Thus the $t = 0$ face carries the negative of the H_k orientation. On $t = \pi$, let $\Phi(u, s) = (-u, s + \pi)$. Then $G(u, s, \pi) = F_k(\Phi(u, s))$ and

$$\Phi^*(dv_1 \wedge \cdots \wedge dv_k \wedge d\sigma) = (-1)^k \omega.$$

Since the boundary form there is $+\omega$, the $t = \pi$ face carries $(-1)^k$ times the H_k orientation. Hence, when k is odd, both helicoid faces carry the negative of the H_k orientation.

In the following, we write $[\![A]\!]$ for the current of integration over an oriented set A, and use the standard notations ∂ and $\mathbb{M}$ for boundary and mass in geometric measure theory; see [15, Chapter 6]. Define $T_R = G_\#([\![Q_R]\!])$, where $G_\#$ denotes the pushforward of currents under G. Since G is smooth on Q_R, $\partial T_R = G_\#(\partial[\![Q_R]\!])$, with no boundary term from $u = 0$. For odd k, the preceding computation gives

$$\partial T_R = -[\![P_R]\!] + [\![K_R]\!], \qquad K_R = C_R \cup L_R, \tag{74}$$

where the pieces of K_R carry the orientations induced from Q_R. Thus

$$[\![H_k]\!] + \partial T_R = [\![H_k]\!] - [\![P_R]\!] + [\![K_R]\!] \tag{75}$$

is an integral-current competitor agreeing with $[\![H_k]\!]$ outside a compact set.

6.2. **The area comparison.** The cylinder over B_R has volume

$$\mathrm{Vol}(C_R) = 2\pi\, \mathrm{Vol}(S^{k-1}) \int_0^R \rho^{k-1}\, d\rho = \frac{2\pi}{k} \mathrm{Vol}(S^{k-1}) R^k. \tag{76}$$

On L_R, use (s, w, z) as coordinates. A unit tangent vector to S^{k-1} is scaled by R, while the coordinate vectors in the s and z directions have lengths R and 1; these directions are mutually orthogonal. Therefore

$$\mathrm{Vol}(L_R) = \mathrm{Vol}(S^{k-1}) \int_0^\pi \int_s^{s+\pi} R^k\, dz\, ds = \pi^2\, \mathrm{Vol}(S^{k-1}) R^k. \tag{77}$$

The inserted pieces intersect one another only in sets of zero $(k+1)$-dimensional measure: $C_R \cap L_R$ is k-dimensional. Hence, combining (76) and (77), we obtain

$$\mathrm{Vol}(K_R) = \mathrm{Vol}(S^{k-1}) \left(\pi^2 + \frac{2\pi}{k}\right) R^k. \tag{78}$$

The inserted pieces also meet the retained part of H_k only in sets of zero $(k+1)$-dimensional measure. Indeed, $C_R \subset \{y = 0\}$, while $H_k \cap \{y = 0\}$ is contained in the axis together with the slices $s \in \pi\mathbb{Z}$ and is at most k-dimensional. Also,

$$H_k \cap \{|(x, y)| = R\} = F_k(RS^{k-1} \times \mathbb{R})$$

is k-dimensional, so the same holds for $L_R \cap H_k$. Consequently, there is no overlap or cancellation in the mass comparison.

By (73) and $\sqrt{1+\rho^2} \geq \rho$, for $R > 0$ we have

$$\mathrm{Vol}(P_R) \geq \frac{2\pi}{k+1} \mathrm{Vol}(S^{k-1}) R^{k+1}. \tag{79}$$

Comparing (79) and (78), if

$$R > (k+1)\left(\frac{\pi}{2} + \frac{1}{k}\right), \tag{80}$$

then

$$\mathrm{Vol}(K_R) < \mathrm{Vol}(P_R). \tag{81}$$

Proof of Theorem 6.1. Choose R as in (80) and set $S_R = \partial T_R$. This is a compactly supported integral $(k+1)$-cycle. For every ball B containing its support, (74), (75), the preceding observations on the intersections, and (81) give

$$\mathbb{M}((\llbracket H_k \rrbracket + S_R) \llcorner B) - \mathbb{M}(\llbracket H_k \rrbracket \llcorner B) = \mathrm{Vol}(K_R) - \mathrm{Vol}(P_R) < 0.$$

Thus H_k is not area-minimizing. No smoothing is required, since piecewise-smooth integral currents are admissible competitors. $\square$

Remark 6.2. The construction is especially easy to visualize when $k = 1$; see Figures 1 and 2. The removed portion lies in the solid cylinder

$$\{(x, y, z) : \sqrt{x^2 + y^2} \leq R,\ 0 \leq z \leq 2\pi\}.$$

The replacement $K_R = C_R \cup L_R$ consists of three connected pieces: one rectangle C_R and two bands on the cylinder

$$\{(x, y, z) : \sqrt{x^2 + y^2} = R,\ 0 \leq z \leq 2\pi\}.$$

Writing $w \in S^0 = \{+1, -1\}$ and θ for the polar angle in the (x, y)-plane, one has $\theta = s$ on the component $w = +1$ and $\theta = s + \pi$ on the component $w = -1$. Thus the two bands in L_R are

$$\{0 \leq \theta \leq \pi,\ \theta \leq z \leq \theta + \pi\} \cup \{\pi \leq \theta \leq 2\pi,\ \theta - \pi \leq z \leq \theta\}.$$

As $R \to \infty$, $\mathrm{Vol}(P_R) \sim 2\pi R^2$, whereas

$$\mathrm{Vol}(K_R) = 4\pi R + 2\pi^2 R = O(R).$$

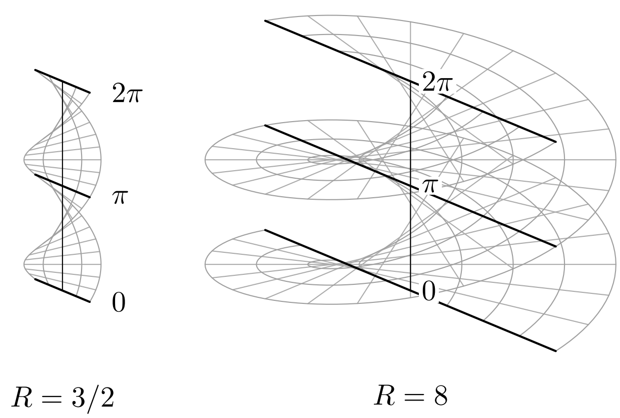
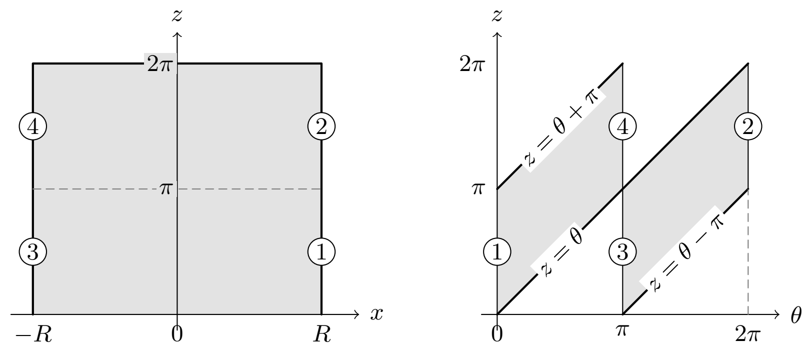


FIGURE 1. Wireframe views of P_R in (72) for $k = 1$, with $R = 3/2$ and $R = 8$ at the same scale. The vertical axis is interior for $0 < z < 2\pi$.



(a) the rectangle C_R (b) the cylinder, $\rho = R$

FIGURE 2. The replacement $K_R = C_R \cup L_R$ for $k = 1$: (a) the rectangle; (b) the two bands on the cylinder, with $\theta = 0$ and $\theta = 2\pi$ identified. Matching circled numbers mark glued boundary segments. The dashed line in (a) is the interior seam $z = \pi$.

Remark 6.3. For even k, one has $(-1)^k = +1$, so the two helicoid faces occur in ∂T_R with opposite H_k orientations; the construction is therefore not an integral comparison. After reducing coefficients modulo 2, the signs disappear, and the same construction shows that H_k is not area-minimizing modulo 2 for any k. The parity is also visible under the orthogonal projection $\mathbb{R}^{2k+1} \to \mathbb{R}^{2k}$ forgetting z, which identifies (u, s) with $(-u, s + \pi)$ and maps the two consecutive turns to the same rank-one cone.

Remark 6.4. In [11], Lawlor's directed-slicing argument shows that the portion of the helicoid

$$\{F_1(u, s) : 0 \le u \le R,\ 0 \le s \le 6\pi\}$$

is area-minimizing among oriented competitors with the same boundary. Our choice of the competitor P_R in (72) is inspired by Lawlor's work. In Lawlor's example, the axis $u = 0$ is part of the boundary. In contrast, it is interior to $P_R = F_1([-R, R] \times [0, 2\pi])$ for $0 < s < 2\pi$. Thus these are two problems with different boundaries.

Remark 6.5. The competitor also works for certain generalized helicoids (2): Suppose that $\alpha > 0$, $m_j \in \mathbb{Z}$, not all zero, and $\sum_{j=1}^{k} m_j$ is odd. Let $\boldsymbol{\lambda} = \alpha(m_1, \ldots, m_k)$. Then, $\mathcal{H}_{\boldsymbol{\lambda}}$ is not area-minimizing.

7. Stability of generalized helicoids in slope space

We return to the generalized helicoids $\mathcal{H}_{\boldsymbol{\lambda}}$ defined by the parametrization in (2). Proposition 2.2 reduces their stability to nonnegativity of $q_{\boldsymbol{\lambda}}$ in (3). For helicoids H_k with parameters $\boldsymbol{\lambda} = (1, \ldots, 1)$, Proposition 3.1 proves stability when $k \ge 3$. We now allow the slopes to vary and derive stable and unstable regions in the parameter space $(0, \infty)^k$.

7.1. **Anisotropic stable regions.** Let

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k), \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k), \qquad \lambda_i > 0.$$

Under the change of variables

$$\tilde{f}(u) = f(\Lambda u), \qquad x = \Lambda u,$$

$q_{\boldsymbol{\lambda}}$ in (3) becomes

$$\Big(\prod_{i=1}^{k}\lambda_i\Big)q_{\boldsymbol{\lambda}}[\tilde f]=\int_{\mathbb{R}^k}h\sum_{j=1}^{k}\lambda_j^2\left[|\partial_{x_j}f|^2+\big|\partial_{x_j}(x\cdot f)\big|^2-\frac{3(f^j)^2}{h^2}\right]dx,\qquad h=\sqrt{1+|x|^2}. \tag{82}$$

Hence $\mathcal{H}_{\boldsymbol{\lambda}}$ is stable if the right-hand side is nonnegative for every $f\in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$.

For unequal slopes, Proposition 3.1 cannot be applied directly to the integral in (82). Set

$$\lambda_{\min}=\min_j\lambda_j,\qquad \lambda_{\max}=\max_j\lambda_j.$$

For $0\le\sigma\le\lambda_{\min}^2$, we split the terms involving derivatives into an isotropic part of size σ and remaining terms. The isotropic part is controlled by Proposition 3.1, while the residual part is estimated through the scalar quantity introduced in the proof below. Optimizing over σ gives the following criterion.

Theorem 7.1. *Let $k\ge 3$ and $a_1,\dots,a_k\ge 0$, and set*

$$\mu(a_1,\dots,a_k)=\sup_{\tau\ge 0}\left\{\frac12\sum_{j=1}^{k}\Big(\sqrt{a_j^4+4a_j^2\tau}-a_j^2\Big)-\tau\right\}. \tag{83}$$

If $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_k)\in(0,\infty)^k$ satisfies

$$\sup_{0\le\sigma\le\lambda_{\min}^2}\left\{\frac{(k-1)^2+4}{4}\,\sigma+\mu\Big(\sqrt{\lambda_1^2-\sigma},\dots,\sqrt{\lambda_k^2-\sigma}\Big)\right\}\ \ge\ 2\lambda_{\max}^2, \tag{84}$$

then $\mathcal{H}_{\boldsymbol{\lambda}}$ is stable.

Proof. Fix $\sigma\in[0,\lambda_{\min}^2]$, so that $\lambda_j^2-\sigma\ge 0$ for every j. For any $f\in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$, set $F=(f,x\cdot f)\in\mathbb{R}^{k+1}$, $n=\frac{(-x,1)}{h}\in\mathbb{R}^{k+1}$ and $\rho=|F|$. Since $F\cdot n=0$, $|n|=1$ and $\partial_{x_j}F\cdot n=\frac{f^j}{h}$,

$$|\partial_{x_j}f|^2+|\partial_{x_j}(x\cdot f)|^2=|\partial_{x_j}F|^2\ \ge\ \left|\partial_{x_j}F\cdot\frac{F}{\rho}\right|^2+|\partial_{x_j}F\cdot n|^2=|\partial_{x_j}\rho|^2+\frac{(f^j)^2}{h^2}. \tag{85}$$

Here ρ is Lipschitz and (85) holds at every point of $\{F\ne 0\}$. On the other hand, on $\{F=0\}$ one has $f=0$ and $\nabla_x\rho=0$ almost everywhere, so (85) holds a.e. on $\mathbb{R}^k$, and the integrations by parts below are performed on the compactly supported Lipschitz function ρ. Splitting the gradient sum at the level σ and applying (85) with the nonnegative weights $\lambda_j^2-\sigma$,

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j^2\Big(|\partial_{x_j}f|^2+|\partial_{x_j}(x\cdot f)|^2\Big)\ &\ge\sigma\Big(|\nabla_x f|^2+|\nabla_x(x\cdot f)|^2\Big)\\ &\quad+\sum_{j=1}^{k}(\lambda_j^2-\sigma)|\partial_{x_j}\rho|^2+\sum_{j=1}^{k}(\lambda_j^2-\sigma)\frac{(f^j)^2}{h^2}.\end{aligned}$$

Multiplying by h, subtracting $\frac{3}{h}\sum_j\lambda_j^2(f^j)^2$, and using $\sum_j\big(2\lambda_j^2+\sigma\big)(f^j)^2\le(2\lambda_{\max}^2+\sigma)|f|^2$, the integrand of (82) is bounded below by

$$h\,\sigma\Big(|\nabla_x f|^2+|\nabla_x(x\cdot f)|^2\Big)+h\sum_{j=1}^{k}(\lambda_j^2-\sigma)|\partial_{x_j}\rho|^2-\big(2\lambda_{\max}^2+\sigma\big)\frac{|f|^2}{h}.$$

By Proposition 3.1, the first term integrates to at least $c_k\sigma\int_{\mathbb{R}^k}\frac{|f|^2}{h}\,dx$. For the second, let $\tau\geq 0$ and, for those j with $\lambda_j^2>\sigma$, choose $\gamma_j\geq 0$ with $(\lambda_j^2-\sigma)(\gamma_j^2+\gamma_j)=\tau$, that is $\gamma_j=\frac{\sqrt{(\lambda_j^2-\sigma)^2+4(\lambda_j^2-\sigma)\tau}-(\lambda_j^2-\sigma)}{2(\lambda_j^2-\sigma)}$. Using $\partial_j\big(\frac{x_j}{h}\big)=\frac{1}{h}-\frac{x_j^2}{h^3}$ and integrating by parts,

$$\begin{aligned}0&\leq\int_{\mathbb{R}^k}h\sum_{j=1}^k(\lambda_j^2-\sigma)\left|\partial_{x_j}\rho+\frac{\gamma_jx_j}{h^2}\rho\right|^2dx\\&=\int_{\mathbb{R}^k}h\sum_{j=1}^k(\lambda_j^2-\sigma)|\partial_{x_j}\rho|^2\,dx-\left(\sum_{j=1}^k(\lambda_j^2-\sigma)\gamma_j-\tau\right)\int_{\mathbb{R}^k}\frac{\rho^2}{h}\,dx-\tau\int_{\mathbb{R}^k}\frac{\rho^2}{h^3}\,dx,\end{aligned}$$

so that, after taking the supremum over $\tau\geq 0$ and recalling (83),

$$\int_{\mathbb{R}^k}h\sum_{j=1}^k(\lambda_j^2-\sigma)|\partial_{x_j}\rho|^2\,dx\ \geq\ \mu\Big(\sqrt{\lambda_1^2-\sigma},\ldots,\sqrt{\lambda_k^2-\sigma}\Big)\int_{\mathbb{R}^k}\frac{\rho^2}{h}\,dx.$$

Since $\mu\geq 0$ and $\rho^2\geq|f|^2$, the right-hand side of (82) is therefore at least

$$\left[\frac{(k-1)^2+4}{4}\sigma+\mu\Big(\sqrt{\lambda_1^2-\sigma},\ldots,\sqrt{\lambda_k^2-\sigma}\Big)-2\lambda_{\max}^2\right]\int_{\mathbb{R}^k}\frac{|f|^2}{h}\,dx,$$

where we used $c_k\sigma-\sigma=\frac{(k-1)^2+4}{4}\sigma$. Taking the supremum over σ and invoking (84) completes the proof. □

Corollary 7.2. *Let $k\geq 3$ and $\boldsymbol{\lambda}\in(0,\infty)^k$. If*

$$8\,\lambda_{\max}^2\ \leq\ \big[(k-1)^2+4\big]\lambda_{\min}^2,\tag{86}$$

then $\mathcal{H}_{\boldsymbol{\lambda}}$ is stable.

Proof. Take $\sigma=\lambda_{\min}^2$ in (84) and use $\mu\geq 0$. □

Thus, for $k\geq 4$, stability persists in an explicit neighborhood of the ray $\{\lambda_1=\cdots=\lambda_k>0\}$ in slope space. At $k=3$, condition (86) forces $\lambda_{\max}=\lambda_{\min}$, while Remark 7.4 below shows that every non-isotropic parameter is unstable. We next construct the unstable test fields and prove the openness of the unstable set.

7.2. **Unstable perturbations.** We first complete the instability half of Theorem 1.1 by treating H_2. Proposition 2.2 reduces the problem to finding a compactly supported field f with $q_{(1,1)}[f]<0$. The required field is tangent to the circles and supported on a large annulus.

For $x\in\mathbb{R}^2\setminus\{0\}$ write $x=r\omega$, $\omega=(\cos\theta,\sin\theta)$, and $e_\theta=(-\sin\theta,\cos\theta)$. The field on S^1 is

$$Y(\theta)=\nabla_S\omega_1=e_1-\omega_1\omega=-\sin\theta\,e_\theta.$$

Proposition 7.3. *H_k is unstable for $k\leq 2$. More precisely,* $\operatorname{ind}H_2=\infty$.

Proof. We first consider $k=2$. Fix $0\neq\phi\in C_c^\infty(\mathbb{R})$ and, for $R>1$, define

$$f_R(x)=r^{-1/2}\phi\left(\log\frac{r}{R}\right)Y(\theta).\tag{87}$$

Since $Y\cdot\omega=0$, we have $x\cdot f_R=0$. Set $\sigma=\log(r/R)$. Using

$$\partial_{x_1}r=\cos\theta,\qquad\partial_{x_2}r=\sin\theta,\qquad\partial_{x_1}\theta=-\frac{\sin\theta}{r},\qquad\partial_{x_2}\theta=\frac{\cos\theta}{r},$$

together with
$$\frac{dY}{d\theta} = -\cos\theta\, e_\theta + \sin\theta\,\omega,$$
direct differentiation gives
$$\partial_{x_1} f_R = r^{-3/2}\left[\left(-\sin\theta\cos\theta\,\phi' + \frac{3}{2}\sin\theta\cos\theta\,\phi\right) e_\theta - \sin^2\theta\,\phi\,\omega\right],$$
$$\partial_{x_2} f_R = r^{-3/2}\left[\left(-\sin^2\theta\,\phi' + \left(\frac{1}{2}\sin^2\theta - \cos^2\theta\right)\phi\right) e_\theta + \sin\theta\cos\theta\,\phi\,\omega\right].$$
Since $\{\omega, e_\theta\}$ is orthonormal,
$$|\partial_{x_1} f_R|^2 + |\partial_{x_2} f_R|^2 = r^{-3}\left[\sin^2\theta\,(\phi')^2 - \sin^2\theta\,\phi\phi' + \left(\frac{5}{4} - \frac{1}{4}\cos^2\theta\right)\phi^2\right].$$
For H_2, formula (82) therefore gives, using $dx = r\,dr\,d\theta$ and $dr = r\,d\sigma$,
$$q_{(1,1)}[f_R] = \int_{\mathbb{R}}\int_0^{2\pi} \frac{h}{r}\left[\sin^2\theta\,(\phi')^2 - \sin^2\theta\,\phi\phi' + \left(\frac{5}{4} - \frac{1}{4}\cos^2\theta\right)\phi^2\right] d\theta\, d\sigma$$
$$- 3\int_{\mathbb{R}}\int_0^{2\pi} \frac{r}{h}\sin^2\theta\,\phi^2\, d\theta\, d\sigma, \qquad h = \sqrt{1 + R^2 e^{2\sigma}}.$$
On the support of ϕ, both h/r and r/h converge uniformly to 1 as $R \to \infty$. Since
$$\int_0^{2\pi} \sin^2\theta\, d\theta = \pi, \qquad \int_0^{2\pi}\left(\frac{5}{4} - \frac{1}{4}\cos^2\theta\right) d\theta = \frac{9\pi}{4},$$
and $\int_{\mathbb{R}} \phi\phi'\, d\sigma = 0$, we obtain
$$\lim_{R\to\infty} q_{(1,1)}[f_R] = \pi\int_{\mathbb{R}} (\phi')^2\, d\sigma - \frac{3\pi}{4}\int_{\mathbb{R}} \phi^2\, d\sigma. \tag{88}$$
Now choose $0 \neq \psi \in C_c^\infty(\mathbb{R})$ and set $\phi(\sigma) = \psi(\sigma/L)$. Then
$$\int_{\mathbb{R}} (\phi')^2\, d\sigma = \frac{1}{L}\int_{\mathbb{R}} (\psi')^2\, d\sigma, \qquad \int_{\mathbb{R}} \phi^2\, d\sigma = L\int_{\mathbb{R}} \psi^2\, d\sigma.$$
Hence the right-hand side of (88) is negative for all sufficiently large L. Fixing such an L, we obtain $q_{(1,1)}[f_R] < 0$ for all sufficiently large R. Therefore Proposition 2.2 gives $\operatorname{ind} H_2 = \infty$.

For $k = 1$, H_1 is the classical helicoid, a complete non-planar minimal surface in $\mathbb{R}^3$, hence unstable by [5, 6]. □

Together, Theorem 3.4 and Proposition 7.3 prove Theorem 1.1.

The same annular perturbation also gives explicit unstable regions for anisotropic slopes.

Remark 7.4. The same annular computation applies in general dimension. For $1 \le p \le k$, set
$$r = |\Lambda u|, \qquad \omega = \frac{\Lambda u}{|\Lambda u|}, \qquad \tilde f_R(u) = r^{-(k-1)/2}\psi\left(\log\frac{r}{R}\right)(e_p - \omega_p\omega).$$
A direct computation in (82) shows that $\operatorname{ind} \mathcal{H}_{\boldsymbol{\lambda}} = \infty$ whenever $(7-k)\lambda_p^2 > (k-1)\sum_{j\neq p}\lambda_j^2$ for some p. Consequently, $k = 2$ is always unstable; for $k = 3$, every non-isotropic parameter is unstable; and for $k = 4, 5, 6$ it suffices, respectively, that $\lambda_p^2 > \sum_{j\neq p}\lambda_j^2$, $\lambda_p^2 > 2\sum_{j\neq p}\lambda_j^2$, or $\lambda_p^2 > 5\sum_{j\neq p}\lambda_j^2$ for some p.

These are explicit sufficient conditions. More generally, the unstable set is open in the set of positive slope parameters.

Proposition 7.5. *For every $k \geq 1$, the set*

$$\{\boldsymbol{\lambda} \in (0,\infty)^k : \mathcal{H}_{\boldsymbol{\lambda}} \text{ is unstable}\}$$

is open in $(0,\infty)^k$.

Proof. Fix $\boldsymbol{\lambda}_0 \in (0,\infty)^k$ such that $\mathcal{H}_{\boldsymbol{\lambda}_0}$ is unstable, and write $\Lambda_0 = \operatorname{diag}(\boldsymbol{\lambda}_0)$. By (4) there exists $\tilde{f}_0 \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ with $q_{\boldsymbol{\lambda}_0}[\tilde{f}_0] < 0$. Set

$$f(x) = \tilde{f}_0(\Lambda_0^{-1}x).$$

For $\boldsymbol{\lambda} \in (0,\infty)^k$ and $\Lambda = \operatorname{diag}(\boldsymbol{\lambda})$, (82) gives

$$\left(\prod_{i=1}^k \lambda_i\right) q_{\boldsymbol{\lambda}}[f\circ\Lambda] = \sum_{j=1}^k \lambda_j^2 \int_{\mathbb{R}^k} h\left[|\partial_{x_j} f|^2 + |\partial_{x_j}(x\cdot f)|^2 - \frac{3(f^j)^2}{h^2}\right] dx.$$

The right-hand side is continuous in $\boldsymbol{\lambda}$ and negative at $\boldsymbol{\lambda}_0$, because $f\circ\Lambda_0 = \tilde{f}_0$. It remains negative in a neighborhood of $\boldsymbol{\lambda}_0$, and the positivity of $\prod_i \lambda_i$ gives $q_{\boldsymbol{\lambda}}[f\circ\Lambda] < 0$ there. $\square$

Finally, instability persists after adjoining zero slopes and hence, by perturbation, sufficiently small positive slopes. Indeed, (3) and Proposition 2.2 remain valid for nonnegative slopes, so $q_{(\boldsymbol{\lambda},0,\ldots,0)}[\cdot] < 0$ still implies infinite index.

Proposition 7.6. *Suppose $\mathcal{H}_{\boldsymbol{\lambda}}$ is unstable. For every $m \geq 1$, the helicoid associated with $(\boldsymbol{\lambda},0,\ldots,0) \in \mathbb{R}^{k+m}$ is unstable. Moreover, there exists $\epsilon_0 > 0$ such that the helicoid associated with $(\boldsymbol{\lambda},\epsilon,\ldots,\epsilon) \in (0,\infty)^{k+m}$ is unstable for every $0 < \epsilon < \epsilon_0$.*

Proof. Choose $f \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ with $q_{\boldsymbol{\lambda}}[f] < 0$ and fix $0 \neq \chi \in C_c^\infty(\mathbb{R}^m)$. For $L > 0$ set

$$\chi_L(v) = \chi(v/L), \qquad \widehat{f}_L(u,v) = \big(\chi_L(v)f(u), 0\big)$$

for $(u,v) \in \mathbb{R}^k \times \mathbb{R}^m$. For $(\boldsymbol{\lambda},0,\ldots,0)$ the function h is independent of v, and direct substitution in (3) gives

$$q_{(\boldsymbol{\lambda},0,\ldots,0)}[\widehat{f}_L] = \|\chi_L\|_2^2 q_{\boldsymbol{\lambda}}[f] + \|\nabla\chi_L\|_2^2 \int_{\mathbb{R}^k} h\left[|f|^2 + \big((\Lambda u)\cdot f\big)^2\right] du. \tag{89}$$

Since

$$\|\chi_L\|_2^2 = L^m\|\chi\|_2^2, \qquad \|\nabla\chi_L\|_2^2 = L^{m-2}\|\nabla\chi\|_2^2,$$

the right-hand side of (89) is negative for all sufficiently large L. Fix such an L. For this fixed compactly supported field,

$$q_{(\boldsymbol{\lambda},\epsilon,\ldots,\epsilon)}[\widehat{f}_L] \longrightarrow q_{(\boldsymbol{\lambda},0,\ldots,0)}[\widehat{f}_L] \qquad \text{as } \epsilon \to 0,$$

and is therefore negative for all sufficiently small $\epsilon > 0$. $\square$

8. Entireness criterion for generalized helicoids

Let $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$ with $\lambda_i > 0$. On $\mathbb{R}^{2k+1} = \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}$ with coordinates (x, y, z), consider the generalized helicoid $\mathcal{H}_{\boldsymbol{\lambda}} \subset \mathbb{R}^{2k+1}$ parametrized by (2), or equivalently

$$F_{\boldsymbol{\lambda}}(u, s) = \big(\cos(\Lambda s)u,\ \sin(\Lambda s)u,\ s\big), \qquad (u, s) \in \mathbb{R}^k \times \mathbb{R}. \tag{90}$$

The sufficient condition obtained in [16, Theorem 1.8] is also necessary.

Theorem 8.1. *The helicoid $\mathcal{H}_{\boldsymbol{\lambda}}$ is an entire graph over a $(k+1)$-plane if and only if there exists $B \in \mathbb{R}^{k\times k}$ such that*

$$\det(\cos(\Lambda s) + B\sin(\Lambda s)) > 0 \qquad \forall s \in \mathbb{R}. \tag{91}$$

Proof. Let $e_z = (0, 0, 1)^T \in \mathbb{R}^{2k+1}$ and let $P_\Pi : \mathbb{R}^{2k+1} \to \Pi$ be the orthogonal projection onto a $(k+1)$-plane Π. The submanifold $\mathcal{H}_{\boldsymbol{\lambda}}$ is an entire graph over Π if and only if $\Phi_{\Pi,\boldsymbol{\lambda}} = P_\Pi \circ F_{\boldsymbol{\lambda}} : \mathbb{R}^k \times \mathbb{R} \to \Pi$ is a global diffeomorphism.

Setting $E(s) = (\cos(\Lambda s), \sin(\Lambda s), 0)^T \in \mathbb{R}^{(2k+1)\times k}$, the Jacobian matrix

$$D\Phi_{\Pi,\boldsymbol{\lambda}}(u, s) = \big[P_\Pi E(s),\ P_\Pi e_z + P_\Pi E'(s)u\big]$$

is affine in u. Since $\Phi_{\Pi,\boldsymbol{\lambda}}$ is a diffeomorphism, $D\Phi_{\Pi,\boldsymbol{\lambda}}(u, s)$ is non-singular everywhere, forcing

$$P_\Pi E'(s)u \in \mathrm{im}(P_\Pi E(s)), \qquad \forall u \in \mathbb{R}^k.$$

Since $\lambda_i > 0$, the columns of $E(s)$ and $E'(s)$ span $\{z = 0\}$; hence

$$\dim P_\Pi(\{z = 0\}) \le k.$$

If $e_z \notin \Pi$, then $\dim(\Pi^\perp \cap \{z = 0\}) = k-1$, yielding $\dim P_\Pi(\{z = 0\}) = 2k-(k-1) = k+1$, a contradiction. Therefore, $e_z \in \Pi$ and we can write $\Pi = W \times \mathbb{R}$ with $W = \Pi \cap \{z = 0\} \subset \mathbb{R}^{2k}$.

The map $\Phi_{\Pi,\boldsymbol{\lambda}}$ is a diffeomorphism if and only if $P_W \begin{pmatrix} \cos(\Lambda s) \\ \sin(\Lambda s) \end{pmatrix} : \mathbb{R}^k \to W$ is invertible for all $s \in \mathbb{R}$. Invertibility at $s = 0$ implies $W = \{(x, B^T x) \mid x \in \mathbb{R}^k\}$ for a unique $B \in \mathbb{R}^{k\times k}$, whose orthogonal complement in $\{z = 0\}$ is $W^\perp = \{(v_1, v_2) \in \mathbb{R}^{2k} \mid v_1 + Bv_2 = 0\}$.

Thus, $P_W \begin{pmatrix} \cos(\Lambda s) \\ \sin(\Lambda s) \end{pmatrix}$ has a non-trivial kernel if and only if $\begin{pmatrix} \cos(\Lambda s) \\ \sin(\Lambda s) \end{pmatrix} u \in W^\perp$ for some $u \neq 0$, which is equivalent to $(\cos(\Lambda s) + B\sin(\Lambda s))u = 0$. Hence, invertibility holds for all $s \in \mathbb{R}$ if and only if $\det(\cos(\Lambda s)+B\sin(\Lambda s)) \neq 0$. Because $\det I_k = 1 > 0$, continuity implies that non-vanishing is equivalent to positivity for all $s \in \mathbb{R}$. □

Corollary 8.2. *The helicoid H_k is an entire graph over a $(k+1)$-plane if and only if k is even.*

Proof. For H_k, one has $\Lambda = I_k$. If k is even, entireness follows from the construction in [16, Theorem 1.8]. If k is odd, the determinant condition (91) in the preceding theorem fails at $s = \pi$, since $\det(\cos(\pi I_k) + B\sin(\pi I_k)) = \det(-I_k) = (-1)^k = -1$. □

References

[1] J. L. M. Barbosa, M. Dajczer, and L. P. M. Jorge, Minimal ruled submanifolds in spaces of constant curvature, *Indiana Univ. Math. J.* **33** (1984), no. 4, 531–547, doi:10.1512/iumj.1984.33.33028.

[2] J. Bernstein and C. Breiner, Conformal structure of minimal surfaces with finite topology, *Comment. Math. Helv.* **86** (2011), no. 2, 353–381, doi:10.4171/CMH/226.

[3] R. L. Bryant, Some remarks on the geometry of austere manifolds, *Bol. Soc. Brasil. Mat. (N.S.)* **21** (1991), no. 2, 133–157, doi:10.1007/BF01237361.

[4] T. H. Colding and W. P. Minicozzi II, The space of embedded minimal surfaces of fixed genus in a 3-manifold, *Ann. of Math. (2)* **160** (2004): I. Estimates off the axis for disks, no. 1, 27–68, doi:10.4007/annals.2004.160.27; II. Multi-valued graphs in disks, no. 1, 69–92, doi:10.4007/annals.2004.160.69; III. Planar domains, no. 2, 523–572, doi:10.4007/annals.2004.160.523; IV. Locally simply connected, no. 2, 573–615, doi:10.4007/annals.2004.160.573.

[5] M. P. do Carmo and C. K. Peng, Stable complete minimal surfaces in $\mathbb{R}^3$ are planes, *Bull. Amer. Math. Soc. (N.S.)* **1** (1979), no. 6, 903–906, doi:10.1090/S0273-0979-1979-14689-5.

[6] D. Fischer-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, *Comm. Pure Appl. Math.* **33** (1980), no. 2, 199–211, doi:10.1002/cpa.3160330206.

[7] R. Harvey and H. B. Lawson, Jr., Calibrated geometries, *Acta Math.* **148** (1982), 47–157, doi:10.1007/BF02392726.

[8] M. Kerckhove and G. R. Lawlor, A family of stratified area-minimizing cones, *Duke Math. J.* **96** (1999), no. 2, 401–424, doi:10.1215/S0012-7094-99-09612-6.

[9] V. Y. Kraines, Topology of quaternionic manifolds, *Trans. Amer. Math. Soc.* **122** (1966), no. 2, 357–367, doi:10.1090/S0002-9947-1966-0192513-X.

[10] G. R. Lawlor, A sufficient criterion for a cone to be area-minimizing, *Mem. Amer. Math. Soc.* **91** (1991), no. 446, vi+111 pp., doi:10.1090/memo/0446.

[11] G. R. Lawlor, Proving area minimization by directed slicing, *Indiana Univ. Math. J.* **47** (1998), no. 4, 1547–1592, doi:10.1512/iumj.1998.47.1341.

[12] P. Li, *Geometric Analysis*, Cambridge University Press, Cambridge, 2012.

[13] W. H. Meeks III and H. Rosenberg, The uniqueness of the helicoid, *Ann. of Math. (2)* **161** (2005), no. 2, 727–758, doi:10.4007/annals.2005.161.727.

[14] F. Morgan, Calibrations and new singularities in area-minimizing surfaces: A survey, in *Variational Methods*, Progr. Nonlinear Differential Equations Appl., vol. 4, Birkhäuser, Boston, 1990, pp. 329–342, doi:10.1007/978-1-4757-1080-9_23.

[15] L. Simon, *Lectures on Geometric Measure Theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[16] C.-J. Tsai, M.-P. Tsui, J. Wan, and M.-T. Wang, Constructing entire minimal graphs by evolving planes, *Calc. Var. Partial Differential Equations* **65** (2026), no. 6, article no. 196, doi:10.1007/s00526-026-03372-8.

Department of Mathematics, National Taiwan University, and National Center for Theoretical Sciences, Math Division, Taipei 10617, Taiwan
Email address: cjtsai@ntu.edu.tw

Department of Mathematics, National Taiwan University, and National Center for Theoretical Sciences, Math Division, Taipei 10617, Taiwan
Email address: maopei@math.ntu.edu.tw

Laboratoire Jacques-Louis Lions de Sorbonne Université, 4 place Jussieu, Paris 75005, France
Email address: jingbo.wan@sorbonne-universite.fr

Department of Mathematics, Columbia University, New York, NY 10027, USA
Email address: mtwang@math.columbia.edu