ENTIRE SOLUTIONS OF TWO-CONVEX LAGRANGIAN MEAN CURVATURE FLOWS

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ABSTRACT. Given an entire C^2 function u on $\mathbb{R}^n$, we consider the graph of Du as a Lagrangian submanifold of $\mathbb{R}^{2n}$, and deform it by the mean curvature flow in $\mathbb{R}^{2n}$. This leads to the special Lagrangian evolution equation, a fully nonlinear Hessian type PDE. We prove long-time existence and convergence results under a 2-positivity assumption of $(I + (D^2u)^2)^{-1}D^2u$. Such results were previously known only under the stronger assumption of positivity of D^2u.

1. INTRODUCTION

Fundamental to a fully nonlinear Hessian type partial differential equation is the convexity assumption, which guarantees ellipticity or parabolicity and facilitates essential regularity estimates. Among these equations, the special Lagrangian equation is rather unique in that the variational structure always implies ellipticity or parabolicity without the need of any convexity assumption. Still most previous results were obtained under the convexity condition or its equivalence. In [16], a substantial improvement was achieved by removing the convexity assumption and replacing it by the two-convexity assumption. The main purpose of this paper is to extend results in [16] from the compact setting to the non-compact global setting. Essential difficulties of this extension were dealt with by several new ingredients in this article. These include two new evolution equations (Proposition 4.1 and Proposition 4.2) which allow us to localize estimates in [16] in the global setting. There is also a new regularization procedure that works without the convexity assumption. In particular, new non-convex self-expanders of Lagrangian mean curvature flows were discovered.

Given an entire C^2 function u_0 on $\mathbb{R}^n$, we consider the graph of Du_0 as a Lagrangian submanifold of $\mathbb{R}^{2n}$ and deform it by the mean curvature flow, which corresponds to the following

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initial value problem for the potential function u:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\sqrt{-1}} \log \frac{\det(\mathbf{I} + \sqrt{-1} D^2 u)}{\sqrt{\det(\mathbf{I} + (D^2 u)^2)}} \\ \text{with } u(x,0) &= u_0(x) \ . \end{aligned} \tag{1.1}$$

As remarked above, this is a fully nonlinear Hessian type parabolic equation. We assume that u_0 is *2-convex*, which is a 2-positivity assumption in terms of $(I + (D^2 u_0)^2)^{-1} D^2 u_0$(see Definition 2.1 for the definition in terms of the eigenvalues of $D^2 u_0$). This is a natural quantity as it corresponds to the Hessian of u as measured with respect to the induced metric on the corresponds Lagrangian submanifold.

The main result of this paper is the following long-time existence result.

Theorem 1.1 (Theorem 5.6)**.** *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x \in \mathbb{R}^n} |D^2 u_0|^2 \leq c$ for some $c > 0$. Then, (1.1) admits a unique solution $u(x,t)$ in the space $C^0(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \times (0,\infty))$ such that*

- *for any $t > 0$, u is 2-convex;*
- *there exists $c_\ell = c_\ell(c) > 0$ for any $\ell \geq 2$ such that $\sup_{x \in \mathbb{R}^n} |D^\ell u|^2 \leq c_\ell t^{2-\ell}$ for any $t > 0$.*

Depending on the asymptotic behavior of the initial data, we prove the following convergence theorems.

Theorem 1.2 (Theorem 6.1)**.** *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x \in \mathbb{R}^n} |D^2 u_0|^2 \leq c$ for some $c > 0$. Denote by $u(x,t)$ the solution to (1.1) with $u(x,0) = u_0(x)$ given by Theorem 1.1. Suppose that there exists a constant $c_1 > 0$ so that $|Du_0|^2 \leq c_1$ on $\mathbb{R}^n$. Then, there exists an $\vec{a} \in \mathbb{R}^n$ such that $Du(x,t)$ converges to the constant map from $\mathbb{R}^n$ to $\vec{a}$ in $C^\infty_{loc}(\mathbb{R}^n, \mathbb{R}^n)$.*

In other words, $L_u = \{(x, Du(x,t)) : x \in \mathbb{R}^n\}$ converges locally smoothly to $\mathbb{R}^n \times \{\vec{a}\}$ as $t \to \infty$.

Theorem 1.3 (Theorem 6.2)**.** *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x \in \mathbb{R}^n} |D^2 u_0|^2 \leq c$ for some $c > 0$, and*

$$\lim_{\mu \to \infty} \frac{u_0(\mu x)}{\mu^2} = U_0(x) \ ,$$

for some $U_0(x)$. Let $u(x,t)$ be the solution to (1.1) given by Theorem 1.1. Then, $u(\mu x, \mu^2 t)/\mu^2$ converges to a smooth self-expanding solution $U(x,t)$ to (1.1) in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$ as $\mu \to \infty$. As $t \to 0$, $U(x,t)$ converges to $U_0(x)$ in $C^0_{loc}(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we consider the geometry of a Lagrangian submanifold in terms of its potential function. In Section 3, we review some known results about

Lagrangian mean curvature flows that are needed in the article. In Section 4, we derive two important evolution equations that play critical roles in the proof of long-time existence. The long-time existence results are established in Section 5 and the convergence results in Section 6.

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2. The Lagrangian Geometry in Potential

Endow $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ with the standard metric, symplectic form, and complex structure. Given a function $u : \mathbb{R}^n \to \mathbb{R}$, the graph of its gradient is a Lagrangian submanifold in $\mathbb{R}^{2n}$. Denote it by $L_u = \{(x, Du(x)) : x \in \mathbb{R}^n\}$.

At any $x \in \mathbb{R}^n$, one may find a orthonormal basis to diagonalize the Hessian of u, D^2u. Specifically, $D^2u = \lambda_i \delta_{ij}$ with respect to an orthonormal basis $\{a_i\}_{i=1,\dots,n}$ for $\mathbb{R}^n$. It follows that the tangent space of $\Gamma(Du)$ has the orthonormal basis

$$e_i = \frac{1}{\sqrt{1+\lambda_i^2}} \left(a_i + \lambda_i J(a_i)\right) \tag{2.1}$$

for $i = 1, \dots, n$. In terms of the parametrization $x \in \mathbb{R}^n \to (x, Du(x)) \in \mathbb{R}^n \oplus \mathbb{R}^n$ (the so-called non-parametric form in the minimal graph theory), the induced metric has metric coefficients

$$g_{ij} = (1+\lambda_i^2)\delta_{ij} \ . \tag{2.2}$$

We shall study two parallel tensors on $\mathbb{R}^{2n}$. The first tensor is the volume form of $\mathbb{R}^n \oplus \{0\} \subset \mathbb{R}^{2n}$. It is an n-form on $\mathbb{R}^{2n}$, and denote it by Ω. The restriction of Ω on L_u is equivalent to the scalar-valued function $*\Omega$, where $*$ is the Hodge star of the induced metric on L_u. By using the frame (2.1),

$$*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}} \ . \tag{2.3}$$

It is clear that $*\Omega$ takes value in $(0, 1]$.

The second one is a $(0, 2)$-tensor:

$$S(X, Y) = \langle J\pi_1(X), \pi_2(Y)\rangle \ . \tag{2.4}$$

With respect to the frame (2.1), the restriction of S on L_u is

$$S_{ij} = \frac{\lambda_i}{1+\lambda_i^2}\delta_{ij} \ .$$

In particular, it is positive definite if and only if u is convex. The main interest of this paper is the 2-positivity case. Namely,

$$S_{ii} + S_{jj} = \frac{(\lambda_i + \lambda_j)(1 + \lambda_i\lambda_j)}{(1 + \lambda_i^2)(1 + \lambda_j^2)} > 0 \ . \tag{2.5}$$

for any $i \neq j$. Note that $(\lambda_i + \lambda_j)(1 + \lambda_i\lambda_j) > 0$ does not correspond to a connected region in the $\lambda_i\lambda_j$-plane.

Definition 2.1. A C^2-function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be 2-*convex* if the eigenvalues of its Hessian satisfy everywhere

$$\lambda_i + \lambda_j \geq 0 \quad \text{and} \quad 1 + \lambda_i\lambda_j \geq 0$$

for any $i \neq j$. It is said to be *strictly* 2-*convex* if both inequalities are strict.

As in [1], we introduce a symmetric endomorphism on $\Lambda^2 TL_u$ to study the 2-convexity of u. It is denoted by $S^{[2]}$, and is given by

$$S^{[2]}_{(ij)(k\ell)} = S_{ik}\delta_{j\ell} + S_{j\ell}\delta_{ik} - S_{i\ell}\delta_{jk} - S_{jk}\delta_{i\ell}$$

with respect to an orthonormal frame of L_u. The 2-convexity (2.5) condition corresponds to the positivity of $S^{[2]}$, and thus it is useful to study the scalar valued function $\det S^{[2]}$. In terms of (2.1),

$$\det S^{[2]} = \prod_{i<j}(S_{ii} + S_{jj}) = \prod_{i<j} \frac{(\lambda_i + \lambda_j)(1 + \lambda_i\lambda_j)}{(1 + \lambda_i^2)(1 + \lambda_j^2)} \ . \tag{2.6}$$

It is not hard to see that if u is strictly 2-convex, $\det S^{[2]}$ takes value within $(0, 1]$.

When $\varepsilon_1 \leq *\Omega \leq 1$ and $\varepsilon_2 \leq \det S^{[2]} \leq 1$, one can deduce that

$$\sum_i \lambda_i^2 \leq \varepsilon_1^{-2} - 1 \ , \tag{2.7}$$

$$1 + \lambda_i\lambda_j \geq \frac{\varepsilon_2}{\sqrt{2(\varepsilon_1^{-2} - 1)}} \ , \tag{2.8}$$

$$\lambda_i + \lambda_j \geq \frac{2\varepsilon_2}{\varepsilon_1^{-2} + 1} \tag{2.9}$$

for any $i \neq j$ (under the strict 2-convexity assumption); see [16, section 2.3].

3. The Lagrangian Mean Curvature Flow

Given any $u_0 : \mathbb{R}^n \to \mathbb{R}$, it is known that the Lagrangian mean curvature flow (up to a tangential diffeomorphism) equation on L_{u_0} reduces to (1.1) for the potential function u.

The uniqueness of the solution to (1.1) was established by Chen and Pang in [4].

Remark 3.1. Consider the function

$$f(B) = \frac{1}{\sqrt{-1}} \log \frac{\det(\mathbf{I} + \sqrt{-1}B)}{\sqrt{\det(\mathbf{I} + B^2)}}$$

on the space of symmetric matrices. A direct computation shows that its derivative is

$$\mathrm{d}f(B) = \mathrm{tr}((g_B)^{-1}\mathrm{d}B)$$

where $g_B = \mathbf{I} + B^2$.

We recall some preliminary results about short-time existence and finite time singularity established in [2, 3].

Proposition 3.2 ([3, Proposition 2.1])**.** *Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $\sup |D^\ell u_0| < \infty$ for any $\ell \geq 2$. Then, (1.1) admits a smooth solution $u(x,t) : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ for some $T > 0$. Moreover, $\sup\{|D^\ell u(x,t)| : x \in \mathbb{R}^n\} < \infty$ for any $\ell \geq 2$ and $t \in [0,T)$.*

Lemma 3.3 ([2, Lemma 4.2])**.** *Let u be a smooth solution to (1.1) on $\mathbb{R}^n \times [0,T)$ for some $T > 0$. Suppose that*

- *$\sup\{|D^\ell u(x,t)| : x \in \mathbb{R}^n\} < \infty$ for any $\ell \geq 2$ and $t \in [0,T)$;*
- *$D^2 u$ and $D^3 u$ are uniformly bounded on $\mathbb{R}^n \times [0,T)$. Namely, there exist $c_2, c_3 > 0$ such that $|D^2 u(x,t)| \leq c_2$ and $|D^3 u(x,t)| \leq c_3$ on $\mathbb{R}^n \times [0,T)$.*

Then, there exists $c_\ell = c_\ell(c_2, c_3) > 0$ for any $\ell \geq 4$ such that $|D^\ell u(x,t)| \leq c_\ell$ on $\mathbb{R}^n \times [0,T)$.

Corollary 3.4. *Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $\sup |D^\ell u_0| < \infty$ for any $\ell \geq 2$. Let $u(x,t) : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be the solution to (1.1) given by Proposition 3.2, where T is the maximal existence time. If $T < \infty$ and $|D^2 u(x,t)| \leq c_2$ for some $c_2 > 0$ on $\mathbb{R}^n \times [0,T)$, then*

$$\lim_{t \to T} \sup\{|D^3 u(x,t')| : x \in \mathbb{R}^n,\ t' \leq t\} = \infty \ .$$

Proof. Suppose that the above limit is finite. Due to Lemma 3.3, all the higher order ($\ell \geq 2$) derivatives of u are uniformly bounded on $\mathbb{R}^n \times [0,T)$. In particular, Proposition 3.2 applies to the family of initial data $\tilde{u}_k(x) = u(x, (1-2^{-k})T)$, and there is a $T' > 0$ such that the solution starting from $\tilde{u}_k$ exists on the time interval $[0,T')$. It follows that the solution from u_0 can be extended over time T, which contradicts to $T < \infty$. □

We will also need the Liouville theorem and a priori estimate established by Nguyen and Yuan in [12]. Denote $\mathbb{R}^n \times (-\infty, 0]$ by Q_∞. For any $r > 0$, denote $B_r(0) \times [-r^2, 0] \subset Q_\infty$ by Q_r.

Proposition 3.5 ([12, Proposition[1] 2.1])**.** *Let u be a smooth solution to* (1.1) *in Q_∞. Suppose that D^2u is uniformly bounded over Q_∞. Then, u is stationary. In other words, the right hand side of* (1.1) *vanishes, and L_u is a static special Lagrangian submanifold.*

Theorem 3.6 ([12, Theorem 1.1])**.** *There exists a constant $c > 0$ with the following significance. Let u be a smooth solution to* (1.1) *in Q_1. Suppose that $1 + \lambda_i\lambda_j \geq 0$ for any $i \neq j$ everywhere in Q_1. Then,*

$$[\partial_t u]_{1,\frac{1}{2};Q_{\frac{1}{2}}} + [D^2u]_{1,\frac{1}{2};Q_{\frac{1}{2}}} \leq c\,||D^2u||_{L^\infty(Q_1)}\ . \tag{3.1}$$

The notation on the left hand side is the semi-norm:

$$[f]_{1,\frac{1}{2};Q_r} = \sup\left\{\frac{f(x',t') - f(x,t)}{\max\{|x'-x|, |t'-t|^{\frac{1}{2}}\}} : (x',t'),(x,t) \in Q_r \text{ and } (x',t') \neq (x,t)\right\}\ .$$

In particular, Theorem 3.6 implies that $||D^3u||_{L^\infty(Q_{\frac{1}{2}})} \leq c\,||D^2u||_{L^\infty(Q_1)}$.

4. Two Evolution Equations

We derive the evolution equations for $\log(*\Omega)$ and $\log\det S^{[2]}$, defined in (2.3) and (2.6), respectively. They will be considered in the parametric form: the parametrization given by $\frac{\partial}{\partial t}F = H$. That is to say, suppose that $u : \mathbb{R}^n \times [0,T)$ is a solution to (1.1), then F is $(x, Du(x,t))$ composing with a time-dependent diffeomorphism.

Proposition 4.1. *Let $u : \mathbb{R}^n \times [0,T)$ be a solution to* (1.1)*. Suppose that u is 2-convex for all $t \in [0,T)$. Then,* $\log(*\Omega)$ *satisfies*

$$(\frac{\partial}{\partial t} - \Delta)\log(*\Omega) \geq \frac{1}{n}\,|\nabla \log(*\Omega)|^2 \tag{4.1}$$

in the parametric form of the mean curvature flow.

Proof. By [17, 2nd equation on P.532],

$$\nabla_k(*\Omega) = -(*\Omega)(\sum_i \lambda_i h_{iik})$$

with respect to the orthonormal frame (2.1). By the Cauchy–Schwarz inequality,

$$\begin{aligned}|\nabla \log(*\Omega)|^2 &= \sum_k \left|\sum_i \lambda_i h_{kii}\right|^2 \\ &\leq n\sum_{i,k}\lambda_i^2 h_{kii}^2 = n\left[\sum_i \lambda_i^2 h_{iii}^2 + \sum_{i\neq j}\lambda_i^2 h_{iij}^2\right]\ .\end{aligned}$$

[1]Specifically, this is the first step in that proposition, which uses on the Krylov–Safonov Hölder inequality.

According to [16, Proposition 2.2],

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\log(*\Omega)\\
&\geq \sum_i (1+\lambda_i^2)h_{iii}^2+\sum_{i\neq j}(3+\lambda_i^2+2\lambda_i\lambda_j)h_{iij}^2+\sum_{i<j<k}(6+2\lambda_i\lambda_j+2\lambda_j\lambda_k+2\lambda_k\lambda_i)h_{ijk}^2\\
&\geq \sum_i h_{iii}^2+\sum_{i\neq j}(3+2\lambda_i\lambda_j)h_{iij}^2+\sum_{i<j<k}(6+2\lambda_i\lambda_j+2\lambda_j\lambda_k+2\lambda_k\lambda_i)h_{ijk}^2+\frac{1}{n}\left|\nabla\log(*\Omega)\right|^2 \ .
\end{aligned}
$$

Since $1+\lambda_i\lambda_j\geq 0$, it finishes the proof of this proposition. □

Proposition 4.2. *Let $u:\mathbb{R}^n\times[0,T)$ be a solution to (1.1). Suppose that u is strictly 2-convex for all $t\in[0,T)$. Then. $\log\det S^{[2]}$ satisfies*

$$
(\frac{\partial}{\partial t}-\Delta)\log\det S^{[2]}\geq 2|A|^2+\frac{1}{n(n-1)}\left|\nabla\log\det S^{[2]}\right|^2 \tag{4.2}
$$

in the parametric form of the mean curvature flow.

Proof. According to [16, Proposition 2.1],

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\log\det S^{[2]}\\
&\geq \sum_k\sum_{i<j}\left[4h_{kij}^2+\frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2+\frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2\right] \ .
\end{aligned}
$$

Since

$$
\begin{aligned}
(1+\lambda_i^2)(1+\lambda_j^2)&=(\lambda_i+\lambda_j)^2+(1-\lambda_i\lambda_j)^2\\
&=(1+\lambda_i\lambda_j)^2+(\lambda_i-\lambda_j)^2 \ ,
\end{aligned}
$$

one finds that

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\log\det S^{[2]}\\
&\geq \sum_k\sum_{i<j}\big[4h_{kij}^2+(h_{kii}+h_{kjj})^2+(h_{kii}-h_{kjj})^2\\
&\qquad\qquad +\frac{(1-\lambda_i\lambda_j)^2}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2+\frac{(\lambda_i-\lambda_j)^2}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2\Big]\\
&\geq 2|A|^2+\sum_k\sum_{i<j}\left[\frac{(1-\lambda_i\lambda_j)^2}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2+\frac{(\lambda_i-\lambda_j)^2}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2\right] \ .
\end{aligned} \tag{4.3}
$$

With respect to the orthonormal frame (2.1),

$$
\nabla_k(S_{ii}+S_{jj})=\left(\frac{1-\lambda_i^2}{1+\lambda_i^2}h_{kii}+\frac{1-\lambda_j^2}{1+\lambda_j^2}h_{kjj}\right) \ .
$$

By the Cauchy–Schwarz inequality,

$$|\nabla \log\det S^{[2]}|^2 = \sum_k \left| \sum_{i<j} (S_{ii}+S_{jj})^{-1} \nabla_k (S_{ii}+S_{jj}) \right|^2$$
$$\leq \frac{n(n-1)}{2} \sum_k \sum_{i<j} (S_{ii}+S_{jj})^{-2} |\nabla_k (S_{ii}+S_{jj})|^2 \ . \tag{4.4}$$

Rewrite $\nabla_k(S_{ii}+S_{jj})$ as follows:

$$\begin{aligned}
&\frac{1-\lambda_i^2}{1+\lambda_i^2} h_{kii} + \frac{1-\lambda_j^2}{1+\lambda_j^2} h_{kjj} \\
&= \frac{1}{2}\left(\frac{1-\lambda_i^2}{1+\lambda_i^2} + \frac{1-\lambda_j^2}{1+\lambda_j^2} \right)(h_{kii}+h_{kjj}) + \frac{1}{2}\left(\frac{1-\lambda_i^2}{1+\lambda_i^2} - \frac{1-\lambda_j^2}{1+\lambda_j^2} \right)(h_{kii}-h_{kjj}) \\
&= \frac{1-\lambda_i^2\lambda_j^2}{(1+\lambda_i^2)(1+\lambda_j^2)}(h_{kii}+h_{kjj}) - \frac{\lambda_i^2-\lambda_j^2}{(1+\lambda_i^2)(1+\lambda_j^2)}(h_{kii}-h_{kjj}) \ .
\end{aligned}$$

Together with (2.5), it leads to

$$\begin{aligned}
&(S_{ii}+S_{jj})^{-2} |\nabla_k (S_{ii}+S_{jj})|^2 \\
&= \left[\frac{1-\lambda_i\lambda_j}{\lambda_i+\lambda_j}(h_{kii}+h_{kjj}) - \frac{\lambda_i-\lambda_j}{1+\lambda_i\lambda_j}(h_{kii}-h_{kjj}) \right]^2 \\
&\leq 2\frac{(1-\lambda_i\lambda_j)^2}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2 + 2\frac{(\lambda_i-\lambda_j)^2}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2 \ .
\end{aligned} \tag{4.5}$$

Putting (4.4), (4.5) and (4.3) together finishes the proof of this proposition. □

5. Long-Time Existence

5.1. **A Maximum Principle.** We first establish a maximum principle, whose proof is based on the argument in [6, Theorem 2.1].

Lemma 5.1. *Let $F : L \times [0,T) \to \mathbb{R}^N$ be a solution to the mean curvature flow[2]. Suppose that v is a positive smooth function satisfying*

$$(\frac{\partial}{\partial t} - \Delta) \log v \leq -q |\nabla \log v|^2 \tag{5.1}$$

for some constant $q > 0$, and $\sup_{L\times\{t\}} v < \infty$ for every $t \in [0,T)$. Then,

$$\sup_{L\times\{t\}} v \leq \sup_{L\times\{0\}} v \tag{5.2}$$

for every $t \in (0,T)$.

[2] in the parametric form, $\frac{\partial}{\partial t} F = H$

Proof. Let $\dim L = n$. It follows from $\frac{\partial}{\partial t} F = H$ that

$$(\frac{\partial}{\partial t} - \Delta)|F|^2 = -2n \ . \tag{5.3}$$

Fix $R > 0$, and consider the function $\varphi = R^2 - |F|^2 - 2nt$. It follows from (5.3) that

$$\nabla \varphi^2 = 2\varphi \, \nabla \varphi \ , \tag{5.4}$$

$$(\frac{\partial}{\partial t} - \Delta)\varphi^2 = -2 \, |\nabla \varphi|^2 \tag{5.5}$$

By (5.1), the functions $w = v^q = \exp(q \log v) > 0$ obeys

$$(\frac{\partial}{\partial t} - \Delta) w = qw \left[(\frac{\partial}{\partial t} - \Delta) \log v \right] - w^{-1} |\nabla w|^2 \leq -2w^{-1} |\nabla w|^2 \ .$$

Together with (5.4) and (5.5),

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)(\varphi^2 w^2) \leq & -6|\nabla w| \varphi^2 - 2 \, |\nabla \varphi|^2 \, w^2 - 2 \langle \nabla \varphi^2, \nabla w^2 \rangle \\ & + c \left(\varphi^{-1} \langle \nabla \varphi, \nabla (\varphi^2 w^2) \rangle - 2w^2 |\nabla \varphi|^2 - \varphi \langle \nabla \varphi, \nabla w^2 \rangle \right) \\ = & \, c \, \varphi^{-1} \langle \nabla \varphi, \nabla (\varphi^2 w^2) \rangle \\ & - 6|\nabla w| \varphi^2 - 2(1+c) \, |\nabla \varphi|^2 \, w^2 - (8+2c) \langle \varphi \nabla w, w \nabla \varphi \rangle \end{aligned} \tag{5.6}$$

for any $c \in \mathbb{R}$.

By taking $c = 2$, (5.6) becomes $-6 \, |\varphi \nabla w + w \nabla \varphi|^2$, and hence

$$(\frac{\partial}{\partial t} - \Delta)(\varphi^2 w^2) \leq 2 \varphi^{-1} \langle \nabla \varphi, \nabla (\varphi^2 w^2) \rangle \ . \tag{5.7}$$

By replacing φ with $\varphi_+ = \max\{\varphi, 0\}$, the computation remains valid. Due to the maximum principle,

$$\sup_{L \times \{t\}} (\varphi_+ \cdot w) \leq \sup_{L \times \{0\}} (\varphi_+ \cdot w) \ . \tag{5.8}$$

By letting $R \to \infty$, it implies that $\sup_{L \times \{t\}} w \leq \sup_{L \times \{0\}} w$. □

5.2. **Smooth Initial Condition.**

Proposition 5.2. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\sup |D^\ell u_0| < \infty$ for any $\ell \geq 2$. Suppose that*

$$u_0 \text{ is strictly 2-convex}, \quad * \Omega \geq \varepsilon_1 \quad \text{and} \quad \det S^{[2]} \geq \varepsilon_2 \tag{5.9}$$

for some $\varepsilon_1, \varepsilon_2 \in (0,1)$. Then, (1.1) admits a unique smooth solution $u(x,t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ which has the following properties.

(i) *The solution $u(x,t)$ is strictly 2-convex for all t. Specifically, (5.9) is preserved along the flow.*

(ii) *For any $\ell > 2$, there exists a constant $c_\ell = c_\ell(\varepsilon_1) > 0$ such that $\sup_{x\in\mathbb{R}^n} |D^\ell u|^2 \le c_\ell t^{2-\ell}$ for any $t > 0$.*

Proof. Step 1. Preservation of (5.9). Denote by T the maximal existence time. Since

$$\mathcal{P} = \{t \in [0, T) : (5.9) \text{ holds true everywhere on } L_{u(\cdot,t)}\}$$

is a closed subset of $[0, T)$, it remains to show that if $t_0 \in \mathcal{P}$, $[t_0, t_0 + \delta) \subset \mathcal{P}$ for some $\delta = \delta(t_0) > 0$.

It suffices to do it for $t_0 = 0$. Note that (2.7), (2.8) and (2.9) hold true everywhere at $t = 0$. According to Proposition 3.2, $|D^\ell u(x, t)|$ is bounded over $\mathbb{R}^n \times [0, T/2]$ for any $\ell \ge 2$. By (1.1) and Remark 3.1, $D^2u(x, t)$ is a small[3] perturbation of $D^2u(x, 0)$ for $t << 1$. Recall that the spectrum of a symmetric matrix is stable with respect to small perturbations; see for instance [15, section 1.3] and the references therein. Together with (2.7), (2.8) and (2.9) at $t = 0$, one infers that there exists a $\delta \in (0, T/2]$ such that u remains 2-convex for $t \in [0, \delta)$.

According to Proposition 4.2,

$$(\frac{\partial}{\partial t} - \Delta) \log(\det S^{[2]})^{-1} \le -\frac{1}{n(n-1)} \left|\nabla \log(\det S^{[2]})^{-1}\right|^2$$

for $t \in [0, \delta)$. By applying Lemma 5.1 for $v = (\det S^{[2]})^{-1}$, we find that $\det S^{[2]} \ge \varepsilon_2$ for $t \in [0, \delta)$.

Similarly, Proposition 4.1 says that

$$(\frac{\partial}{\partial t} - \Delta) \log(*\Omega)^{-1} \le \frac{1}{n} \left|\nabla \log(*\Omega)^{-1}\right|^2$$

for $t \in [0, \delta)$. It follows from Lemma 5.1 for $v = (*\Omega)^{-1}$ that $(*\Omega) \ge \varepsilon_1$ for $t \in [0, \delta)$. Hence, $\mathcal{P} = [0, T)$.

Step 2. Long Time Existence. The next step is to show the maximal existence time T is infinity. Suppose not. By Corollary 3.4,

$$A(t) = \sup\{|D^3u(x, t')| : x \in \mathbb{R}^n, t' \le t\}$$

is unbounded as $t \to T$. It follows that there exists a sequence $(x_k, t_k) \in \mathbb{R}^n \times [0, T)$ such that

- $t_k \to T$ as $k \to \infty$;
- $A(t_k) \to \infty$ as $k \to \infty$;
- $|D^3u(x_k, t_k)| \ge A(t_k)/2$ for all k.

Denote $A(t_k)$ by ρ_k. Let

$$\tilde{u}_k(y, s) = \rho_k^2 \left[u(\frac{y}{\rho_k}, t_k + \frac{s}{\rho_k^2}) - u(0, t_k) - Du(0, t_k) \cdot \frac{y}{\rho_k}\right]$$

[3] uniformly in space

for $y \in \mathbb{R}^n$ and $s \in [-\rho_k^2 t_k, 0]$. Then one has $\tilde{u}_k(0,0) = 0 = D\tilde{u}_k(0,0)$, $|D^3\tilde{u}(0,0)| \geq 1/2$, and $|D^3\tilde{u}(y,t)| \leq 1$ on $\mathbb{R}^n \times [-\rho_k^2 t_k, 0]$. It is straightforward to see that $\tilde{u}_k$ solves (1.1) on $\mathbb{R}^n \times [-\rho_k^2 t_k, 0]$. Note that the eigenvalues of $D_y^2 \tilde{u_k}$ is the same as those of $D_x^2 u$.

By using Lemma 3.3 and the Arzelà–Ascoli theorem, $\tilde{u}_k$ admits a subsequence which converges to $\tilde{u} : \mathbb{R}^n \times (-\infty, 0]$ in C^∞_{loc}. Hence, $\tilde{u}$ is an ancient solution to (1.1) with $|D^3\tilde{u}(0,0)| \geq 1/2$, and the eigenvalues of $D^2\tilde{u}$ satisfies (2.7), (2.8) and (2.9). Due to Proposition 3.5, $\tilde{u}$ is stationary. Since $1 + \lambda_i\lambda_j \geq 0$ and λ_i's are bounded, [17, Theorem A] asserts that $L_{\tilde{u}}$ is an affine n-plane. This contradicts to $|D^3\tilde{u}(0,0)| \geq 1/2$. Thus, the maximal existence time T cannot be finite.

Step 3. Estimates. The argument for assertion (ii) follows from the a priori estimate [12, Theorem 1.1] and the scaling argument; see [3, p.173]. The $\ell = 3$ case is included here for completeness. Fix $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Let

$$\hat{u}(y,s) = \frac{1}{t} u(x + t^{\frac{1}{2}} y, t(1+s)) \ .$$

Then, $\hat{u}$ is a solution to (1.1) on $Q_1 = B_1(0) \times [-1, 0]$. Since the second order derivative remains unchanged in the rescaling, it follows from Theorem 3.6 that $||D^3\hat{u}||_{L^\infty}(Q_{\frac{1}{2}}) \leq c||D^2\hat{u}||_{L^\infty}(Q_1)$. Due to (2.7), $|D^2\hat{u}|$ is uniformly bounded. Therefore,

$$|D^3 u(x',t')| \leq c(\varepsilon_1)\, t^{-\frac{1}{2}}$$

for any (x',t') with $|x' - x| \leq t^{\frac{1}{2}}/2$ and $3t/4 \leq t' \leq t$. The estimate holds true for any (x,t), and it finishes the proof for $\ell = 3$. The estimate for $\ell > 3$ follows a scaling argument; see the last part in the proof of [2, Lemma 5.2]. □

5.3. C^2 **Initial Condition.** With Proposition 5.2, we can now prove the main theorem of this paper. In [2, section 5 and 6] and [3, section 3], the long-time existence was proved by constructing smooth approximations to the initial condition. Unlike their situation, the convolution between the standard mollifier and a 2-convex function needs not to be 2-convex. It requires some extra work to handle this issue.

The following lemma says that for a *strict* 2-convex region, one can find a positive cone such that the corresponding translation leaves the region invariant. In fact, the slope of the cone is determined by the corner points of the boundary of the region. See Figure 1 for the 2-convex region, and Figure 2 for a strictly 2-convex region.

Lemma 5.3. *For any $\delta_1 \in (0,1)$ and $\delta_2 > 0$, let $Q_{\delta_1,\delta_2} = \{(x,y) \in \mathbb{R}^2 : 1 + xy \geq \delta_1,\ x + y \geq \delta_2\}$. There exists a $\tau > 1$ depending on δ_1, δ_2 such that*

$$Q_{\delta_1,\delta_2} + C_\tau \subset Q_{\delta_1,\delta_2}$$

where $C_\tau = \{(x,y) : x \geq 0,\ x/\tau \leq y \leq \tau x\}$. The above sum of two sets is the Minkowski sum.

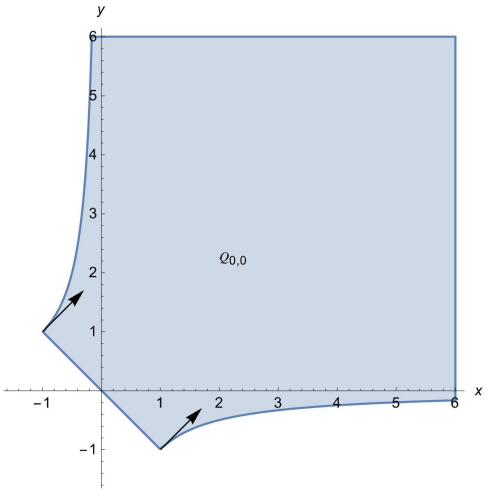

FIGURE 1. Q_{δ_1,δ_2} for $\delta_1 = 0 = \delta_2$

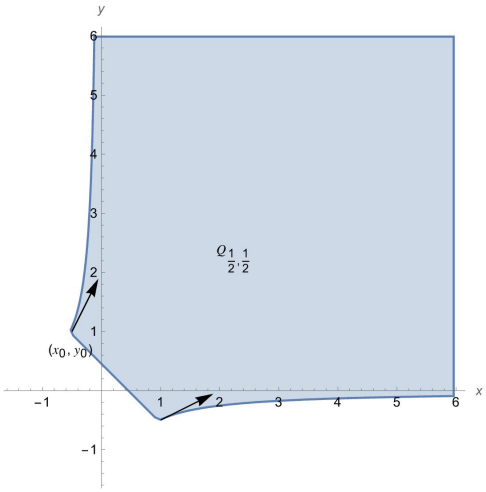

FIGURE 2. Q_{δ_1,δ_2} for $\delta_1 = 1/2 = \delta_2$

Proof. Since $\delta_1 \in (0,1)$ and $\delta_2 > 0$, $1 + xy = \delta_1$ and $x + y = \delta_2$ have exactly two intersection point; one in the second quadrant, and the other in the fourth quadrant. Denote the intersection point in the second quadrant by (x_0, y_0). It must belong to the region $\Omega = \{(x,y) \in \mathbb{R}^2 : -1 < x < 0,\ -x < y < -1/x\}$. It is not hard to see that $(\delta_1, \delta_2) \in (0,1) \times (0,\infty) \mapsto (x_0, y_0) \in \Omega$ is a one-to-one correspondence. The inverse map is given by $\delta_1 = 1 + x_0 y_0$ and $\delta_2 = x_0 + y_0$.

Let

$$\tau = -\frac{y_0}{x_0}\ .$$

Pick any $(u,v) \in C_\tau \setminus \{(0,0)\}$. It is clear that $(x+u)+(y+v) > x_0 + y_0$ if $x + y \geq x_0 + y_0$. It remains to verify that $(x+u)(y+v) \geq x_0 y_0$ if $xy \geq x_0 y_0$ and $x + y \geq x_0 + y_0$. Note that the condition implies that $x \geq x_0$. Assume that $x < 0$,

$$\begin{aligned}(x+u)(y+v) - (x_0 y_0) &> u\left(y + x\frac{v}{u}\right) \geq u\,(y + x\tau)\\ &\geq u\left(y - x\frac{y_0}{x_0}\right)\\ &\geq u\left(x_0 + y_0 - x - x\frac{y_0}{x_0}\right) = u(x_0+y_0)\frac{x - x_0}{-x_0} \geq 0\ .\end{aligned}$$

The argument for $y < 0$ is similar. If $x \geq 0$ and $y \geq 0$, it is obvious. □

Lemma 5.3 allows to construct some functions F_k such that $u_0 + F_k$ is still 2-convex, and becomes *convex* outside $B_k(0)$.

Lemma 5.4. *Given any $\tau > 1$, there exists a sequence of smooth functions $\{F_k(x)\}_{k\in\mathbb{N}}$ with the following significance.*

(i) $D^2F_k = \mathbf{I}$ *when* $|x| \geq k$.
(ii) $0 < D^2F_k \leq \tau$ *everywhere.*
(iii) *At any* $x \in \mathbb{R}^n$*, the ratio between any two eigenvalues of* $D^2F_k(x)$ *belongs to* $[1/\tau, \tau]$.
(iv) *For any* $R > 0$*,* F_k *converges to* 0 *uniformly over* $\overline{B_R(0)}$ *as* $k \to \infty$*, so do their derivatives.*

Proof. Let r be the distance to the origin. For $F = F(r)$, its Hessian has two eigenvalues, F'' and $r^{-1}F'$. The first one is simple, and the second one has multiplicity $n-1$. Denote $F'(r)$ by $f(r)$. Let

$$u(r) = f'(r)\frac{r}{f(r)} \ , \tag{5.10}$$

and then

$$f(r) = f(r_0)\exp\left(\int_{r_0}^{r} \frac{u(\rho)}{\rho}\mathrm{d}\rho\right) \ . \tag{5.11}$$

Thus, one only needs to specify $u(r)$, r_0, and $f(r_0)$ to construct $f(r)$. By requiring $F(0) = 0$, the function $F(r)$ is uniquely determined.

Fix a constant $\theta \in (0, 1/10)$. For any $k \in \mathbb{N}$, choose a smooth function $u_k(r)$ for $r \geq 0$ with

- $u_k(r) = 1$ if $r \leq \theta$ or $r \geq k$;
- $u_k(r) = \tau$ if $2\theta \leq r \leq k/2$;
- $u_k'(r) \geq 0$ if $\theta \leq r \leq 2\theta$;
- $u_k'(r) \leq 0$ if $k/2 \leq r \leq k$.

The base point r_0 is set to be k, and the base value $f_k(k)$ is set to be k. One can see Figure 3 for an illustration of some $f_k(r)$'s (by using piecewise constant u_k).

It follows that $f_k(r) = r$ for $r \geq k$. When $r \leq k$,

$$f_k(r) = k\exp\left(-\int_r^k \frac{u(\rho)}{\rho}\mathrm{d}\rho\right) \leq k\exp\left(-\int_r^k \frac{1}{\rho}\mathrm{d}\rho\right) = r \ .$$

When $2\theta \leq r \leq k/2$,

$$f_k(r) = f_k(k/2)\exp\left(-\int_r^{k/2} \frac{u(\rho)}{\rho}\mathrm{d}\rho\right) \leq \left(\frac{2}{k}\right)^{\tau-1} r^\tau \ .$$

With a similar argument, $f_k(r) = c_k r$ when $r \leq \theta$, where

$$c_k = \frac{f_k(\theta)}{\theta} \leq \frac{f_k(2\theta)}{2\theta} \leq \left(\frac{4\theta}{k}\right)^{\tau-1} \ .$$

It is not hard to see that the corresponding $F_k(r)$ satisfies the assertions of this lemma. □

Note that Lemma 5.4 (ii) and (iii) implies that $(\langle (D^2F_k)\mathbf{u}, \mathbf{u}\rangle, \langle (D^2F_k)\mathbf{v}, \mathbf{v}\rangle)$ belongs to C_τ defined in Lemma 5.3, for any unit vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

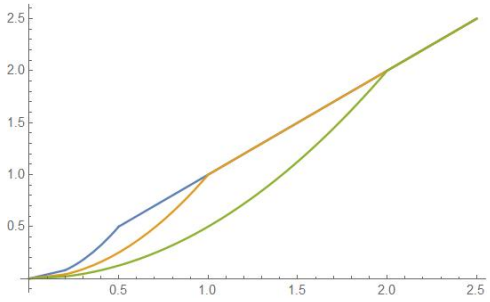

FIGURE 3. $F_k'(r)$

Theorem 5.5. *Let $u_0 \in C^2(\mathbb{R}^n)$ satisfying (5.9) for some $\varepsilon_1, \varepsilon_2 \in (0,1)$. Then, (1.1) admits a unique solution $u(x,t)$ in the space $C^0(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \times (0,\infty))$ such that*

- *(5.9) is preserved along the flow;*
- *there exists $c_\ell = c_\ell(\varepsilon_1) > 0$ for any $\ell > 2$ such that $\sup_{x\in\mathbb{R}^n} |D^\ell u|^2 \le c_\ell t^{2-\ell}$ for any $t > 0$.*

Proof. Let

$$\delta_1 = \min\left\{ \frac{\varepsilon_2}{\sqrt{2(\varepsilon_1^{-2}-1)}}, \varepsilon_2 \right\} \quad \text{and} \quad \delta_2 = \frac{2\varepsilon_2}{\varepsilon_1^{-2}+1} \ .$$

By (2.8) and (2.9), the eigenvalues of $D^2 u_0$ satisfy $1 + \lambda_i\lambda_j \ge \delta_1$ and $\lambda_i + \lambda_j \ge \delta_2$ for any $i \ne j$. Let $\tau = \tau(\delta_1, \delta_2)$ be given by Lemma 5.3. Apply Lemma 5.4 for this τ to get a sequence of functions, $\{F_k(x)\}_{k\in\mathbb{N}}$.

It follows from Lemma 5.3, Lemma 5.4 and (2.7) that the eigenvalues of $D^2(u_0 + F_k)$ satisfy

$$\begin{aligned} &1 + \lambda_i\lambda_j \ge \delta_1 \ , \ \lambda_i + \lambda_j \ge \delta_2 \ \text{ for any } i \ne j \ , \\ &\text{and } \ \sum_i |\lambda_i|^2 \le 2(\varepsilon_1^{-2} - 1 + n\tau^2) \ . \end{aligned} \tag{5.12}$$

The reason is that the spectrum of a symmetric matrix is stable with respect to small perturbations; see for instance [15, section 1.3] and the references therein. A complete discussion on the perturbation theory of eigenvalues of matrices can be found in [11, ch.1 and 2]. By (5.12), there exist $\varepsilon_1', \varepsilon_2' \in (0,1)$ depending[4] on $\varepsilon_1, \varepsilon_2$ such that

$$*\Omega \ge \varepsilon_1' \quad \text{and} \quad \det S^{[2]} \ge \varepsilon_2' \tag{5.13}$$

for $u_0 + F_k$. Let A be a symmetric matrix whose eigenvalues satisfy (5.12). It is not hard to see that there exists $\delta_3 > 0$ such that $\delta_3 \le A + \mathbf{I} \le 1/\delta_3$. Note that $\{B : \text{ symmetric } n \times n \text{ matrix} :$

[4]The dependence on the dimension n is always omitted in this paper.

$\delta_3 \le B \le 1/\delta_3\}$ is a convex set, and there exists $\varepsilon_1'', \varepsilon_2'' \in (0,1)$ such that

$$*\Omega \ge \varepsilon_1'' \quad \text{and} \quad \det S^{[2]} \ge \varepsilon_2'' \tag{5.14}$$

for any symmetric matrix[5] B with $\delta_3 \le B \le 1/\delta_3$.

Let $\{\eta_\sigma\}_{\sigma>0}$ be the standard mollifiers; see for instance [7, Appendix C.4]. They have the following properties: $\eta_\sigma \ge 0$, $\operatorname{supp}\eta_\sigma \subset \overline{B_\sigma(0)}$, $\int_{\mathbb{R}^n} \eta_\sigma = 1$ for all $\sigma > 0$, and $\eta_\sigma(x)$ depends only on $|x|$. For any $k \in \mathbb{N}$ and $\sigma > 0$, let

$$\tilde{w}_0^{k,\sigma}(x) = \int_{\mathbb{R}^n} \left(u_0(y) + F_k(y)\right)\, \eta_\sigma(x-y)\, \mathrm{d}y \ . \tag{5.15}$$

We claim that there exists a sequence of positive numbers $\{\sigma_k\}_{k\in\mathbb{N}}$ with $\sigma_k \to 0$ as $k \to \infty$ such that $\tilde{w}_0^{k,\sigma_k}$ obeys

$$*\Omega \ge \min\{\frac{1}{2}\varepsilon_1', \varepsilon_1''\} \quad \text{and} \quad \det S^{[2]} \ge \min\{\frac{1}{2}\varepsilon_2', \varepsilon_2''\} \ . \tag{5.16}$$

Note that

$$D_x^2 \tilde{w}_0^{k,\sigma}(x) = \int_{\mathbb{R}^n} D_y^2 \left(u_0(y) + F_k(y)\right)\, \eta_\sigma(x-y)\, \mathrm{d}y \ .$$

When $|x| \ge k+1$, it follows from Lemma 5.4 (i) and (5.14) that for $\tilde{w}_0^{k,\sigma}$, $*\Omega|_x < \varepsilon_1''$ and $\det S^{[2]}|_x \ge \varepsilon_2''$ for any $\sigma_k < 1$. When $|x| \le k+1$, it follows from (5.12), (5.13) and the uniform continuity of $D^2(u_0 + F_k)$ on $\overline{B_{k+2}(0)}$ that (5.16) holds true for $\tilde{w}_0^{k,\sigma_k}$ on $B_{k+1}(0)$, provided that σ_k is sufficiently small. This finishes the proof of the claim.

Now, set w_0^k to be $\tilde{w}_0^{k,\sigma_k}$. With (5.16) and

$$D_x^\ell w_0^k(x) = \int_{\mathbb{R}^n} D_y^2 \left(u_0(y) + F_k(y)\right)\, D_x^{\ell-2} \eta_{\sigma_k}(x-y)\, \mathrm{d}y \ ,$$

for any $\ell \ge 2$, one can apply Proposition 5.2 to the initial conditions w_0^k. Denote the solution by w^k, and it obeys (5.16) for all t.

It follows from (5.16) and (2.7) that $D^2 w^k$ is uniformly bounded on $\mathbb{R}^n \times [0,\infty)$. Since w^k solves (1.1), $\frac{\partial w^k}{\partial t}$ is also uniformly bounded over $\mathbb{R}^n \times [0,\infty)$. Fix $R > 0$ and $T > 0$. Let $K_{R,T} = \overline{B_R(0)} \times [0,T] \subset \mathbb{R}^n \times [0,\infty)$. It follows from (5.15) and Lemma 5.4 (iv) that w_0^k is uniformly bounded over $\overline{B_R(0)}$. With the uniform boundedness of the time derivative, w^k is uniformly bounded over $K_{R,T}$. Together with the uniform boundedness of $D^2 w^k$, Dw^k is uniformly bounded over $K_{R,T}$. Hence, there exists a subsequence of w^k which converges uniformly on any compact subset of $\mathbb{R}^n \times [0,\infty)$. By Lemma 5.4 (iv) and $\sigma_k \to 0$ as $k \to \infty$, $\lim_{k\to\infty} w_0^k(x) = u_0(x)$.

This together with assertion (ii) of Proposition 5.2 implies that (a subsequence of) $w^k(x,t)$ converges smoothly on any compact subset of $\mathbb{R}^n \times (0,\infty)$. The limit u is a solution to (1.1) with initial condition u_0. The decay estimate on the derivatives of u follows from that of w^k.

[5]The functions $\log(*\Omega)$ and $\log\det S^{[2]}$ are defined for a symmetric matrix by using (2.3) and (2.6), respectively.

Since the constant c_ℓ depends on the $|D^2 w^k|$, it is not hard to see from the last item of (5.12) that c_ℓ depends only on ε_1.

By construction, u satisfies (5.16) for all $t > 0$. To see that u still satisfies (5.9), we will need the local estimate (5.8) in Lemma 5.1. Fix an $\epsilon_0 > 0$. For any $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, let K_0 be its compact neighborhood $\overline{B_1(x_0)} \times [t_0/2, 2t_0] \subset \mathbb{R}^n \times (0, \infty)$. Since Dw^k is uniformly bounded over K_0, there exists an $R > 0$ such that

$$\varphi = R^2 - |x|^2 - |Dw^k|^2 - 2nt \geq \frac{R^2}{1 + \epsilon_0}$$

on K_0. Note that this φ is equal to the one in the proof of Lemma 5.1. By Proposition 4.1 and Lemma 5.1, (5.8) for $v = (*\Omega)^{-1}$ gives that

$$\max_{(x,t)\in K_0} \left[(*\Omega) \text{ of } w^k\right]^{-\frac{1}{n}} \leq (1 + \epsilon_0) \max_{x \in \overline{B_R(0)}} \left[(*\Omega) \text{ of } w_0^k\right]^{-\frac{1}{n}} .$$

Due to the uniform continuity of $D^2 u_0$ over $\overline{B_R(0)}$, Lemma 5.4 (iv), and $\sigma_k \to 0$,

$$\limsup_{k\to\infty} \max_{x \in \overline{B_R(0)}} \left[(*\Omega) \text{ of } w_0^k\right]^{-1} \leq \frac{1}{\varepsilon_1} .$$

It follows that $(*\Omega)^{-1}$ of u over K_0 is no greater than $(1 + \epsilon_0)^n/\varepsilon_1$. Since it is true for any $\epsilon_0 > 0$, $*\Omega$ of u is always greater than or equal to ε_1. The argument for $\det S^{[2]}$ is similar. □

Since a 2-convex initial can be approximated by strictly 2-convex functions, we can extend Theorem 5.5 to 2-convex initial condition.

Theorem 5.6. *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x\in\mathbb{R}^n} |D^2 u_0|^2 \leq c$ for some $c > 0$. Then, (1.1) admits a unique solution $u(x, t)$ in the space $C^0(\mathbb{R}^n \times [0, \infty)) \cap C^\infty(\mathbb{R}^n \times (0, \infty))$ such that*

- *for any $t > 0$, u is 2-convex and $*\Omega \geq \varepsilon_1$;*
- *there exists $c_\ell = c_\ell(\varepsilon_1) > 0$ for any $\ell > 2$ such that $\sup_{x\in\mathbb{R}^n} |D^\ell u|^2 \leq c_\ell t^{2-\ell}$ for any $t > 0$.*

Proof. Consider $w_0^k = u_0 + e^{-k}|x|^2$ for $k \in \mathbb{N}$. Since $|D^2 u_0| \leq C$, it is not hard to see that there exist $\varepsilon_1' = \varepsilon_1'(\varepsilon_1) > 0$ and $\varepsilon_{2,k}' = \varepsilon_{2,k}'(\varepsilon_1, k) > 0$ such that $*\Omega \geq \varepsilon_1'$ and $\det S^{[2]} \geq \varepsilon_{2,k}'$ for w_0^k. Denote by w^k the solution to (1.1) with initial condition w_0^k given by Theorem 5.5. Note that the bound of the derivatives given by Theorem 5.5 only depends on ε_1. By the same argument as that in the proof of Theorem 5.5, there exists a subsequence of w^k which converges in $C^0_{\text{loc}}(\mathbb{R}^n \times [0, \infty))$ and in $C^\infty(\mathbb{R}^n \times (0, \infty))$. The properties asserted by this theorem follows from $\liminf_{k\to\infty}(*\Omega \text{ of } w_0^k) \geq \varepsilon_1$ and the smooth convergence of w^k in $C^\infty(\mathbb{R}^n \times (0, \infty))$. □

6. Some Convergence Results

6.1. **Potentials of Bounded Gradient.**

Theorem 6.1. *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x\in\mathbb{R}^n} |D^2 u_0|^2 \le c$ for some $c > 0$. Denote by $u(x,t)$ the solution to (1.1) with $u(x,0) = u_0(x)$ given by Theorem 5.6. Then, there exists an $\vec{a} \in \mathbb{R}^n$ such that $Du(x,t)$ converges to the constant map from $\mathbb{R}^n$ to $\vec{a}$ in $C^\infty_{loc}(\mathbb{R}^n, \mathbb{R}^n)$. In other words, $L_u = \{(x, Du(x,t)) : x \in \mathbb{R}^n\}$ converges locally smoothly to $\mathbb{R}^n \times \{\vec{a}\}$ as $t \to \infty$.*

Proof. The key step is to show that the bounded gradient condition is preserved along the flow. Denote $\partial u/\partial x^k$ by u_k, $\partial^2 u/\partial x^i \partial x^j$ by u_{ij}, etc. By (1.1) and Remark 3.1,

$$\frac{\partial}{\partial t} u_k = g^{ij} u_{ijk} \tag{6.1}$$

where g^{ij} is the inverse of $g_{ij} = \delta_{ij} + \sum_k u_{ik} u_{jk}$. It follows that

$$\begin{aligned} \frac{\partial}{\partial t}\big[c - \sum_k (u_k)^2\big] &= -2 \sum_k u_k g^{ij} u_{ijk} \\ &= g^{ij}\partial_i \partial_j \big[c - \sum_k (u_k)^2\big] + 2 \sum_k g^{ij} u_{ki} u_{kj} \ , \end{aligned}$$

and hence

$$\frac{\partial}{\partial t}\big[c - \sum_k (u_k)^2\big] - g^{ij}\partial_i \partial_j \big[c - \sum_k (u_k)^2\big] \ge 0 \ . \tag{6.2}$$

Since $*\Omega \ge \varepsilon_1$ along the flow, g^{ij} is uniformly bounded over $\mathbb{R}^n \times [0,\infty)$. According to Theorem 5.6, $t|D^3 u|^2$ is uniformly bounded. With (6.1), there exits for any $T > 0$ a constant C_T such that $|u_k| \le C_T$ on $\mathbb{R}^n \times [0,T]$. Thus, there is a $C'_T > 0$ such that $\big[c - \sum_k (u_k)^2\big] \ge -C'_T$ on $\mathbb{R}^n \times [0,T]$. By the maximum principle in [8, Theorem 9 on p.43], $c - \sum_k (u_k)^2 \ge 0$ on $\mathbb{R}^n \times [0,T]$.

With $|D^\ell u| \le c_\ell t^{2-\ell}$ for any $\ell > 2$, $Du(x,t)$ converges in $C^\infty_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n)$ to some $p(x)$ as $t \to \infty$. Since the convergence is $C^\infty_{\rm loc}$, the image of $(x, p(x))$ is still a minimal Lagrangian submanifold. Thus, $p(x) = D\bar{u}(x)$ for some $\bar{u}(x) \in C^\infty(\mathbb{R}^n)$ satisfying $|D^2 \bar{u}| \le K$ for some $K > 0$ and $1 + \lambda_i \lambda_j \ge 0$. According to the Bernstein theorem [17, Theorem A], $(x, D\bar{u}(x))$ must be an affine subspace. Since $\sum_k (\partial_k \bar{u})^2 \le c$, the affine subspace must be $\mathbb{R}^n \times \{a\}$. □

6.2. **Convergence to Self-Expanders.**

A solution $F : L \times (0,\infty) \to \mathbb{R}^n$ to the mean curvature flow is called a self-expander if the submanifold $F(L,t)$ is the dilation of $F(L,1)$ with the factor $\sqrt{t}$. It is a natural model for immortal solutions to the mean curvature flow, and can also be

used to study the mean curvature flow of conical singularities. In our setting $F(x,t) = (x, Du)$, one finds that being a self-expander means that

$$u(x,1) = t \cdot u\left(\frac{x}{\sqrt{t}}, 1\right) \tag{6.3}$$

for any $t > 0$. Therefore, for the entire Lagrangian mean curvature flow (1.1), $u : L \times (0,\infty) \to \mathbb{R}^n$ is a self-expander if and only if $u_1 = u(\,\cdot\,, 1)$ obeys

$$\frac{1}{\sqrt{-1}} \frac{\det(\mathbf{I} + \sqrt{-1} D^2 u_1)}{\sqrt{\det(\mathbf{I} + (D^2 u_1)^2)}} - u_1 + \frac{1}{2}\langle Du_1, x\rangle = 0 \tag{6.4}$$

In [2, Theorem 1.2], it is shown that if the initial condition u_0 is strictly distance-decreasing, and $(x, Du_0(x))$ is asymptotic to a cone at spatially infinity, then the rescaled flow converges to a self-expander. In the following proposition, we prove that the result holds true in the 2-convex case.

Theorem 6.2. *Let $u_0 \in C^2(\mathbb{R}^n)$ be a 2-convex function with $\sup_{x\in\mathbb{R}^n} |D^2 u_0|^2 \le c$ for some $c > 0$, and*

$$\lim_{\mu\to\infty} \frac{u_0(\mu x)}{\mu^2} = U_0(x) \ ,$$

for some $U_0(x)$. Let $u(x,t)$ be the solution to (1.1) given by Theorem 5.6. Then, $u(\mu x, \mu^2 t)/\mu^2$ converges to a smooth self-expanding solution $U(x,t)$ to (1.1) in $C^\infty_{loc}(\mathbb{R}^n \times (0,\infty))$ as $\mu \to \infty$. As $t \to 0$, $U(x,t)$ converges to $U_0(x)$ in $C^0_{loc}(\mathbb{R}^n)$.

Proof. It is straightforward to see that for any $\mu > 0$,

$$u_\mu(x,t) = \frac{1}{\mu^2} u(\mu x, \mu^2 t)$$

is a solution to (1.1) with initial condition $u_\mu(x,0) = \mu^{-2} u_0(\mu x)$. Since the spatial Hessian remains unchanged in this rescaling, it follows from Theorem 5.6 that

$$\left| D^\ell u_\lambda(x,t) \right| \le c_\ell \, t^{1-\frac{\ell}{2}}$$

for any $\lambda > 0$, $\ell > 2$ and on $\mathbb{R}^n \times (0,\infty)$. With Remark 3.1, there exists for any $m \ge 1$ and $\ell \ge 0$ a constant $c_{m,\ell} > 0$ such that

$$\left| \frac{\partial^m}{\partial t^m} D^\ell u_\mu \right| \le c_{m,\ell} \, t^{1-m-\frac{\ell}{2}} \ .$$

It is clear that $u_\mu(0,0)$ and $Du_\mu(0,0)$ are uniformly bounded for $\mu \ge 1$. These estimates together with the Arzelà–Ascoli theorem implies that $u_\mu(x,t)$ converges to some $U(x,t)$ in $C^\infty_{\rm loc}(\mathbb{R}^n \times (0,\infty))$ as $\mu \to \infty$. Moreover, $U(x,t)$ satisfies the equation (1.1), is 2-convex for all t, and $*\Omega \ge \varepsilon_1$ for all t. It follows from the construction that $U(x,t)$ satisfies $U(\mu x, \mu^2 t) = \mu^2 U(x,t)$ for any $\mu > 0$, and thus $U(x,t)$ is a self-expander.

Since $|\partial U/\partial t| \le c_{1,0}$, $U(x,t)$ converges to a function $U(x,0)$ as $t \to 0$. Again by $|\partial u_\mu/\partial t| \le c_{1,0}$, one finds that

$$U(x,0) = \lim_{t\to 0}\lim_{\mu\to\infty} u_\mu(x,t) = \lim_{\mu\to\infty} u_\mu(x,0) = U_0(x) \ .$$

It finishes the proof of this proposition. □

On the other hand, it is shown in [2, Lemma 7.1] that for any $U_0(x)$ which is strictly distance-decreasing and homogeneous of degree 2, there exists a self-expander to (1.1) which is strictly distance-decreasing for all time. A function is said to be homogeneous of degree 2 if

$$U_0(x) = \frac{1}{\lambda^2} U_0(\lambda x)$$

for all $\lambda > 0$. If U_0 is C^2 over $\mathbb{R}^n$, we can invoke Theorem 5.5. It is more interesting to assume that U_0 is C^2 only on $\mathbb{R}^n \setminus \{0\}$, and it requires some more work to construct the solution.

Proposition 6.3. *Suppose that $U_0 : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 2, is C^2 on $\mathbb{R}^n\setminus\{0\}$, and satisfy (5.9) on $\mathbb{R}^n\setminus\{0\}$. Then, (1.1) admits a unique self-expanding solution $u(x,t)$ in the space $C^0(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \times (0,\infty))$ with initial condition U_0 such that $u(x,t) = t \cdot u(x/\sqrt{t}, 1)$ and (5.9) is preserved along the flow.*

Moreover, there exists $c_\ell = c_\ell(\varepsilon_1)$ for any $\ell > 2$ such that $\sup_{x\in\mathbb{R}^n} |D^\ell u|^2 \le c_\ell t^{2-\ell}$ for any $t > 0$.

It follows from the assumption that DU_0 is uniformly Lipschitz. The following lemma is analogous to Lemma 5.4, and will be used to handle U_0 near the origin.

Lemma 6.4. *Given any $\tau > 1$, there exists a sequence of smooth functions $\{E_k(x)\}_{k\in\mathbb{N}}$ with the following significance.*

(i) *$D^2 E_k = \mathbf{I}$ when $|x| \le 1/k$.*
(ii) *$0 < D^2 E_k \le \tau$ everywhere.*
(iii) *At any $x \in \mathbb{R}^n$, the ratio between any two eigenvalues of $D^2 E_k(x)$ belongs to $[1/\tau, \tau]$.*
(iv) *For any $R > 0$, E_k converges to 0 uniformly over $\overline{B_R(0)}$ as $k \to \infty$, so do their derivatives.*

Proof. As in the proof of Lemma 5.4, let $E = E(r)$ where $r = |x|$. Denote $E'(r)$ by $e(r)$. Let

$$u(r) = e'(r)\frac{r}{e(r)} \ , \tag{6.5}$$

and then

$$e(r) = e(r_0)\exp\left(\int_{r_0}^{r} \frac{u(\rho)}{\rho}\mathrm{d}\rho\right) \ . \tag{6.6}$$

For any $k \in \mathbb{N}$, choose a smooth function $u_k(r)$ for $r \ge 0$ with

- $u_k(r) = 1$ if $r \leq 1/k$;
- $u_k(r) = 1/\tau$ if $r \geq 2/k$;
- $u_k'(r) \leq 0$ for all r.

Set r_0 to be $1/k$, and set $e_k(1/k)$ to be $1/k$. It follows that $e_k(r) = r$ for $r \leq k$. When $r \geq 1/k$,

$$e_k(r) = \frac{1}{k} \exp\left(\int_{1/k}^{r} \frac{u(\rho)}{\rho} \mathrm{d}\rho\right) \leq \frac{1}{k} \exp\left(\int_{1/k}^{r} \frac{1}{\rho} \mathrm{d}\rho\right) = r \ .$$

When $r \geq 2/k$,

$$e_k(r) = e_k(2/k) \exp\left(\int_{k/2}^{r} \frac{u(\rho)}{\rho} \mathrm{d}\rho\right) \leq \left(\frac{2}{k}\right)^{1-\frac{1}{\tau}} r^{\frac{1}{\tau}} \ .$$

The function $E_k(r)$ is determined by setting $E_k(0) = 0$. It is not hard to see that the corresponding $E_k(r)$ satisfies the assertions of this lemma. □

Proof of Proposition 6.3. To start, note that there exists a $\delta > 0$ such that $\delta \leq D^2(U_0 + E_k) \leq 1/\delta$ on where $0 < |x| \leq 1/k$. With this observation, the argument of Theorem 5.5 works for the initial condition $U_0 + E_k$. Denote the solution by $\tilde{u}^k$. By the same argument as that in Theorem 5.5, $\tilde{u}^k$ (subsequentially) converges on compact subsets of $\mathbb{R}^n \times [0, \infty)$, and converges smoothly on compact subsets of $\mathbb{R}^n \times (0, \infty)$. The limit, u, is a solution to (1.1) with initial condition U_0.

Since U_0 is homogeneous of degree 2, $u_\mu(x,t) = \mu^{-2}\, u(\mu x, \mu^2 t)$ (for any $\mu > 0$) is also a solution to (1.1) with initial condition U_0. It follows from the uniqueness theorem in [4] that $u_\mu(x,t) = u(x,t)$ for any $\mu > 0$. Hence, the solution is a self-expander. □

With the same argument as that in Theorem 5.6, Proposition 6.3 can be pushed to the 2-convex case.

Proposition 6.5. *Let $U_0 : \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree 2. Suppose that on $\mathbb{R}^n \setminus \{0\}$, U_0 is C^2, satisfies $*\Omega \geq \varepsilon_1$ for some $\varepsilon_1 > 0$, and is 2-convex. Then,* (1.1) *admits a unique solution $u(x,t)$ in the space $C^0(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \times (0,\infty))$ with initial condition U_0 such that $u(x,t) = t \cdot u(x/\sqrt{t}, 1)$. Moreover, the two conclusions of Theorem 5.6 hold true for $u(x,t)$.*

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