AN ANSATZ FOR CONSTRUCTING EXPLICIT SOLUTIONS OF HESSIAN EQUATIONS

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ABSTRACT. We introduce a (variation of quadrics) ansatz for constructing explicit, real-valued solutions to broad classes of complex Hessian equations on domains in $\mathbb{C}^{n+1}$ and real Hessian equations on domains in $\mathbb{R}^{n+1}$. In the complex setting, our method simultaneously addresses the deformed Hermitian–Yang–Mills/Leung–Yau–Zaslow (dHYM/LYZ) equation, the Monge–Ampère equation, and the J-equation. Under this ansatz each PDE reduces to a second-order system of ordinary differential equations admitting explicit first integrals. These ODE systems integrate in closed form via abelian integrals, producing wide families of explicit solutions together with a detailed description. In particular, on $\mathbb{C}^3$, we construct entire dHYM/LYZ solutions of arbitrary subcritical phase, and on $\mathbb{R}^3$ we produce entire special Lagrangian solutions of arbitrary subcritical phase. More generally, in any complex or real dimension, our ansatz yields entire solutions of certain subcritical phases for both the dHYM/LYZ and special Lagrangian equations. Some of these solutions develop singularities on compact regions. In the special Lagrangian case we show that, after a natural extension across the singular locus, these blow-up solutions coincide with previously known complete special Lagrangian submanifolds obtained via a different ansatz.

1. Introduction

Let $X \subset \mathbb{C}^{n+1}$ be a domain and let $u \in C^2(X)$ be a real-valued function. We study the complex Hessian equation:

$$c_n\sigma_{n+1}(\partial\bar{\partial}u) + c_{n-1}\sigma_n(\partial\bar{\partial}u) + \cdots + c_0\sigma_1(\partial\bar{\partial}u) + c_{-1} = 0 \tag{1.1}$$

where $c_{-1}, c_0, \cdots, c_n$ are real constants, and $\sigma_k(\partial\bar{\partial}u), k = 1, \cdots, n+1$ are the k-th symmetric functions of the complex Hessian of u. The coefficients $c_{-1}, c_0, \cdots, c_n$ of (1.1) determine two

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polynomials F and G in the variables $p_1, \cdots, p_n$:

$$F(p_1, \cdots, p_n) = \sum_{k=0}^{n} c_k \sigma_k(p) \quad \text{and} \quad G(p_1, \ldots, p_n) = \sum_{k=0}^{n} c_{k-1} \sigma_k(p) \tag{1.2}$$

where $\sigma_k(p), k = 0, \cdots n$ are the k-the symmetric functions of $p_i, i = 1, \cdots, n$.

By introducing a suitable ansatz, we reduce the PDE (1.1) to a system of second-order ordinary differential equations determined by F and G. Concretely, if $p_i(s)$, $i = 1, \ldots, n$ are C^2 functions of s that satisfy the ODE system

$$\begin{aligned} &\frac{p_i''}{2} \cdot F(p_1, \ldots, p_n) = (p_i')^2 \frac{\partial F}{\partial p_i}(p_1, \ldots, p_n) \quad \text{for } i = 1, \ldots, n \quad , \\ &r'' \cdot F(p_1, \ldots, p_n) = -G(p_1, \ldots, p_n) \ , \end{aligned} \tag{1.3}$$

then the function

$$u(z_1, \ldots, z_{n+1}) = 2 \sum_{j=1}^{n} p_j(\operatorname{Re} z_{n+1}) \, (\operatorname{Re} z_j)^2 + 4 \, r(\operatorname{Re} z_{n+1})$$

solves the complex Hessian equation (1.1) (see Proposition 5.1). Here $(z_1, \cdots, z_{n+1})$ are standard complex coordinates on $\mathbb{C}^{n+1}$ and $\operatorname{Re} z_i$ denotes the real part of $z_i, i = 1 \cdots n+1$.

The same ansatz—and the very same reduction to an ODE system—applies to the corresponding real Hessian equation on $\mathbb{R}^{n+1}$, i.e.

$$f(x_1, \ldots, x_n, x_{n+1}) = \frac{1}{2} \sum_{j=1}^{n} p_j(x_{n+1}) \, x_j^2 + r(x_{n+1})$$

satisfies the equation

$$c_n \sigma_{n+1}(\nabla^2 f) + c_{n-1} \sigma_n(\nabla^2 f) + \cdots + c_0 \sigma_1(\nabla^2 f) + c_{-1} = 0 \tag{1.4}$$

where $\sigma_k(\nabla^2 f), k = 1, \cdots, n+1$ are the k-th symmetric functions of the real Hessian of f.

We then focus on Hessian equations whose coefficients $c_0, \cdots, c_n$ satisfy a *recursive relation*.

Definition 1.1. Let a_0 and a_1 be real numbers. The complex Hessian equation (1.1)/real Hessian equation (1.4) is said to be of *r*ecursive type (a_0, a_1) if the coefficients $c_0, c_1, \cdots c_n$ satisfy the recursive relation:

$$c_{k-1} = a_1 c_k - a_0 c_{k+1} \quad \text{for } k = 1, \ldots, n-1$$

In particular, the coefficients $c_0, \cdots, c_n$ are determined by a_0, a_1, c_{n-1} and c_n. Many classical nonlinear PDEs lie in this class—including the deformed Hermitian–Yang–Mills/Leung–Yau–Zaslow (dHYM/LYZ) equation, the real and complex Monge–Ampère equation, the J-equation, and the special Lagrangian equation—so that one may recover each by choosing the appropriate recursive relation (see Proposition 5.4 for a complete classification of recursive-type equations).

For any such *recursive-type* equation, we show that the associated second-order ODE system is *completely integrable*: in particular, it admits enough first integrals to reduce the dynamics to quadratures.

Theorem 1.2. *Suppose* (1.1)/(1.4) *is of recursive type* (a_0, a_1)*. Then its associated 2nd order ODE system* (1.3) *is completely integrable. In fact, denoting*

$$\xi_i = \frac{p_i^2 + a_1 p_i + a_0}{p_i'}, \quad i = 1, \ldots, n \ ,$$

then

$$\frac{F^2}{\prod_{i=1}^n p_i'} \quad \text{and} \quad \xi_i - \xi_1, \quad i = 2, \ldots, n$$

are first integrals of the system. Moreover, $\xi_i, i = 1 \cdots n$ *satisfy the following ODE system:*

$$(\xi_i')^2 = k_1 + k_2 \prod_{j=1}^n \xi_j$$

for explicit real constants k_1, k_2 *depending only on* a_0, a_1, c_{n-1} *and* c_n*.*

Theorem 1.2 can be used to construct non-polynomial entire solutions for both dHYM/LYZ equation on $\mathbb{C}^{n+1}$ and special Lagrangian equation on $\mathbb{R}^{n+1}$, when $n \geq 3$. The equations are of recursive type $(a_0 = 1, a_1 = 0)$ and take the following form:

$$\cos\theta(\sigma_1 - \sigma_3 + \cdots + (-1)^{k-1}\sigma_{2k-1} + \cdots) - \sin\theta(1 - \sigma_2 + \cdots + (-1)^{k-1}\sigma_{2k-2} + \cdots) = 0$$

for some $\theta \in \mathbb{R}$. Note that θ and $\theta + \pi$ give equivalent equations. On the other hand, for a C^2 function u defined on a domain of $\mathbb{C}^{n+1}$, one can consider $\Theta = \sum_{j=1}^{n+1} \arctan \lambda_j$, where λ_j's are eigenvalues of $\partial\bar{\partial}u$. This real-valued function Θ is said to be the *phase of* u. If u solves the dHYM/LYZ equation, Θ is a constant, and $\Theta - \theta \in \mathbb{Z}\pi$. However, for a priori estimate and relevant PDE techniques, the value of Θ matters. If $|\Theta| = (n-1)\frac{\pi}{2}$, the function is said to be of *critical phase*; the range $|\Theta| > (n-1)\frac{\pi}{2}$ is called *supercritical phase*; the range $|\Theta| < (n-1)\frac{\pi}{2}$ is referred as the *subcritical phase*. Known results are primarily concentrated in the critical and supercritical phases; see [8, 3, 4, 2, 14, 15] for dHYM/LYZ equation and [19, 16, 20] for special Lagrangian equations. For the relation between these conditions and a priori estimates of Hessian equations, we refer to [10, 6, 1]. We apply Theorem 1.2 to subcritical, entire solutions to both dHYM/LYZ equation and special Lagrangian equation.

Theorem 1.3 (Theorem 3.2 and Theorem 4.2)**.** *For any integer* $n \geq 3$ *and any* $\Theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$*,*

- *there exist subcritical, non-polynomial entire solutions to the dHYM/LYZ equation on* $\mathbb{C}^{n+1}$ *with phase* Θ*;*
- *there exist subcritical, non-polynomial entire solutions to the special Lagrangian equation on* $\mathbb{R}^{n+1}$ *with phase* Θ*.*

Entire solutions to the special Lagrangian equation on $\mathbb{R}^3$ were previously constructed by Warren [17, 18] (with phase $\pi/2$) and by Li [13] (with phase 0).

The non-entire solutions produced by Theorem 1.2 develop singularities on compact regions. In the special Lagrangian case we show in Section 4 that, after a natural extension across the singular locus, these blow-up solutions coincide with previously known complete special Lagrangian submanifolds obtained via a different ansatz studied by Harvey-Lawson [7], Lawlor [11], and Joyce [9].

Section 2 is devoted to the dHYM/LYZ equation. In Section 3, we investigate entire solutions to the dHYM/LYZ equation on $\mathbb{C}^{n+1}$. In Section 4, we turn to the special Lagrangian equation, demonstrating that the blow-up solutions obtained earlier can be extended to complete special Lagrangian submanifolds. In Section 5, we deal with general equations of recursive type. The appendix contains two important calculation lemmas.

2. The deformed Hermitian–Yang–Mills/Leung–Yau–Zaslow Equation

The Leung–Yau–Zaslow (LYZ) equation, also known as the deformed Hermitian–Yang–Mills (dHYM) equation in the literature (see Collins–Xie–Yau [5, 8]) is a fully nonlinear partial differential equation. It governs a Hermitian metric on a line bundle over a Kähler manifold, or more generally for a real $(1,1)$-form. Suppose (X,ω) is a Kähler manifold and $[\alpha] \in H^{1,1}(X,\mathbb{R})$ is a $(1,1)$ class. The case of a line bundle consists of setting $[\alpha] = c_1(L)$ where $c_1(L)$ is the first Chern class of a holomorphic line bundle $L \to X$. Suppose that the complex dimension of X is $n+1$ and consider the topological constant

$$\hat{z}([\omega],[\alpha]) = \int_X (\omega + i\alpha)^{n+1}.$$

Notice that $\hat{z}$ depends only on the class of ω and α. Suppose that $\hat{z} \neq 0$. Then this is a complex number

$$\hat{z}([\omega],[\alpha]) = re^{i\theta}$$

for some real $r > 0$ and angle $\theta \in (-\pi, \pi]$ which is uniquely determined.

Fix a smooth representative differential form α in the class $[\alpha]$. For a smooth function $u : X \to \mathbb{R}$, the dHYM/LYZ equation for (X,ω) with respect to $[\alpha]$ is

$$\begin{cases} \mathrm{Im}(e^{-i\theta}(\omega + i(\alpha + \frac{i}{2}\partial\bar{\partial}u))^{n+1}) = 0 \\ \mathrm{Re}(e^{-i\theta}(\omega + i(\alpha + \frac{i}{2}\partial\bar{\partial}u))^{n+1}) > 0. \end{cases}$$

Take X to be a domain of $\mathbb{C}^{n+1}$, $\alpha = 0$, and $\omega = \frac{i}{2}\sum_{j=1}^{n+1} dz_j \wedge d\bar{z}_k$ for the standard complex coordinates $z_1, \ldots, z_{n+1}$ of $\mathbb{C}^{n+1}$, the LYZ equation for $u : X \to \mathbb{R}$ becomes

$$\mathrm{Im}\Big(e^{-i\theta}\det\Big(\mathbf{I}_{n+1} + i\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big]_{1\le j,k\le n+1}\Big)\Big) = 0\ , \tag{2.1}$$

$$\mathrm{Re}\Big(e^{-i\theta}\det\Big(\mathbf{I}_{n+1} + i\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big]_{1\le j,k\le n+1}\Big)\Big) > 0\ . \tag{2.2}$$

Recall that the sum of the arctangent of the eigenvalues of $\big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\big]_{1\le j,k\le n+1}$ is called the phase of u, and belongs to $(-\frac{n+1}{2}\pi, \frac{n+1}{2}\pi)$. If u satisfies (2.1), its phase is a constant, and is equal to θ modulo $\pi\mathbb{Z}$.

Our ansatz assumes the potential function u is of the form:

$$u(z_1, \ldots, z_n, z_{n+1}) = \sum_{j=1}^{n} 2p_j(s)(x_j)^2 + 4r(s) \tag{2.3}$$

where $p_j(s)$ and $r(s)$ are real-valued functions in $s = \mathrm{Re}\, z_{n+1}$, and $x_j = \mathrm{Re}\, z_j$, $j = 1, \ldots, n$.

We compute

$$\begin{aligned}
\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} &= p_j(s)\delta_{jk} \quad \text{for } j,k = 1,\ldots,n\ ,\\
\frac{\partial u}{\partial z_j \partial \bar{z}_{n+1}} &= p_j'(s)x_j \quad \text{for } j = 1,\ldots,n\ ,\\
\frac{\partial^2 u}{\partial z_{n+1} \partial \bar{z}_{n+1}} &= \sum_{j=1}^{n} \frac{1}{2}p_j''(s)(x_j)^2 + r''(s)\ .
\end{aligned}$$

It follows that the coefficient matrix of $\partial\bar{\partial}u$ is

$$\begin{bmatrix}
p_1 & 0 & \cdots & 0 & x_1p_1' \\
0 & p_2 & \cdots & 0 & x_2p_2' \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p_n & x_np_n' \\
x_1p_1' & x_2p_2' & \cdots & x_np_n' & \Big(\sum_{j=1}^{n}\frac{1}{2}(x_j)^2p_j''(s) + r''(s)\Big)
\end{bmatrix}. \tag{2.4}$$

With Lemma A.1 in the appendix, we compute

$$\begin{aligned}
&\det\Big(\mathbf{I}_{n+1} + i\Big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\Big]_{1\le j,k\le n+1}\Big) \\
&= (1+ip_1)\cdots(1+ip_n)\Big(1 + i\Big(\sum_{j=1}^n \frac{1}{2}(x_j)^2 p_j''(s) + r''(s)\Big)\Big) \\
&\quad + \sum_{j=1}^n (x_j)^2 (p_j'(s))^2 (1+ip_1)\cdots\widehat{(1+ip_j)}\cdots(1+ip_n) \\
&= \mathfrak{F}\cdot\Big(1 + i\Big(\sum_{j=1}^n \frac{1}{2}(x_j)^2 p_j''(s) + r''(s)\Big)\Big) - i\sum_{j=1}^n (x_j)^2 (p_j'(s))^2 \frac{\partial \mathfrak{F}}{\partial p_j}
\end{aligned} \tag{2.5}$$

where

$$\mathfrak{F} = (1+ip_1)\cdots(1+ip_n)\ . \tag{2.6}$$

Denoting

$$F_\theta = \mathrm{Re}(e^{-i\theta}\mathfrak{F}), \tag{2.7}$$

we obtain:

Definition 2.1. For any $\theta \in (-\pi, \pi]$, $p_1(s), \cdots, p_n(s)$ and $r(s)$ are said to satisfy the θ-angle ODE system on the interval $I \subset \mathbb{R}$ if

$$F_\theta(p_1,\ldots,p_n)\,\frac{p_j''}{2} = \frac{\partial F_\theta}{\partial p_j}\,(p_j')^2 \quad \text{for } j\in\{1,\ldots,n\} \quad \text{and} \tag{2.8}$$

$$F_\theta(p_1,\ldots,p_n)\,r'' = -F_{\theta+\frac{\pi}{2}}(p_1,\ldots,p_n)\ , \tag{2.9}$$

for $F_\theta = \mathrm{Re}(e^{-i\theta}(1+ip_1)\cdots(1+ip_n))$ and any $s \in I$. As we will see in Lemma 2.3, θ and $\theta+\pi$ indeed correspond to equivalent ODE systems.

Proposition 2.2. *Suppose $p_1(s), \cdots, p_n(s)$ and $r(s)$ satisfy the ODE system* (2.8) (2.9) *with $p_j'(s) \neq 0$ and $F_\theta(p_1,\ldots,p_n) > 0$. The function u formed by* (2.3) *satisfies the dHYM/LYZ equation* (2.1) *on the domain X.*

It follows that

$$\Big(\frac{1}{p_j'}\Big)' = -2\partial_{p_j}(\log F_\theta) \tag{2.10}$$

and

$$r'' = -\frac{F_{\theta+\frac{\pi}{2}}}{F_\theta}\ .$$

Lemma 2.3. *For any $\theta \in \mathbb{R}$, the polynomial F_θ defined by* (2.7) *has the following properties.*

(i) $F_\theta = -F_{\theta+\pi}$.
(ii) $(F_\theta)^2 + (F_{\theta+\frac{\pi}{2}})^2 = \prod_{j=1}^n (1+p_j^2)$.

(iii) *For any* $j \in \{1, \ldots, n\}$,

$$p_j\, F_\theta - (1+p_j^2)\,\frac{\partial F_\theta}{\partial p_j} = F_{\theta+\frac{\pi}{2}} \ .$$

Proof. Properties (i) and (ii) follow directly from the definition. We compute

$$p_j \mathfrak{F} - (1+p_j^2)\frac{\partial \mathfrak{F}}{\partial p_j} = (p_j - i(1-ip_j))\,\mathfrak{F} = -i\mathfrak{F} \ ,$$

and property (iii) follows. □

2.1. **First integrals of the ODE system.** In this section, we show that the ODE system (2.8) admits n first integrals and the system is integrable.

Suppose that p_j' and $F_\theta(p_1, \ldots, p_n)$ are nonzero. By (2.8),

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\prod_{j=1}^n p_j'}{(F_\theta)^2}\right) = \frac{\prod_{k=1}^n p_k'}{(F_\theta)^2} \sum_{j=1}^n \left(\frac{p_j''}{p_j'} F_\theta - 2\frac{\partial F_\theta}{\partial p_j} p_j'\right) = 0 \ .$$

Thus, there exists a constant $c_0 \neq 0$ such that

$$\frac{\prod_{j=1}^n p_j'}{(F_\theta)^2} = c_0 \ . \tag{2.11}$$

For $j \in \{1, \ldots, n\}$, let

$$\xi_j = \frac{1+p_j^2}{p_j'} \ . \tag{2.12}$$

According to (2.8) and Lemma 2.3 (iii),

$$\begin{aligned} \xi_j' &= 2p_j - (1+p_j^2)\frac{p_j''}{(p_j')^2} \\ &= \frac{2}{F_\theta}\left(p_j\, F_\theta - (1+p_j^2)\frac{\partial F_\theta}{\partial p_j}\right) = \frac{2F_{\theta+\frac{\pi}{2}}}{F_\theta} \end{aligned} \tag{2.13}$$

for any $j \in \{1, \ldots, n\}$. Hence, there exist constants $c_2, \ldots, c_n$ such that

$$\xi_1 - \xi_j = c_j \tag{2.14}$$

for $j \in \{2, \ldots, n\}$. It is convenient to set c_1 to be 0.

By (2.13), Lemma 2.3 (ii), (2.12) and (2.11),

$$\begin{aligned} \frac{(\xi_j')^2}{4} &= \left(\frac{F_{\theta+\frac{\pi}{2}}}{F_\theta}\right)^2 = \frac{\prod_{k=1}^n (1+p_k^2)}{(F_\theta)^2} - 1 = \frac{\prod_{k=1}^n (p_i'\,\xi_k)}{(F_\theta)^2} - 1 \\ &= c_0 \prod_{k=1}^n \xi_k - 1 \end{aligned} \tag{2.15}$$

for $j \in \{1, \ldots, n\}$. Consider $j = 1$, and apply (2.14); one finds that

$$\xi_1' = \pm 2 \sqrt{c_0 \prod_{j=1}^{n} (\xi_1 - c_j) - 1} \ . \tag{2.16}$$

2.2. **Limits of s when $n \geq 3$.** It follows from the definition (2.12) of ξ_j and $p_k' \neq 0$ that ξ_j does not change sign. We would like to argue that $p_k(s)$ cannot be defined for all $s \in \mathbb{R}$ when $n \geq 3$.

At first, suppose that $\xi_i \geq 0$ and is bounded from above, $\xi_i \leq C$ for some $C > 0$. By (2.12), $\arctan p_k = \int \frac{1}{\xi_k} \mathrm{d}s$, and the lower bound of ξ_k implies that p_k must blow up for finite s.

On the other hand, suppose that ξ_1 is unbounded from above. It follows from (2.16) that

$$\int \mathrm{d}s = \frac{\pm 1}{2} \int \frac{\mathrm{d}\xi_1}{\sqrt{c_0 \prod_{j=1}^{n} (\xi_1 - c_j) - 1}} \ .$$

If one considers the improper integral of the right hand side (to $\xi_1 = \infty$), it diverges only when $n = 2$. Therefore, $p_k(s)$ cannot be defined for all $s \in \mathbb{R}$ if $n \geq 3$.

2.3. **The isotropic Case.** In this subsection, we consider the *isotropic* case of Proposition 2.2. That is, $p_1 = \cdots = p_n$. Abbreviate them as p, and let $\varphi = \arctan p$. Equations (2.8) and (2.9) read as follows.

$$(1 + p^2)^{\frac{1}{2}} \ \mathrm{Re}(e^{-i\theta} e^{in\varphi}) \, \frac{p''}{2} = -\, \mathrm{Im}(e^{-i\theta} e^{i(n-1)\varphi}) \, (p')^2 \ , \tag{2.17}$$

$$\mathrm{Re}(e^{-i\theta} e^{in\varphi}) \, r'' = -\, \mathrm{Im}(e^{-i\theta} e^{in\varphi}) \ , \tag{2.18}$$

and we assume that $p' \neq 0$ and $\mathrm{Re}(e^{-i\theta} e^{in\varphi}) \neq 0$. Note that (2.18) implies that

$$r'' = \tan(\theta - n\varphi) \quad \Leftrightarrow \quad \arctan r'' = \theta - n\varphi + k\pi \tag{2.19}$$

where k is the unique integer such that $|\theta - n\varphi + k\pi| < \frac{\pi}{2}$.

For a solution to (2.1), its phase be evaluated at $x_1 = \cdots = x_n = 0$. By (2.19), the phase is

$$n \arctan p + \arctan r'' = n\varphi + \theta - n\varphi + k\pi = \theta + k\pi \ .$$

The above discussion gives the following:

$$(p')^n = c_0 \big(\mathrm{Re}(e^{-i\theta} (1 + ip)^n) \big)^2 \ ,$$

or equivalently, in terms of φ,

$$(\varphi')^n = c_0 \big(\cos(n\varphi - \theta) \big)^2 \ .$$

One infers that

$$(n\varphi - \theta)' = c_0' \big(\cos(n\varphi - \theta) \big)^{\frac{2}{n}} \tag{2.20}$$

for some constant c_0'. By analyzing the linearization at where $n\varphi - \theta - \frac{\pi}{2} \in \mathbb{Z}\pi$, it is not hard to find that for $n \geq 3$, $n\varphi - \theta$ cannot be defined for all s.

Proposition 2.4. *When $n \geq 3$ and $p_1(s) = \cdots = p_n(s)$, there is no non-constant entire solution to* (2.8) *and* (2.9).

3. Entire solutions of dHYM/LYZ

3.1. **On $\mathbb{C}^3$.** When $n = 2$, i.e., on $\mathbb{C}^3$, (2.16) can be solved explicitly and we obtain explicit solutions to the dHYM/LYZ equation. In particular, when the constant c_0 (2.11) is positive, the solution is defined on the whole space.

3.1.1. *When c_0 is positive.* When $c_0 > 0$, the polynomial $c_0\xi_1^2 - c_0c_2\xi_1 - 1$ must have one positive root and one negative root. Denote its roots by α^2 and $-\beta^2$ for $\alpha, \beta > 0$. It follows that $c_0 = (\alpha\beta)^{-2}$, $c_2 = \alpha^2 - \beta^2$, and (2.16) becomes

$$\pm 1 = \frac{\alpha\beta}{2} \frac{\xi_1'}{\sqrt{(\xi_1 - \alpha^2)(\xi_1 + \beta^2)}} \ .$$

We now assume that $\xi_1 > \alpha^2$, and the case where $\xi_1 < -\beta^2$ is similar. By integrating both sides and translating s,

$$\tanh(\frac{s}{\alpha\beta}) = \frac{\sqrt{\xi_1 - \alpha^2}}{\sqrt{\xi_1 + \beta^2}} \qquad \Rightarrow \quad \xi_1 = \alpha^2 \cosh^2(\frac{s}{\alpha\beta}) + \beta^2 \sinh^2(\frac{s}{\alpha\beta}) \ .$$

Together with (2.14),

$$\xi_2 = \beta^2 \cosh^2(\frac{s}{\alpha\beta}) + \alpha^2 \sinh^2(\frac{s}{\alpha\beta}) \ .$$

With ξ_1, p_1 can be found by (2.12):

$$\begin{aligned}
(\arctan p_1)' &= \frac{1}{\xi_1} = \frac{1}{\alpha^2 \cosh^2(\frac{s}{\alpha\beta}) + \beta^2 \sinh^2(\frac{s}{\alpha\beta})} \\
\Rightarrow \quad \arctan p_1 &= \arctan\left(\frac{\beta}{\alpha}\tanh(\frac{s}{\alpha\beta})\right) + \psi_1 \\
\Rightarrow \quad p_1 &= \frac{\alpha \sin\psi_1 \cosh(\frac{s}{\alpha\beta}) + \beta\cos\psi_1 \sinh(\frac{s}{\alpha\beta})}{\alpha\cos\psi_1 \cosh(\frac{s}{\alpha\beta}) - \beta \sin\psi_1 \sinh(\frac{s}{\alpha\beta})} \ ,
\end{aligned} \tag{3.1}$$

for some $\psi_1 \in \mathbb{R}$. Similarly,

$$p_2 = \frac{\beta \sin\psi_2 \cosh(\frac{s}{\alpha\beta}) + \alpha\cos\psi_2 \sinh(\frac{s}{\alpha\beta})}{\beta\cos\psi_2 \cosh(\frac{s}{\alpha\beta}) - \alpha \sin\psi_2 \sinh(\frac{s}{\alpha\beta})} \tag{3.2}$$

for some $\psi_2 \in \mathbb{R}$.

Using (2.11) and Proposition 2.2, one finds that

$$F_\theta = \frac{\alpha\beta}{(\alpha\cos\psi_1\cosh(\frac{s}{\alpha\beta}) - \beta\sin\psi_1\sinh(\frac{s}{\alpha\beta}))(\beta\cos\psi_2\cosh(\frac{s}{\alpha\beta}) - \alpha\sin\psi_2\sinh(\frac{s}{\alpha\beta}))} \ . \tag{3.3}$$

We compute

$$\begin{aligned} \mathfrak{F} &= (1+ip_1)(1+ip_2) \\ &= \frac{F_\theta}{\alpha\beta} e^{i(\psi_1+\psi_2)} \left(\alpha\beta + \frac{i}{2}(\alpha^2+\beta^2)\sinh(\frac{2s}{\alpha\beta})\right) \ . \end{aligned}$$

It follows that

$$\begin{aligned} e^{i\theta} &= e^{i(\psi_1+\psi_2)} \ , \\ F_{\theta+\frac{\pi}{2}} &= \frac{\alpha^2+\beta^2}{2\alpha\beta}\sinh(\frac{2s}{\alpha\beta})\, F_\theta \ . \end{aligned} \tag{3.4}$$

By (2.9),

$$r'' = -\frac{\alpha^2+\beta^2}{2\alpha\beta}\sinh(\frac{2s}{\alpha\beta}) \quad \Rightarrow \quad r = -\frac{\alpha\beta(\alpha^2+\beta^2)}{8}\sinh(\frac{2s}{\alpha\beta}) \ , \tag{3.5}$$

up to adding an affine function in s. With the explicit formulae (3.1), (3.2) and (3.5), the phase of (2.4) is a constant, and is equal to

$$\psi_1 + \psi_2 \tag{3.6}$$

In order for these expressions to be defined for all s, the denominators have to be nonzero for all s. It means that

$$\frac{\alpha}{\beta} \geq \tan\psi_1 \tanh(\frac{s}{\alpha\beta}) \quad \text{and} \quad \frac{\beta}{\alpha} \geq \tan\psi_2 \tanh(\frac{s}{\alpha\beta})$$

for all s, and thus

$$\frac{\alpha}{\beta} \geq |\tan\psi_1| \quad \text{and} \quad \frac{\beta}{\alpha} \geq |\tan\psi_2| \tag{3.7}$$

It follows from $\arctan\frac{\alpha}{\beta} + \arctan\frac{\alpha}{\beta} = \frac{\pi}{2}$ that the phase, $\psi_1+\psi_2$, belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We summarize the discussion in the following proposition.

Proposition 3.1. *For any positive α, β, and $\psi_1, \psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfying* (3.7)*, the potential function $2p_1(s)(x_1)^2 + 2p_2(s)(x_2)^2 + 4r(s)$ is an entire solution to the LYZ equation on $\mathbb{C}^3$, with phase $\psi_1+\psi_2$. Here, $s = \operatorname{Re} z_3$, and the functions are given by* (3.1), (3.2) *and* (3.5). *In particular, the LYZ equation with non-supercritical phase on $\mathbb{C}^3$ admits entire solutions.*

3.1.2. *When c_0 is negative.* Suppose that $c_0 < 0$. From (2.15), the polynomial $c_0\xi_1^2 - c_0c_2\xi_1 - 1$ must be positive somewhere. Therefore, the quadratic polynomial admits two real roots of the same sign. Assume that the roots are $\alpha^2 > \beta^2 > 0$, with $\alpha, \beta > 0$. We leave it for the readers to verify that the case of negative roots corresponds to switching the roles of ξ_1 and ξ_2 in the following discussion.

In this case, $c_0 = -(\alpha\beta)^{-2}$ and $c_2 = \alpha^2 + \beta^2$, and (2.16) becomes

$$\pm 1 = \frac{\alpha\beta}{2}\frac{\xi_1'}{\sqrt{(\alpha^2 - \xi_1)(\xi_1 - \beta^2)}} .$$

With a similar computation,

$$\xi_1 = \alpha^2\cos^2(\frac{s}{\alpha\beta}) + \beta^2\sin^2(\frac{s}{\alpha\beta}) \quad \text{and} \quad \xi_2 = -\beta^2\cos^2(\frac{s}{\alpha\beta}) - \alpha^2\sin^2(\frac{s}{\alpha\beta}) .$$

By integration their reciprocals,

$$\begin{aligned} p_1 &= \frac{\alpha\sin\psi_1\cos(\frac{s}{\alpha\beta}) + \beta\cos\psi_1\sin(\frac{s}{\alpha\beta})}{\alpha\cos\psi_1\cos(\frac{s}{\alpha\beta}) - \beta\sin\psi_1\sin(\frac{s}{\alpha\beta})} , \\ p_2 &= \frac{\beta\sin\psi_2\cos(\frac{s}{\alpha\beta}) - \alpha\cos\psi_2\sin(\frac{s}{\alpha\beta})}{\beta\cos\psi_2\cos(\frac{s}{\alpha\beta}) + \alpha\sin\psi_2\sin(\frac{s}{\alpha\beta})} \end{aligned}$$

for some $\psi_1, \psi_2 \in \mathbb{R}$. The denominators of p_1 and p_2 cannot be nonzero for all s, and the solution cannot be extended to an entire solution in this case.

3.2. **On $\mathbb{C}^{n+1}$ with $n + 1 \geq 4$.** In [13, Theorem 2], Li constructed non-quadratic solutions to the special Lagrangian equation 4.1 with $\theta = 0$ by using the ansatz $f = \frac{1}{2}p(x_3)x_1^2 + q(x_3)x_2 + r(x_3)$.

It suggests that we may obtain entire solutions in higher dimensions by modifying the solutions given by Proposition 3.1 based this type of ansatz. Specifically, when $n \geq 3$, consider

$$\tilde{u}(z_1, z_2, z_3, \ldots, z_n, z_{n+1}) = 2p_1(s)(x_1)^2 + 2p_2(s)(x_2)^2 + 4\sum_{j=3}^{n} q_j(s)x_j + 4r(s) \tag{3.8}$$

where $s = \operatorname{Re} z_{n+1}$ and $x_j = \operatorname{Re} z_j$ for $j = 1, \ldots, n$. After a direct computation, the coefficient matrix of $\partial\bar{\partial}\tilde{u}$ is

$$\begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 & x_1p_1' \\ 0 & p_2 & 0 & \cdots & 0 & x_2p_2' \\ 0 & 0 & 0 & \cdots & 0 & q_3' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & q_n' \\ x_1p_1' & x_2p_2' & q_3' & \cdots & q_n' & \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\tilde{u} \end{bmatrix} . \tag{3.9}$$

It follows that

$$
\begin{aligned}
&\det\left(\mathbf{I}_{n+1} + i\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right]_{1\leq j,k\leq n+1}\right) \\
&= (1+ip_1)(1+ip_2)\left[1 + (x_1)^2\frac{(p_1')^2}{1+ip_1} + (x_2)^2\frac{(p_2')^2}{1+ip_2} + \sum_{j=3}^{n}(q_j')^2\right. \\
&\qquad\qquad \left. + i\Big(\frac{1}{2}p_1''(s)(x_1)^2 + \frac{1}{2}p_2''(s)(x_2)^2 + \sum_{j=3}^{n} q_j''(s)x_j + r''(s)\Big)\right] .
\end{aligned}
$$

Let $\mathfrak{F} = (1+ip_1)(1+ip_2)$, and $F_\theta = \mathrm{Re}(e^{-i\theta}F)$ as before. The LYZ equation (2.1) becomes the following system:

$$
F_\theta(p_1,p_2)\frac{p_j''}{2} = \frac{\partial F_\theta}{\partial p_j}(p_j')^2 \qquad \text{for } j=1,2\ , \tag{3.10}
$$

$$
F_\theta(p_1,p_2)\, q_k'' = 0 \qquad \text{for } k = 3,\ldots,n\ , \tag{3.11}
$$

$$
F_\theta(p_1,p_2)\, r'' = -F_{\theta+\frac{\pi}{2}}(p_1,p_2)\big(1 + \sum_{k=3}^{n}(q_k')^2\big)\ . \tag{3.12}
$$

The equation (3.10) is analyzed in Section 3.1.1. From (3.11), one infers that $q_k(s) = \gamma_k s + \kappa_k$ for some constants γ_k, κ_k. Note that κ_k's do not show up in $\partial\bar\partial\tilde{u}$. By comparing (3.12) with (3.5), it is not hard to find that

$$
r = -\frac{\alpha\beta(\alpha^2+\beta^2)}{8}(1 + \sum_{k=3}^{n}(\gamma_k)^2)\sinh(\frac{2s}{\alpha\beta})\ .
$$

We summarize the discussion in the following theorem.

Theorem 3.2. *Suppose that $n \geq 3$. For any $\alpha, \beta > 0$, $\psi_1, \psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfying (3.7) and $\gamma_3, \ldots, \gamma_n \in \mathbb{R}$, the function $\tilde{u}$ defined by*

$$
\begin{aligned}
&\frac{\alpha\sin\psi_1\cosh(\frac{s}{\alpha\beta}) + \beta\cos\psi_1\sinh(\frac{s}{\alpha\beta})}{\alpha\cos\psi_1\cosh(\frac{s}{\alpha\beta}) - \beta\sin\psi_1\sinh(\frac{s}{\alpha\beta})}2(x_1)^2 + \frac{\beta\sin\psi_2\cosh(\frac{s}{\alpha\beta}) + \alpha\cos\psi_2\sinh(\frac{s}{\alpha\beta})}{\beta\cos\psi_2\cosh(\frac{s}{\alpha\beta}) - \alpha\sin\psi_2\sinh(\frac{s}{\alpha\beta})}2(x_2)^2 \\
&\quad + 4\sum_{k=3}^{n}\gamma_k\, s\, x_k - 4\frac{\alpha\beta(\alpha^2+\beta^2)}{8}\Big(1 + \sum_{k=3}^{n}(\gamma_k)^2\Big)\sinh(\frac{2s}{\alpha\beta})
\end{aligned}
$$

is an entire solution to the LYZ equation (2.1) on $\mathbb{R}^{n+1}$ with phase $\theta = \psi_1+\psi_2$. Here, $x_j = \mathrm{Re}\, z_j$ for $j = 1,\ldots,n$ and $s = \mathrm{Re}\, z_{n+1}$.

In other words, the LYZ equation admits non-polynomial entire solutions on $\mathbb{C}^n$ with any phase within $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark 3.3. More generally, for $u = \sum_{j=1}^n (2p_j(s)(x_j)^2 + 4q_j(s)x_j) + 4r(s)$, the dHYM/LYZ equation (2.1) becomes

$$
\begin{aligned}
F_\theta(p_1,\dots,p_n)\,\frac{p_j''}{2} &= \frac{\partial F_\theta}{\partial p_j}\,(p_j')^2 \qquad \text{for } j=1,\dots,n\ ,\\
F_\theta(p_1,\dots,p_n)\,\frac{q_j''}{2} &= \frac{\partial F_\theta}{\partial p_j}\,p_j'\,q_j' \qquad \text{for } j=1,\dots,n\ ,\\
F_\theta(p_1,\dots,p_n)\,r'' &= \sum_{j=1}^n \frac{\partial F_\theta}{\partial p_j}\,(q_j')^2 - F_{\theta+\frac{\pi}{2}}(p_1,\dots,p_n)\ .
\end{aligned}
$$

4. Solutions to the special Lagrangian equation

According to Leung–Yau–Zaslow in [12], a dHYM connection is mirror to a special Lagrangian sections via the Fourier-Mukai transform under the setting of semi-flat Calabi–Yau metrics. If one works out the transformation with respect to the standard metric on $\mathbb{C}^{n+1}$, each of the solutions of the dHYM/LYZ equation we obtained in previous sections corresponds to a solution of the special Lagrangian equation (4.1). In this section, we explore the geometry of the corresponding special Lagrangian submanifold.

Proposition 4.1. *Let $p_1(s),\cdots,p_n(s)$ and $r(s)$ be solutions of the θ-angle ODE system in Definition 2.1. Consider the following function f defined on a domain $X \subset \mathbb{R}^{n+1}$ by*

$$f(x_1,\dots,x_n,s) = \frac{1}{2}\sum_{j=1}^n p_j(s)x_j^2 + r(s)\ .$$

Then, f satisfies the special Lagrangian equation with angle θ:

$$\operatorname{Im}\left(e^{-i\theta}\det\left(\mathbf{I}_{n+1} + i\left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{1\le j,k\le n+1}\right)\right) = 0\ . \tag{4.1}$$

Proof. Write s as x_{n+1}. The Hessian matrix of f is

$$\left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{1\le j,k\le n+1} = \begin{bmatrix} p_1(s) & 0 & \cdots & 0 & x_1p_1'(s) \\ 0 & p_2(s) & \cdots & 0 & x_2p_2'(s) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_n(s) & x_np_n'(s) \\ x_1p_1'(s) & x_2p_2'(s) & \cdots & x_np_n'(s) & \sum_{j=1}^n \frac{1}{2}p_j''(s)x_j^2 + r''(s) \end{bmatrix}.$$

With this, the computation is the same as that in Section 2. □

The correspondence also holds true for the more general ansatz described in Remark 3.3. To be more precise, suppose that $p_j(s)$, $q_j(s)$ and $r(s)$ obey the system of equations in Remark 3.3. Then, $f = \sum_{j=1}^n (\frac{1}{2}p_j(s)(x_j)^2 + q_j(s)x_j) + r(s)$ satisfies (4.1). Therefore, Proposition 3.1 and Theorem 3.2 lead to the following theorem:

Theorem 4.2. *When $n \geq 3$ and any $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the special Lagrangian equation with angle θ* (4.1) *admits non-polynomial entire solutions on $\mathbb{R}^{n+1}$. The phase of the solution is exactly θ.*

4.1. **Extensions of solutions to the special Lagrangian equation.** According to Proposition 4.1, the graph of ∇f defines a special Lagrangian submanifold in $\mathbb{C}^{n+1}$, which is graphical on a domain $X \subset \mathbb{R}^{n+1}$. However, ∇f may blow up at the boundary of X, and the corresponding special Lagrangian submanifold ceases to be graphical. In this subsection, we demonstrate that the submanifold admits a global extension (cf. [7,11,9]) as a complete (non-graphical), embedded, special Lagrangian in $\mathbb{C}^{n+1}$. It is natural to ask whether similar extension mechanisms can be applied to the dHYM/LYZ formulation on the mirror side.

We begin by recalling the following result of Joyce.

Theorem 4.3 ([9, Theorem[1] 7.1])**.** *Let $w_1, \ldots, w_n : (-\varepsilon, \varepsilon) \to \mathbb{C} \setminus \{0\}$ and $\beta : (-\varepsilon, \varepsilon) \to \mathbb{C} \setminus \{0\}$ be differentiable functions satisfying*

$$\begin{aligned} \frac{\mathrm{d}w_j}{\mathrm{d}t} &= \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_n}, \quad j = 1, \ldots, n \ , \\ \frac{\mathrm{d}\beta}{\mathrm{d}t} &= \overline{w_1 \cdots w_n} \ . \end{aligned} \tag{4.2}$$

Define a subset $N \subset \mathbb{C}^{n+1}$ by

$$N = \left\{ \left(w_1(t)\xi_1, \ldots, w_n(t)\xi_n, -\frac{\xi_1^2 + \cdots + \xi_n^2}{2} + \beta(t) \right) : t \in (-\varepsilon, \varepsilon), \xi_j \in \mathbb{R} \right\}. \tag{4.3}$$

Then N is a special Lagrangian submanifold of $\mathbb{C}^{n+1}$.

To relate our construction with Joyce's theorem, we begin with the expression for the graph of ∇f:

$$(x_1, \ldots, x_n, s) \mapsto \left((1 + ip_1(s))x_1, \ldots, (1 + ip_n(s))x_n, s + i \left(\frac{1}{2} \sum_{j=1}^{n} x_j^2 p_j'(s) + r'(s) \right) \right)$$

where $p_1(s), \ldots, p_n(s)$ and $r(s)$ satisfy (2.8) and (2.9). The angle θ will be specified later. Moreover, assume that the constant (2.11) is positive,

$$\frac{\prod_{j=1}^n p_j'(s)}{(F_\theta)^2} = c_0 > 0 \ , \quad \text{and assume that} \ \ p_j'(s) > 0$$

for $j = 1, \ldots, n$. Other cases can be treated similarly by appropriately adjusting signs, and they correspond to quadrics of other signatures.

[1]The dimension m in Joyce's theorem corresponds to $n + 1$ here. For convenience, we specialize to the case $a = n$. One can also work out the transformation for quadrics of other signature.

We claim that our ansatz corresponds to the solution given by Theorem 4.3 through the following relation:

$$\begin{aligned}&\left((1+ip_1(s))x_1,\ldots,(1+ip_n(s))x_n, s+i\left(\frac{1}{2}\sum_{j=1}^{n}x_j^2p_j'(s)+r'(s)\right)\right)\\&=-i\left(w_1(t)\xi_1,\ldots,w_n(t)\xi_n,-\frac{\xi_1^2+\cdots+\xi_n^2}{2}+\beta(t)\right)\ .\end{aligned}\tag{4.4}$$

To facilitate this, introduce the parameters $(\xi_1,\ldots,\xi_n,t)$ related to $(x_1,\ldots,x_n,s)$ by

$$t=\sqrt{c_0}s\quad\text{and}\quad\xi_j=\sqrt{p_j'(s)}\,x_j$$

for $j=1\ldots,n$. Define the complex-valued functions $\omega_j(t)$ and $\beta(t)$ by

$$\begin{aligned}\omega_j(t)&=i\left(\frac{1+ip_j(s)}{\sqrt{p_j'(s)}}\right)\ ,\\\beta(t)&=i(s+ir'(s))\ .\end{aligned}$$

It remains to verify that this parameterization satisfies the ODE system 4.2. By (2.8),

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+ip_j(s)}{\sqrt{p_j'(s)}}\right)=i\sqrt{p_j'}-(1+ip_j)\sqrt{p_j'}\cdot\frac{\partial_{p_j}F_\theta}{F_\theta}.$$

With the identity $F_\theta-iF_{\theta+\frac{\pi}{2}}=e^{i\theta}\bar{F}$ and the relation $p_jF_\theta-(1+p_j^2)\partial_{p_j}F_\theta=F_{\theta+\frac{\pi}{2}}$ (see Lemma 2.3), it can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+ip_j}{\sqrt{p_j'}}\right)=\frac{\sqrt{p_j'}}{1-ip_j}\cdot\frac{ie^{i\theta}\bar{F}}{F_\theta}\ ,$$

and hence

$$\frac{\mathrm{d}\omega_j}{\mathrm{d}t}=\frac{i}{\bar{\omega}_j}\cdot\overline{\left(\frac{e^{-i\theta}F}{F_\theta}\right)}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}\ .$$

Similarly, it follows from (2.9) that

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}=i(1+ir''(s))\frac{\mathrm{d}s}{\mathrm{d}t}=i\overline{\left(\frac{e^{-i\theta}F}{F_\theta}\right)}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}\ .$$

On the other hand,

$$\omega_1\cdots\omega_n=\prod_{j=1}^{n}\left(\frac{i(1+ip_j(s))}{\sqrt{p_j'(s)}}\right)=\frac{i^nF}{\sqrt{c_0}F_\theta}\ .$$

Therefore, $\omega_j(t)$ and $\beta(t)$ satisfy

$$\frac{\mathrm{d}\omega_j}{\mathrm{d}t} = i\overline{\left(\frac{e^{-i\theta}\omega_1\cdots\omega_n}{i^n\omega_j}\right)} \quad \text{and} \quad \frac{\mathrm{d}\beta}{\mathrm{d}t} = i\overline{\left(\frac{e^{-i\theta}\omega_1\cdots\omega_n}{i^n}\right)} .$$

Finally, by choosing the angle θ such that $e^{i\theta}i^n = -i$, we recover the ODE system (4.2). This verifies the correspondence 4.4.

5. General equations of recursive type

In this section, we fix real constants $c_n, \ldots, c_{-1}$ and study solutions of a general real Hessian equation of the form

$$c_n\sigma_{n+1}(\nabla^2 f) + c_{n-1}\sigma_n(\nabla^2 f) + \cdots + c_0\sigma_1(\nabla^2 f) + c_{-1} = 0. \tag{5.1}$$

Denote the coordinate of $\mathbb{R}^{n+1}$ by $(x_1, \ldots, x_n, s)$, and consider the following ansatz for f:

$$f = \frac{1}{2}\sum_{j=1}^{n} p_j(s)x_j^2 + r(s) . \tag{5.2}$$

We also consider solutions of a general complex Hessian equation of the form

$$c_n\sigma_{n+1}(\partial\bar\partial u) + c_{n-1}\sigma_n(\partial\bar\partial u) + \cdots + c_0\sigma_1(\partial\bar\partial u) + c_{-1} = 0. \tag{5.3}$$

Denote the coordinate of $\mathbb{C}^{n+1}$ by $(z_1, \ldots, z_n, z_{n+1})$, and consider the following ansatz for u:

$$u = 2\sum_{j=1}^{n} p_j(\operatorname{Re} z_{n+1})(\operatorname{Re} z_i)^2 + 4r(\operatorname{Re} z^{n+1}) . \tag{5.4}$$

Proposition 5.1. *Suppose $p_i(s), i = 1 \ldots n$ and $r(s)$ satisfy the following ODE system*

$$\frac{p_i''}{2}\cdot F(p_1, \ldots, p_n) = (p_i')^2\frac{\partial F}{\partial p_i}(p_1, \ldots, p_n) \quad \textit{for } i = 1, \ldots, n \textit{ and} \tag{5.5}$$

$$r''\cdot F(p_1, \ldots, p_n) = -G(p_1, \ldots, p_n), \tag{5.6}$$

where

$$F(p_1, \ldots, p_n) = \sum_{k=0}^{n} c_k\sigma_k(p) \quad \textit{and} \quad G(p_1, \ldots, p_n) = \sum_{k=0}^{n} c_{k-1}\sigma_k(p) .$$

Then a function u of the form (5.4) *satisfies the complex Hessian equation* (5.3)*, and a function f of the form* (5.2) *satisfies the real Hessian equation* (5.1)*.*

Proof. The proof extends the argument of Proposition 2.2 (the dHYM/LYZ case) and Proposition 4.1 (the special Lagrangian case). We only deal with the real case (5.1) here and the complex case (5.3) can be dealt with similarly. By Lemma A.1, (5.1), under the ansatz (5.2), becomes

$$0 = \Xi_0(s) + \sum_{i=1}^{n} (x^i)^2 \Xi_i(s) \tag{5.7}$$

where

$$\Xi_0(s) = (c_{n-1}\sigma_n(p) + \cdots + c_0\sigma_1(p) + c_{-1}) + (c_n\sigma_n(p) + \cdots + c_1\sigma_1(p) + c_0)\, r'' \quad \text{and}$$
$$\Xi_i(s) = (c_n\sigma_n(p) + \cdots + c_1\sigma_1(p) + c_0)\, \frac{p_i''}{2} - (c_n\sigma_{n-1}(p|i) + \cdots + c_1\sigma_1(p|i) + c_1)\, (p_i')^2 \ .$$

As before, $\sigma_k(p|i) = \sigma_k(p_1, \ldots, \widehat{p_i}, \ldots, p_n)$. Note that (5.7) is equivalent to $\Xi_i(s) = 0 = \Xi_0(s)$ for $i = 1, \ldots, n$. $\square$

Again, the main task is to solve (5.5) for p_i, and then r can be found by integrating (5.6) twice. Let $R_i(s)$ be $1/p_i'(s)$, and (5.5) becomes the following first order system

$$\begin{cases} p_i' &= \frac{1}{R_i} \ , \\ R_i' &= -2\frac{\partial \log F}{\partial p_i} \end{cases} \tag{5.8}$$

for $i = 1, \ldots, n$.

Remark 5.2. It is not hard to see that with the symplectic form $\sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}R_i$, the ODE system is Hamiltonian with respect to $H(p_i, R_i) = \sum_{i=1}^{n} \log R_i + 2\log F$ and thus $R_1 \cdots R_n F^2$ is a first integral. However, since there is no other continuous symmetry of H for general F, this perspective is not particularly useful.

Definition 5.3. The ODE system (5.5) is said to be of recursive type (a_0, a_1) if there exist real numbers a_0 and a_1 such that the coefficients of F satisfy the recursive relation:

$$c_{k-1} = c_k a_1 - c_{k+1} a_0 \quad \text{for } k = 1, \ldots, n-1 \ .$$

In particular, $c_0, c_1, \cdots, c_{n-2}$ are determined by a_0, a_1, c_{n-1} and c_n. All recursive types F can be classified according to the following proposition (c_n and c_{n-1} are not necessarily real in this proposition):

Proposition 5.4. *Let $n \geq 1$ and let $a_0, a_1 \in \mathbb{R}$. Suppose*

$$F(p_1, \ldots, p_n) = \sum_{k=0}^{n} c_k\, \sigma_k(p_1, \ldots, p_n)$$

is a symmetric polynomial in $(p_1, \ldots, p_n)$, where σ_k denotes the k-th elementary symmetric function. Assume the coefficients $\{c_k\}$ satisfy the recurrence

$$c_{k-1} = a_1\, c_k - a_0\, c_{k+1} \ , \qquad k = 1, 2, \ldots, n-1 \ .$$

Let r_1, r_2 be the (not necessarily distinct) roots of the quadratic equation

$$r^2 - a_1 r + a_0 = 0 \ .$$

Then F must take one of the following forms:

Case 1: $r_1 \neq r_2$.

$$F(p_1,\ldots,p_n) = A\prod_{i=1}^{n}(p_i + r_1) + B\prod_{i=1}^{n}(p_i + r_2) \ ,$$

where the constants A, B are given by

$$A = \frac{c_{n-1} - c_n r_2}{r_1 - r_2} \quad \text{and} \quad B = \frac{-(c_{n-1} - c_n r_1)}{r_1 - r_2} \ .$$

Case 2: $r_1 = r_2 = u \neq 0$.

$$F(p_1,\ldots,p_n) = A\prod_{i=1}^{n}(p_i + u) + Bu \cdot \frac{\mathrm{d}}{\mathrm{d}u}\Big(\prod_{i=1}^{n}(p_i + u)\Big) \ ,$$

where $A = c_n$ and $B = \frac{c_{n-1}}{u} - c_n$.

Case 3: $r_1 = r_2 = 0$.

$$F(p_1,\ldots,p_n) = c_n\,\sigma_n(p_1,\ldots,p_n) + c_{n-1}\,\sigma_{n-1}(p_1,\ldots,p_n) \ .$$

Proof. Case 1: $r_1 \neq r_2$. One can verify directly that the sequence $B_k = Ar_1^{n-k} + Br_2^{n-k}$ satisfies the recurrence $c_{k-1} = a_1\,c_k - a_0\,c_{k+1}$ for $k = 1, 2, \ldots, n-1$, with initial conditions $B_n = c_n$, $B_{n-1} = c_{n-1}$. Moreover, the identity

$$\sum_{k=0}^{n} B_k\,\sigma_k(p_1,\ldots,p_n) = A\prod_{i=1}^{n}(p_i + r_1) + B\prod_{i=1}^{n}(p_i + r_2)$$

follows from the generating function for elementary symmetric polynomials.

Case 2: $r_1 = r_1 = u \neq 0$. In this case, the recurrence becomes $c_{k-1} = 2u\,c_k - u^2\,c_{k+1}$. It is not hard to verify that $B_k = Au^{n-k} + B(n-k)u^{n-k}$ satisfies the recurrence, with initial conditions $B_n = c_n$, $B_{n-1} = c_{n-1}$. Furthermore,

$$\sum_{k=0}^{n} B_k\,\sigma_k(p_1,\ldots,p_n) = A\prod_{i=1}^{n}(p_i + u) + Bu \cdot \frac{\mathrm{d}}{\mathrm{d}u}\Big(\prod_{i=1}^{n}(p_i + u)\Big)$$

follows from term-wise differentiation of the generating polynomial.

Case 3: $r_1 = r_2 = 0$. In this case, the recurrence becomes $c_{k-1} = 0$, implying $c_0 = \cdots = c_{n-2} = 0$. Only c_{n-1} and c_n may be nonzero, and thus $F(p) = c_n\sigma_n(p) + c_{n-1}\sigma_{n-1}(p)$. □

In case 1, by setting $a_0 = 1, a_1 = 0$, we have $r_1 = i, r_2 = -i$, and this corresponds to the dHYM/LYZ equation in the complex case and the special Lagrangian equation in the real case. Case 3 gives the Monge–Ampère equation and the J-equation.

Theorem 5.5. *Suppose the ODE system* (5.5) *is of recursive type* (a_0, a_1). *Define*

$$\xi_i = \frac{p_i^2 + a_1 p_i + a_0}{p_i'} \ , \quad i = 1, \ldots, n \ ,$$

then

$$\xi_i - \xi_1 \ , \quad i = 2, \ldots, n$$

are first integrals of the system. In addition, ξ_i, $i = 1, \ldots, n$ *satisfy the following ODE system:*

$$(\xi_i)'^2 = (a_1^2 - 4a_0) + \frac{4}{c}(c_{n-1}^2 - a_1 c_{n-1} c_n + a_0 c_n^2) \prod_{j=1}^{n} \xi_j \ ,$$

where c *is the constant such that* $F^2 = c\, p_1' \cdots p_n'$.

Proof. Let $\xi_i = R_i(p_i^2 + a_1 p_i + a_0)$. It follows from a direct computation that

$$\xi_i' = \frac{1}{F}(-2(p_i^2 + a_1 p_i + a_0)\frac{\partial F}{\partial p_i} + (2p_i + a_1)F).$$

By Lemma A.2,

$$(\xi_i')^2 = (a_1^2 - 4a_0) + \frac{4}{F^2}(c_{n-1}^2 - a_1 c_{n-1} c_n + a_0 c_n^2) \prod_{j=1}^{n} (p_j^2 + a_1 p_j + a_0)$$

This together with $F^2 = c\, p_1' \cdots p_n'$ and the definition of ξ_i finishes the proof of this theorem. $\square$

Appendix A. Some Algebraic Calculations

Lemma A.1. *For the* $(n+1) \times (n+1)$ *Hermitian matrix*

$$H_{n+1} = \begin{bmatrix} P_1 & 0 & \cdots & 0 & Q_1 \\ 0 & P_2 & \cdots & 0 & Q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_n & Q_n \\ \bar{Q}_1 & \bar{Q}_2 & \cdots & \bar{Q}_n & R \end{bmatrix} \ ,$$

$$\begin{aligned} \det(\lambda \mathbf{I}_{n+1} - H_{n+1}) &= (\lambda - P_1)(\lambda - P_2) \cdots (\lambda - P_n)(\lambda - R) \\ &\quad - \sum_{i=1}^{n} |Q_i|^2 (\lambda - P_1) \cdots (\widehat{\lambda - P_i}) \cdots (\lambda - P_n) \ . \end{aligned}$$

Proof. When $n = 2$ or 3, the assertion can be proved by a direct computation.

Suppose this lemma is true when the size is no greater than n. When the size is $n+1$, expand $\det(\lambda \mathbf{I}_{n+1} - H_{n+1})$ along the first column.

$$\det \begin{bmatrix} \lambda - P_1 & 0 & \cdots & 0 & -Q_1 \\ 0 & \lambda - P_2 & \cdots & 0 & -Q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda - P_n & -Q_n \\ -\bar{Q}_1 & -\bar{Q}_2 & \cdots & -\bar{Q}_n & \lambda - R \end{bmatrix} = (\lambda - P_1) \cdot \det \begin{bmatrix} \lambda - P_2 & \cdots & 0 & -Q_2 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda - P_n & -Q_n \\ -\bar{Q}_2 & \cdots & -\bar{Q}_n & \lambda - R \end{bmatrix}$$
$$+ (-1)^n(-\bar{Q}_1) \cdot \det \begin{bmatrix} 0 & \cdots & 0 & -Q_1 \\ \lambda - P_2 & \cdots & 0 & -Q_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda - P_n & -Q_n \end{bmatrix}$$

It follows from the induction hypothesis that the first term on the right hand side is equal to

$$(\lambda - P_1) \cdot \left[(\lambda - P_2) \cdots (\lambda - P_n)(\lambda - R) - \sum_{j=2}^{n} |Q_j|^2 (\lambda - P_2) \cdots (\widehat{\lambda - P_j}) \cdots (\lambda - P_n) \right] .$$

A direct computation on the determinant shows that the second term on the right hand side is equal to

$$(-1)^n(-\bar{Q}_1) \cdot \left[(-1)^{n-1}(-Q_1)(\lambda - P_2) \cdots (\lambda - P_n) \right] .$$

Putting these together finishes the proof of this lemma. □

Lemma A.2. *Let a_0 a_1, c_{n-1}, and c_n be real numbers. Let $q(x)$ be the quadratic polynomial $q(x) = x^2 + a_1 x + a_0$ and $F(p_1, \ldots, p_n)$ be the symmetric polynomial in $p_1, \ldots, p_n$ given by*

$$F(p_1, \ldots, p_n) = \sum_{k=0}^{n} c_k \sigma_k$$

with

$$c_{k-1} = a_1 c_k - a_0 c_{k+1}, k = 1, \ldots, n-1$$

and σ_k the k-th symmetric function in $p_1, \cdots, p_n$. Then for every $i = 1, \ldots, n$

$$(q'(p_i) F - 2q(p_i) \partial_{p_i} F)^2 = (a_1^2 - 4a_0) F^2 + 4(c_{n-1}^2 - a_1 c_{n-1} c_n + a_0 c_n^2) \prod_{j=1}^{n} q(p_j) . \tag{A.1}$$

Proof. Note that F only depends on the coefficients $a_0, \cdots, a_n$. For the sake of the proof, we introduce a temporary constant

$$c_{-1} = a_1 c_0 - a_0 c_1$$

which is distinct from the earlier c_{-1} and does not appear in the final formula.

The first step is to prove that

$$q'(p_i)F - 2q(p_i)\partial_{p_i}F = -a_1F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k \ . \tag{A.2}$$

For $i = 1, \dots, n$ and $k = 0, \dots, n-1$, denote by $\sigma_k(p|i)$ the k-th symmetric function of $p_1, \cdots, p_n$ with p_i excluded. We have

$$\sigma_k(p) = p_i\sigma_{k-1}(p|i) + \sigma_k(p|i) \ , \quad k = 0, \dots, n \tag{A.3}$$

where we adopt the convention that $\sigma_{-1}(p|i) = 0$ and $\sigma_n(p|i) = 0$. In particular, $\partial_{p_i}F = \sum_{k=0}^{n} c_k\sigma_{k-1}(p|i)$.

With these, we compute that $q'(p_i)F - 2q(p_i)\partial_{p_i}F$ is given by

$$\begin{aligned}
&(2p_i + a_1)\sum_{k=0}^{n} c_k(p_i\sigma_{k-1}(p|i) + \sigma_k(p|i)) - 2(p_i^2 + a_1p_i + a_0)\sum_{k=0}^{n} c_k\sigma_{k-1}(p|i) \\
&= (-a_1p_i - 2a_0)\sum_{k=0}^{n} c_k\sigma_{k-1}(p|i) + (2p_i + a_1)\sum_{k=0}^{n} c_k\sigma_k(p|i) \\
&= (-a_1p_i - 2a_0)\sum_{k=0}^{n-1} c_{k+1}\sigma_k(p|i) + (2p_i + a_1)\sum_{k=0}^{n-1} c_k\sigma_k(p|i) \\
&= \sum_{k=0}^{n-1}\Big[p_i(2c_k - a_1c_{k+1}) + c_ka_1 - 2c_{k+1}a_0\Big]\sigma_k(p|i) \\
&= \sum_{k=0}^{n} p_i(2c_{k-1} - a_1c_k)\sigma_{k-1}(p|i) + \sum_{k=0}^{n-1}(c_ka_1 - 2c_{k+1}a_0)\sigma_k(p|i),
\end{aligned}$$

where the indexes are shifted after the second and the fourth equalities. With the recursive relation and the definition of c_{-1}, we have $c_ka_1 - 2c_{k+1}a_0 = 2c_{k-1} - a_1c_k$ for $k = 0, \cdots n-1$. Regrouping terms yields:

$$\sum_{k=0}^{n}(2c_{k-1} - a_1c_k)(p_i\sigma_{k-1}(p|i) + \sigma_k(p|i)) \ .$$

Applying (A.3), we obtain the desired expression

$$-a_1F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k(p)$$

and complete the proof of (A.2).

We now verify the identity (A.1) case-by-case by using Proposition 5.4.

<u>Case 1</u>: $r_1 \neq r_2$. Suppose $r_1 \neq r_2$ are distinct (real or complex conjugate) roots of the characteristic equation $r^2 - a_1r + a_0 = 0$. Then, $a_1 = r_1 + r_2$ and $a_0 = r_1r_2$. By Proposition 5.4,

the function F must be of the form

$$F = \sum_{k=0}^{n} (Ar_1^{n-k} + Br_2^{n-k})\sigma_k = A\,P + B\,Q, \tag{A.4}$$

where

$$P = \sum_{k=0}^{n} r_1^{n-k}\sigma_k = \prod_{j=1}^{n}(p_j + r_1) \quad \text{and} \quad Q = \sum_{k=0}^{n} r_2^{n-k}\sigma_k = \prod_{j=1}^{n}(p_j + r_2)$$

for some constants A, B. (These constants may be complex if r_1 and r_2 are complex conjugates.)

The coefficients of F are therefore given by $c_k = Ar_1^{n-k} + Br_2^{n-k}$. Using this, we compute

$$\sum_{k=0}^{n} c_{k-1}\sigma_k = Ar_1 P + Br_2 Q\ . \tag{A.5}$$

Substituting (A.4) and (A.5), we find

$$\left(-a_1 F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k\right)^2 = (r_1 - r_2)^2 \left[F^2 - 4AB\prod_{j=1}^{n} q(p_j)\right]\ ,$$

where $q(p_j) = (p_j + r_1)(p_j + r_2)$. Expressing $(r_1 - r_2)^2$ and AB in terms of a_0, a_1, c_{n-1}, and c_n, we obtain

$$\left(-a_1 F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k\right)^2 = (a_1^2 - 4a_0)F^2 + 4(c_{n-1}^2 - a_1 c_{n-1} c_n + a_0 c_n^2)\prod_{j=1}^{n} q(p_j)\ .$$

Case 2: $r_1 = r_2 = u \neq 0$. Suppose the characteristic equation has a repeated root u, so that $a_1 = 2u$ and $a_0 = u^2$. Then F takes the form

$$F = \sum_{k=0}^{n} (A + B(n-k))\,u^{n-k}\sigma_k = A\,P + B\,Q\ , \tag{A.6}$$

where

$$P = \sum_{k=0}^{n} u^{n-k}\sigma_k = \prod_{j=1}^{n}(p_j + u) \quad \text{and} \quad Q = \sum_{k=0}^{n}(n-k)u^{n-k}\sigma_k = u\frac{\mathrm{d}P}{\mathrm{d}u}\ .$$

Thus, the coefficients of F are

$$c_k = Au^{n-k} + B(n-k)u^{n-k}\ . \tag{A.7}$$

From this, we compute

$$\begin{aligned}\sum_{k=0}^{n} c_{k-1}\sigma_k &= \sum_{k=0}^{n}\Big[Au^{n-k+1} + B(n-k+1)u^{n-k+1}\Big]\sigma_k \\ &= u\sum_{k=0}^{n}\Big[Au^{n-k} + B(n-k)u^{n-k} + Bu^{n-k}\Big]\sigma_k \\ &= u(A+B)P + uBQ\ .\end{aligned} \tag{A.8}$$

Combining (A.6) and (A.8), we have

$$-a_1F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k = a_1BP\ ,$$

and hence

$$\Big(-a_1F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k\Big)^2 = a_1^2B^2\prod_{j=1}^{n} q(p_j)\ .$$

Since $a_1 = 2u$,

$$B = -\frac{2}{a_1}\Big(c_{n-1} - \frac{a_1}{2}c_n\Big)\ ,$$

so the identity becomes

$$\Big(-a_1F + 2\sum_{k=0}^{n} c_{k-1}\sigma_k\Big)^2 = 4\Big(c_{n-1} - \frac{a_1}{2}c_n\Big)^2\prod_{j=1}^{n} q(p_j)\ .$$

Case 3: $r_1 = r_2 = 0$. In this case, $a_0 = a_1 = 0$, and the recurrence implies $c_k = 0$ for all $k < n-1$. Therefore, $F = c_n\sigma_n + c_{n-1}\sigma_{n-1}$, and the identity (A.1) follows immediately by a direct substitution. $\square$

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