

DYNAMICAL STABILITY OF MINIMAL LAGRANGIANS IN KÄHLER-EINSTEIN MANIFOLDS OF NON-POSITIVE CURVATURE

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ABSTRACT. It is known that minimal Lagrangians in Kähler–Einstein manifolds of non-positive scalar curvature are linearly stable under Hamiltonian deformations. We prove that they are also stable under the Lagrangian mean curvature flow, and therefore establish the equivalence between linear stability and dynamical stability.

Specifically, if one starts the mean curvature flow with a Lagrangian which is C^1 -close and Hamiltonian isotopic to a minimal Lagrangian, the flow exists smoothly for all time, and converges to that minimal Lagrangian. Due to the work of Neves [12], this cannot be true for C^0 -closeness.

1. INTRODUCTION

1.1. Stability of Minimal Submanifold and Mean Curvature Flow. Given a submanifold in a Riemannian manifold $F_0 : \Sigma \hookrightarrow (M, g)$, the mean curvature flow deforms it according to

$$\frac{\partial F_t}{\partial t} = H_t$$

where H_t is the mean curvature vector of $F_t(L)$. A natural question is the dynamical stability at a stationary state: if one starts the mean curvature flow with a submanifold close to a minimal one, will the flow exist for all time, and converge to a minimal submanifold? As one may expect, it depends on the precise meaning of closeness.

Since the flow is the negative gradient flow of the volume functional, its second variation shall give some insights. The Jacobi operator on a minimal submanifold Σ is $(\bar{\nabla}^\perp)^* \bar{\nabla}^\perp - (\mathcal{R} + \mathcal{A})$, where \mathcal{R} is constructed from the curvature of (M, g) , and \mathcal{A} is a quadratic expression in the second fundamental form of Σ . It immediately implies that when $\mathcal{R} + \mathcal{A} \in \text{End}(N\Sigma)$ is negative definite everywhere, Σ is strictly stable in the linear sense. Namely, $\delta_W^2 \text{Vol}(\Sigma) > 0$ for any non-trivial variation W . The negativity of $\mathcal{R} + \mathcal{A}$ is referred to as *strong stability*. In [21], the second-named author and M.-T. Wang proved that a strongly stable minimal submanifold is dynamically stable in C^1 .

Theorem 1.1 ([21, Theorem B]). *Let $\Sigma^n \subset (M, g)$ be a compact, oriented, minimal submanifold which is strongly stable. Then, if Γ is an n -dimensional submanifold which is close to Σ in C^1 , the mean curvature flow Γ_t with $\Gamma_0 = \Gamma$ exists for all time, and Γ_t converges smoothly to Σ as $t \rightarrow \infty$.*

In [8], Lotay and Schulze extended this theorem to a considerably weaker setting.

1.2. Lagrangian Submanifold. When the ambient manifold comes with additional structures, one can refine the discussion by considering submanifolds and variations with additional properties. This paper concerns about the stability of minimal Lagrangian submanifolds. Recall that a half-dimensional submanifold L in a Kähler manifold (M, g, J, ω) is called a Lagrangian submanifold if the restriction of ω on L vanishes. Motivated by the study of moduli space of Calabi–Yau manifolds, minimal Lagrangian and Lagrangian mean curvature flow have received much attention.

Suppose that L is a compact, oriented, minimal Lagrangian. To incorporate the Lagrangian condition into the second variation of the volume, one has to understand the variation fields of nearby Lagrangians. Note that a vector field W normal to L can be identified with the 1-form $\xi_W = \omega(W, \cdot)|_L$ on L . It turns out that the infinitesimal condition of Lagrangian deformations is ξ_W being d-closed. Since harmonic 1-forms are finite dimensional, one focuses on exact variations: $\xi_W = du$ for $u \in C^\infty(L; \mathbb{R})$, which are called *Hamiltonian variations*.

The second variation of the volume functional along Hamiltonian variations is derived by Oh [13, 14]. When the ambient manifold M is Kähler–Einstein $\text{Ric}_g = \kappa g$, the formula reads

$$\delta_{du}^2 \text{Vol}(L) = \int_L (-\langle d(\Delta_L u), du \rangle - \kappa |du|^2) dV_L, \quad (1.1)$$

where Δ_L is the Laplace–Beltrami operator of the induced metric. In particular, a minimal Lagrangian in a Kähler–Einstein manifold of non-positive scalar curvature is always stable under Hamiltonian variations.

On the other hand, if one starts the mean curvature flow with a Lagrangian submanifold, the Lagrangian condition needs not to be preserved along the flow. As proved by Smoczyk [17] and Oh [15], the ambient space being Kähler–Einstein is a natural sufficient condition for the preservation of the Lagrangian condition. For general symplectic manifolds, there are some attempts to modify the mean curvature flow in order to preserve the Lagrangian condition; see for example [19].

1.3. Dynamical Stability of Minimal Lagrangians. Given a Hamiltonian variation du , one can easily construct a nearby Lagrangian as follows. The Weinstein theorem says that the tubular neighborhood of a Lagrangian L in a symplectic manifold (M, ω) is always modeled on the cotangent bundle of L with the canonical symplectic form ω_L . To be more precise, there

exist an open neighborhood U of L in T^*L and an embedding $\varphi : U \rightarrow M$ such that $\varphi^*\omega = \omega_L$ and the restriction of φ on the zero section is the given inclusion. With the help of the Weinstein neighborhood, $(\varphi \circ du)(L)$ is a Lagrangian submanifold. Since its position is given by du , the induced metric is determined by the second order derivative D^2u .

We are now ready to state the main result of this paper.

Theorem 1.2 (Theorem 6.1). *Suppose that (M, g, J, ω) is a Kähler–Einstein manifold of non-positive scalar curvature. Let $L \subset M$ be a compact, oriented, minimal Lagrangian, and fix a Weinstein neighborhood $\varphi : U \subset T^*L \rightarrow M$. Then, there exists $\varepsilon > 0$ such that for any $u_0 \in C^\infty(L; \mathbb{R})$ with $|u_0| + |du_0| + |D^2u_0| < \varepsilon$, the Lagrangian mean curvature flow starting from $\varphi \circ du_0 : L \hookrightarrow M$ exists for all time. Moreover, as $t \rightarrow \infty$, the flow converges smoothly to L .*

The above norm is the static norm. To say more, endow $L^n \subset M$ with the induced metric σ . The notation D^2u means the Riemannian Hessian in σ , and all the norms are taken by using σ . For the generalized Lagrangian mean curvature flow in the cotangent bundle of Smoczyk, Tsui and Wang [18], a similar result was proved by Jin and Liu [6].

Here is the key ingredient of the proof. An $(n, 0)$ -form is introduced in Construction 2.4. It is not holomorphic when $\kappa \neq 0$, but still induces a well-behaved angle function on Lagrangians. This angle function helps us to improve the technique of [21] and to identify a monotone quantity along the flow when $\kappa \leq 0$, which is Proposition 5.3.

We make a comparison between Theorem 1.1 and 1.2. As explained in [21, section 3.2], a minimal Lagrangian L in a Kähler–Einstein manifold is strongly stable if $\text{Ric}_\sigma - \kappa\sigma$ is positive definite. In particular, (L, σ) has positive Ricci curvature when $\kappa = 0$, and thus has topological constraints. For example, Theorem 1.2 is applicable for a minimal Lagrangian subtorus in a flat torus, but Theorem 1.1 is not. In a nutshell, Theorem 1.1 requires a stronger condition, while its conclusion works not only for Hamiltonian variations.

Note that adding a constant to u_0 leaves du_0 unchanged. By requiring u_0 to vanish at some point, the smallness of $|du_0|$ implies the smallness of $|u_0|$. This observation leads to the following rephrasing of Theorem 1.2 in a more symplectic geometric flavor.

Corollary 1.3. *Suppose that L is a compact, oriented, minimal Lagrangian in a Kähler–Einstein manifold of non-positive scalar curvature. If one starts the mean curvature flow of a Lagrangian which is close to L in C^1 and is Hamiltonian isotopic to L , then the flow exists for all time, and converges smoothly to L as $t \rightarrow \infty$.*

Due to the work of Neves [12], one may construct a Lagrangian \tilde{L} Hamiltonian isotopic to L which is arbitrarily close to L in C^0 , but the Lagrangian mean curvature flow starting at \tilde{L} will develop a singularity in finite time. Therefore, it is not possible to improve Corollary 1.3

to C^0 . If one adopts a weaker notion of the mean curvature flow, it is possible to flow through the singularity for strongly stable L . See the work [8, section 3.1] by Lotay and Schulze.

Acknowledgement. The authors are grateful to Wei-Bo Su, Valentino Tosatti, and Mao-Pei Tsui for helpful discussions. The authors thank Mu-Tao Wang for suggestions on the earlier draft of this paper. This research is supported in part by the Taiwan NSTC grants 112-2636-M-002-003 and 112-2628-M-002-004-MY4.

2. PRELIMINARIES

2.1. Kähler–Einstein metric. Let (M, g, J, ω) be a Kähler manifold of complex dimension n . It is known (see for instance [11, ch. 3]) that the Levi-Civita connection naturally induces a connection on the canonical line bundle K_M , whose curvature form $\sqrt{-1}\rho$ is equivalent to the Ricci curvature of g . Specifically,

$$\rho(X, Y) = \text{Ric}_g(JX, Y) \tag{2.1}$$

for any tangent vectors X, Y . The metric is Kähler–Einstein if and only if

$$\rho = \kappa \omega \tag{2.2}$$

for some constant $\kappa \in \mathbb{R}$.

2.2. Lagrangian submanifold. A half-dimensional submanifold $L \subset M$ is called a *Lagrangian* submanifold if $\omega|_L \equiv 0$.

Denote by τ_L the *tautological* 1-form on T^*L . A local coordinate $\{x^i\}_{i=1}^n$ for L induces a coordinate $\{y_i\}_{i=1}^n$ for the fibers of T^*L over the coordinate chart, and $\tau_L = \sum_{i=1}^n y_i dx^i$. Its negative exterior derivative, $\omega_L = -d\tau_L$, is a canonically defined symplectic form on T^*L .

The Weinstein neighborhood theorem ([9, ch. 3]) asserts that the tubular neighborhood of a Lagrangian submanifold is symplectomorphic to its cotangent bundle. Such a parametrized neighborhood is not unique, and any choice suffices for the purpose of this paper.

Theorem 2.1. *Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Then, there exists an open neighborhood U of the zero section in T^*L , and an embedding $\varphi : U \hookrightarrow M$ such that the restriction of φ on the zero section is the given inclusion $L \subset M$, and $\varphi^*\omega = \omega_L$.*

With the Weinstein neighborhood, a 1-form ξ on L gives an embedding of L in T^*L . It is straightforward to verify that $\xi^*\tau_L = \xi$. Thus, $\xi(L)$ is Lagrangian in (T^*L, ω_L) if and only if $d\xi = 0$. In particular, for any function u on L , $du(L)$ is always Lagrangian.

2.3. Extrinsic geometry of Lagrangians. Let $L \subset (M, g, J, \omega)$ be a Lagrangian submanifold in a Kähler manifold. One has the natural bundle isomorphisms:

$$NL: \text{ normal bundle of } L \xrightarrow{J} TL \xrightarrow{\flat} T^*L, \quad (2.3)$$

and they are parallel with respect to the connections induced by the Levi-Civita connection. A direct computation shows that the composition of the maps in (2.3) is $W \in NL \mapsto \omega(W, \cdot)|_L$.

Definition 2.2. Let H be the mean curvature vector of a Lagrangian $L \subset M$ in a Kähler manifold. Denote $\omega(H, \cdot)|_L$ by α_L , which is called the *mean curvature 1-form* of L .

The mean curvature 1-form is related to the complex structure of M by the following lemma, which is proved in [3, sec. III.2.D] and [15, Prop. 2.2].

Lemma 2.3. *Suppose that L is an oriented Lagrangian in a Kähler manifold M . Consider the line bundle $K_M|_L$ with the connection induced by the Levi-Civita connection, which is denoted by ∇ . Then, $K_M|_L$ admits a unique section Ω_L whose restriction on the tangent space of L coincides with the volume form dV_L of L . The section Ω_L is of constant length, and*

$$\nabla \Omega_L = \sqrt{-1} \alpha_L \otimes \Omega_L \quad (2.4)$$

Suppose that $\{\sigma^i\}_{i=1}^n$ is a local, oriented, orthonormal frame for T^*L . The section Ω_L is given by $(\sigma^1 + \sqrt{-1}J(\sigma^1)) \wedge \cdots \wedge (\sigma^n + \sqrt{-1}J(\sigma^n))$.

2.4. Mean curvature 1-form of nearby Hamiltonian equivalent Lagrangians.

Construction 2.4. Suppose that $L \subset (M, g, J, \omega)$ is an oriented, *minimal Lagrangian* submanifold in a *Kähler–Einstein* manifold, $\text{Ric}_g = \kappa g$. Fix a Weinstein neighborhood given by Theorem 2.1, $\varphi: U \rightarrow M$.

For any $z \in \varphi(U)$, $\gamma_z(t) = \varphi(t \cdot \varphi^{-1}(z))$ for $t \in [0, 1]$ is a curve connecting z to L , which corresponds to a line segment in T^*L . Take the section Ω_L of $K_M|_L$ produced by Lemma 2.3. Parallel transport Ω_L along γ_z for any $z \in \varphi(U)$ gives a smooth section of K_M over $\varphi(U)$, which will be denoted by Ω_φ . Note that Ω_φ is of constant length, and gives the exponential gauge for K_M along the fibers of T^*L (under φ).

Due to the minimality of L , (2.4) says that Ω_L is parallel along L . Since M is Kähler–Einstein, the curvature of K_M is $\sqrt{-1}\kappa\omega$ (2.2). By the calculation for exponential gauge in [24, sec. 2], one finds that on $\varphi(U)$,

$$\bar{\nabla} \Omega_\varphi = -\sqrt{-1}\kappa \tau_L \otimes \Omega_\varphi, \quad (2.5)$$

where $\bar{\nabla}$ is the connection of K_M . We omit $(\varphi^{-1})^*$ on τ_L for simplicity.

Remark 2.5. Suppose that M is a Calabi–Yau manifold with a holomorphic volume form Ω , and L is calibrated by $\text{Re}\Omega$. The output Ω_φ of the above construction coincides with Ω .

For an oriented Lagrangian submanifold $\Gamma \subset \varphi(U)$, it follows from [3, sec. III.1] that the restriction of Ω_φ on the tangent space of Γ is $e^{\sqrt{-1}\theta}dV_\Gamma$ for some function $e^{\sqrt{-1}\theta} : \Gamma \rightarrow \mathbb{S}^1$. It follows from Lemma 2.3 that $\Omega_\varphi = e^{\sqrt{-1}\theta}\Omega_\Gamma$ on Γ . By (2.4), we find that on Γ ,

$$\nabla\Omega_\varphi = \nabla(e^{\sqrt{-1}\theta}\Omega_\Gamma) = \sqrt{-1}(d\theta + \alpha_\Gamma) \otimes \Omega_\varphi .$$

Comparing it with the restriction of (2.5) on Γ gives

$$\alpha_\Gamma = -d\theta - \kappa \tau_L|_\Gamma . \quad (2.6)$$

For a function $u : L \rightarrow \mathbb{R}$, applying (2.6) to $\Gamma = \varphi \circ du$ gives the following lemma.

Lemma 2.6. *Under the setting of Construction 2.4, suppose the $u : L \rightarrow \mathbb{R}$ is a smooth function so that $du(L) \subset U$. Then, the mean curvature 1-form of $F_u = \varphi \circ du$ is given by the exterior derivative of $-\theta(F_u) - \kappa u$. Here, $\theta(F_u)$ is the argument of $\Omega_\varphi/dV_{(\varphi \circ du)(L)}$, which is a well-defined smooth function.*

2.5. Mean curvature flow in potential. For a smooth function $u : L \times [0, t_0) \rightarrow \mathbb{R}$, denote the spatial exterior derivative by du . Suppose that $F_u = \varphi \circ du$ evolves by the mean curvature flow; the non-parametric form of the equation reads

$$\left(\frac{\partial}{\partial t} F_u \right)^\perp = H(t) \quad (2.7)$$

where the right hand side means the mean curvature vector of $F_u(L)$ at time t . According to the isomorphisms (2.3), Lemma 2.6 and the fact that $F_u(L)$ is Lagrangian, (2.7) becomes

$$(\varphi \circ du)^* \left(\omega \left(\frac{\partial}{\partial t} (\varphi \circ du), \cdot \right) \right) = -d(\theta(F_u) + \kappa u) .$$

Since $\varphi^*\omega = \omega_L$, the left hand side can be computed explicitly in T^*L , and is equal to $-d(\frac{\partial}{\partial t}u)$. Therefore, (2.7) in the current setting is equivalent to

$$\frac{\partial u}{\partial t} = \theta(du) + \kappa u + C(t) \quad (2.8)$$

for some $C(t) : [0, t_0) \rightarrow \mathbb{R}$, aka a time-dependent constant. The function $C(t)$ has no effect on $\varphi \circ du$. In this paper, $C(t)$ will be simply set to be 0.

2.6. Basic Riemannian geometry of the cotangent bundle. Let $\sigma = \sum_{i,j} \sigma_{ij}(x)dx^i \otimes dx^j$ be a Riemannian metric on L . Denote by D the Levi-Civita connection of σ , and let Λ_{ij}^k be the Christoffel symbols of D . The coordinate $\{x^i\}$ induces a coordinate $\{y_i\}$ for the fibers of T^*L . The metric σ naturally gives a Riemannian metric on T^*L :

$$\tilde{\sigma} = \sum_{i,j} \sigma_{ij} dx^i \otimes dx^j + \sigma^{ij} \eta_i \otimes \eta_j \quad (2.9)$$

where $\eta_i = dy_i - \sum_{j,k} \Lambda_{ij}^k y_k dx^j$ and σ^{ij} is the inverse of σ_{ij} . The dual frame for $\{dx^i, \eta_i\}_{1 \leq i \leq n}$ is $\{X_i, \frac{\partial}{\partial y_i}\}_{1 \leq i \leq n}$ where

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j,k} \Lambda_{ij}^k y_k \frac{\partial}{\partial y_j} . \quad (2.10)$$

For a smooth function $u : L \rightarrow \mathbb{R}$, denote $(D^k u)(\frac{\partial}{\partial x^{i_k}}, \dots, \frac{\partial}{\partial x^{i_1}})$ by $u_{;i_1 i_2 \dots i_k}$. In particular,

$$u_{;ij} = \frac{\partial^2 u}{\partial x^j \partial x^i} - \sum_k \Lambda_{ji}^k \frac{\partial u}{\partial x^k} , \text{ and } \Delta_\sigma u = \sum_{i,j} \sigma^{ij} u_{;ij}$$

is the Laplacian of u with respect to the metric σ .

For a function $u : L \rightarrow \mathbb{R}$, let $F_u : L \rightarrow T^*L$ be the embedding given by du . In a local coordinate system, F_u sends (x^1, \dots, x^n) to $(x^1, \dots, x^n, \frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^n})$. Hence,

$$\frac{\partial F_u}{\partial x^i} = \frac{\partial}{\partial x^i} + \sum_j \frac{\partial^2 u}{\partial x^j \partial x^i} \frac{\partial}{\partial y_j} = X_i + \sum_j u_{;ij} \frac{\partial}{\partial y_j} . \quad (2.11)$$

3. ASYMPTOTICS OF THE AMBIENT GEOMETRY

For a Lagrangian L in a Kähler manifold (M, g, J, ω) , fix a Weinstein neighborhood $\varphi : U \subset T^*L \rightarrow M$. Denote by $\langle \cdot, \cdot \rangle$ the metric pairing on U given by φ^*g , and by $\bar{\nabla}$ its Levi-Civita connection, which is equivalent to the Levi-Civita connection of (M, g) .

Let $\{g_{AB}(x, y)\}_{1 \leq A, B \leq 2n}$ be the coefficients of φ^*g in the coframe $\{dx^i, \eta_i\}_{1 \leq i \leq n}$ introduced in section 2.6; namely,

$$\varphi^*g = \sum_{i,j} g_{ij} dx^i \otimes dx^j + g_{i(n+j)}(dx^i \otimes \eta_j + \eta_j \otimes dx^i) + g_{(n+i)(n+j)} \eta_i \otimes \eta_j , \quad (3.1)$$

and set $g_{AB} = g_{BA}$. Equivalently,

$$g_{ij} = \langle X_i, X_j \rangle , \quad g_{i(n+j)} = \langle X_i, \frac{\partial}{\partial y_j} \rangle , \quad g_{(n+i)(n+j)} = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle .$$

Since $\varphi^*\omega = \omega_L$, we abuse the notation and denote ω_L by ω from now on. Similarly, denote φ^*J by J . It follows from $\omega = \sum_i dx^i \wedge dy_i$ and $\Lambda_{ij}^k = \Lambda_{ji}^k$ that

$$\omega(X_i, X_j) = 0 , \quad \omega(X_i, \frac{\partial}{\partial y_j}) = \delta_{ij} , \quad \omega(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = 0 .$$

Assumption 3.1. Suppose that $L \subset (M, g, J, \omega)$ is a compact, oriented, minimal Lagrangian in a Kähler–Einstein manifold. Fix a Weinstein neighborhood $\varphi : U \subset T^*L \rightarrow M$. Let σ be the induced metric on L . It induces an inner product on the fibers of T^*L : $|y|_\sigma^2 = |y_i dx^i|_\sigma^2 = \sum_{i,j} \sigma^{ij}(x) y_i y_j$. Since L is compact, there exists a positive constant $\varepsilon_0 \leq 1$ such that $\{y \in T^*L : |y|_\sigma \leq \varepsilon_0\}$ is contained in U .

We also fix a finite coordinate covering of L with the following significance: there exist constants c_0, c_1, c_2, \dots such that for any x in each coordinate chart, and any y with $|y|_\sigma \leq \varepsilon_0$,

- $\frac{1}{c_0} g_{AB}(x, 0) \leq g_{AB}(x, y) \leq c_0 g_{AB}(x, 0)$ as $(2n) \times (2n)$ positive definite matrices;
- $\frac{1}{c_0} \sigma_{ij}(x) \leq g_{ij}(x, y) \leq c_0 \sigma_{ij}(x)$ as $n \times n$ positive definite matrices;
- $|\partial_x^{(k)} \partial_y^{(\ell)} g_{AB}(x, y)| \leq c_{k+\ell}$ for every $k, \ell \geq 0$.

Definition 3.2. Denote the above finite coordinate covering by $\{O_\beta\}$. Any $x \in O_\beta$ and $Q \in \text{Sym}_n(\mathbb{R})$ naturally define an element $\sum_{ij} Q_{ij} dx^i \otimes dx^j$ in $\text{Sym}^2(T_x^*L)$, whose norm is given by $|Q|_\sigma^2 = \sum_{i,j,k,\ell} \sigma^{ik}(x) \sigma^{j\ell}(x) Q_{ij} Q_{k\ell}$. Let

$$\hat{O}_\beta = \{(x, y, Q) \in O_\beta \times \mathbb{R}^n \times \text{Sym}_n(\mathbb{R}) : |y|_\sigma \leq \varepsilon_0, |Q|_\sigma \leq \varepsilon_0\}. \quad (3.2)$$

A collection of smooth functions $f_\beta(x, y, Q)$ defined on an open neighborhood of \hat{O}_β in $O_\beta \times \mathbb{R}^n \times \text{Sym}_n(\mathbb{R})$ is said to be *bounded in (∞, loc)* if

$$\left| \partial_x^{(i)} \partial_y^{(j)} \partial_Q^{(k)} f_\beta(x, y, Q) \right| \leq c_{i+j+k}$$

for any β , any $i, j, k \geq 0$ and any $(x, y, Q) \in \hat{O}_\beta$.

Example 3.3. It shall be clear that the following functions are defined on each coordinate chart, and the index β will be omitted.

- Due to Assumption 3.1, the Christoffel symbols $\Lambda_{ij}^k(x)$ of $\sigma_{ij}(x) = g_{ij}(x, 0)$ are bounded in (∞, loc) . They only depend on x .
- The metric coefficients $g_{AB}(x, y)$ are bounded in (∞, loc) , so are their Christoffel symbols. They do not depend on Q .

In particular, the functions $\langle X_i, \bar{\nabla}_{X_j} X_k \rangle$ and $\langle X_i, \bar{\nabla}_{\frac{\partial}{\partial y^j}} X_k \rangle$ are bounded in (∞, loc) .

(iii) Let

$$\begin{aligned} \mu_{ij}(x, y, Q) &= g_{ij}(x, y) + \sum_k (g_{i(n+k)}(x, y) Q_{jk} + g_{j(n+k)}(x, y) Q_{ik}) \\ &\quad + \sum_{k,\ell} g_{(n+k)(n+\ell)}(x, y) Q_{ik} Q_{j\ell}. \end{aligned} \quad (3.3)$$

They are bounded in (∞, loc) . They are quadratic in the components of Q .

The functions μ_{ij} defined by (3.3) will be used quite often in the rest of this paper. Their main properties are summarized in the following lemma.

Lemma 3.4. Consider $\mu_{ij}(x, y, Q)$ defined by (3.3). By taking ε_0 smaller if necessary, μ_{ij} as a symmetric $n \times n$ matrix obeys

$$\frac{1}{2} \sigma_{ij}(x) \leq \mu_{ij}(x, y, Q) \leq 2 \sigma_{ij}(x)$$

for any (x, y, Q) with $|y|_\sigma \leq \varepsilon_0$ and $|Q|_\sigma \leq \varepsilon_0$. Moreover,

$$\mu_{ij}(x, y, Q) - \sigma_{ij}(x) = \sum_k y_k \mathfrak{A}_{ij}^k(x, y) + \sum_{k, \ell} Q_{k\ell} \mathfrak{B}_{ij}^{k\ell}(x, y, Q) \quad (3.4)$$

for some $\mathfrak{A}_{ij}^k(x, y)$, $\mathfrak{B}_{ij}^{k\ell}(x, y, Q)$ bounded in (∞, loc) .

Proof. For the first assertion, note that

$$\mu_{ij}(x, y, Q) = \left\langle X_i + \sum_k Q_{ik} \frac{\partial}{\partial y_k}, X_j + \sum_\ell Q_{j\ell} \frac{\partial}{\partial y_\ell} \right\rangle \quad \text{at } (x, y), \quad (3.5)$$

and is thus always positive definite. Its eigenvalues with respect to $\sigma_{ij}(x)$ are continuous functions on the tubular neighborhood of the zero section in $T^*L \oplus \text{Sym}^2(T^*L)$. Since $|y|_\sigma \leq \varepsilon_0$ and $|Q|_\sigma \leq \varepsilon_0$ defines a compact region, the first assertion follows.

The second assertion follows directly from the expression (3.3) and Taylor's theorem on $g_{AB}(x, y)$ in the y -variable. \square

By using the condition that L is a minimal Lagrangian, we deduce the following lemma.

Lemma 3.5. *Let $\mu^{ij}(x, y, Q)$ be the inverse of $\mu_{ij}(x, y, Q)$ (3.3). Under Assumption 3.1, for any $k \in \{1, \dots, n\}$,*

$$\sum_{i,j} \mu^{ij} \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k) = \sum_\ell y_\ell \mathfrak{C}_k^\ell(x, y, Q) + \sum_{\ell, m} Q_{\ell m} \mathfrak{D}_k^{\ell m}(x, y, Q),$$

where $\mathfrak{C}_k^\ell(x, y, Q)$, $\mathfrak{D}_k^{\ell m}(x, y, Q)$ are functions bounded in (∞, loc) .

Proof. To start, consider the Q -independent function $\sum_{i,j} \sigma^{ij} \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k)$. When $y = 0$,

$$\begin{aligned} \sum_{i,j} \sigma^{ij} \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k) &= \sum_{i,j} \sigma^{ij} \cdot \omega\left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \\ &= - \sum_{i,j} \sigma^{ij} \left\langle \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, J\left(\frac{\partial}{\partial x^k}\right) \right\rangle \\ &= - \left\langle \sum_{i,j} \sigma^{ij} \bar{\nabla}_{\frac{\partial}{\partial x^i}}^\perp \frac{\partial}{\partial x^j}, J\left(\frac{\partial}{\partial x^k}\right) \right\rangle = 0, \end{aligned}$$

where we have used the fact that L is a minimal Lagrangian. By writing

$$\sum_{i,j} \mu^{ij} \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k) = \sum_{i,j} (\mu^{ij} - \sigma^{ij}) \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k) + \sum_{i,j} \sigma^{ij} \cdot \omega(\bar{\nabla}_{X_i} X_j, X_k)$$

and applying Lemma 3.4, this lemma follows. \square

4. ASYMPTOTICS OF AN EXACT GRAPH

Under Assumption 3.1, the differential of a function $u : L \rightarrow \mathbb{R}$ with $|du|_\sigma \leq \varepsilon_0$ defines a Lagrangian submanifold $F_u : L \rightarrow U \subset T^*L$. The main purpose of this section is to derive asymptotics of the quantities related to Ω_φ given by Construction 2.4. The constant c in the estimates may change from line to line, but is always independent of u .

As in section 2.6, denote by $D^2u = \sum_{i,j} u_{;ij} dx^i \otimes dx^j$ the Hessian of u with respect to σ . By plugging (2.11) into (3.1), the coefficients of the induced metric by F_u are exactly $\mu_{ij}(x, du, D^2u)$, with μ_{ij} defined by (3.3). It shall be clear that du means the vector $[u_{;i}]$, and D^2u means the coefficient matrix $[u_{;k\ell}]$.

The first task is to establish the expansion of $\theta(F_u)$ given by Lemma 2.6. Note that $\theta(F_u)$ actually depends on du (the position) and D^2u (the tangent space).

Proposition 4.1. *There exists a constant $c > 0$ such that*

$$|\theta(F_u) - \Delta_\sigma u| \leq c(|du|_\sigma^2 + |D^2u|_\sigma^2)$$

for any smooth $u : L \rightarrow \mathbb{R}$ with $|du|_\sigma \leq \varepsilon_0$ and $|D^2u|_\sigma \leq \varepsilon_0$.

Proof. Denote by $\tilde{F}(x, s) : L \times [0, 1] \rightarrow T^*L$ the one-parameter family of Lagrangians given by $s du$ for $s \in [0, 1]$. Since

$$\tilde{F}_i = \frac{\partial \tilde{F}}{\partial x^i} = X_i + s \sum_j u_{;ij} \frac{\partial}{\partial y_j}, \quad (4.1)$$

the induced metric by $\tilde{F}(x, s)$ for any fixed s has coefficients $\mu_{ij}(x, sdu, sD^2u)$. Write the variational field as

$$\frac{\partial}{\partial s} = \frac{\partial \tilde{F}}{\partial s} = \sum_i u_{;i} \frac{\partial}{\partial y_i}. \quad (4.2)$$

The partial derivative of $\theta(\tilde{F}(x, s))$ in s can be computed by

$$\begin{aligned} \frac{\partial}{\partial s} e^{\sqrt{-1}\theta} &= \frac{\partial}{\partial s} \left(\frac{\Omega_\varphi(\tilde{F}_1, \dots, \tilde{F}_n)}{\sqrt{\det \mu}} \right) \\ \Rightarrow \sqrt{-1} \frac{\partial \theta}{\partial s} &= -\frac{1}{2} \frac{\partial \log \det \mu}{\partial s} + \frac{e^{-\sqrt{-1}\theta}}{\sqrt{\det \mu}} \frac{\partial}{\partial s} \Omega_\varphi(\tilde{F}_1, \dots, \tilde{F}_n) \end{aligned} \quad (4.3)$$

Since Ω_φ is constructed by parallel transport along the radial curves of the fibers of T^*L and $\{\tilde{F}(x, s)\}_{s \in [0, 1]}$ is exactly the radial curve for every x , $\bar{\nabla}_{\frac{\partial}{\partial s}} \Omega_\varphi = 0$. It follows that

$$\frac{\partial}{\partial s} \Omega_\varphi(\tilde{F}_1, \dots, \tilde{F}_n) = \Omega_\varphi(\bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_1, \dots, \tilde{F}_n) + \dots + \Omega_\varphi(\tilde{F}_1, \dots, \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_n). \quad (4.4)$$

Because $\tilde{F}(L, s)$ is Lagrangian for every $s \in [0, 1]$,

$$\bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i = \sum_j \mu^{ij} \left(\langle \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i, \tilde{F}_j \rangle \tilde{F}_j + \langle \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i, J(\tilde{F}_j) \rangle J(\tilde{F}_j) \right) .$$

Due to the fact that Ω_φ is an $(n, 0)$ -form, plugging this into (4.4) gives

$$\frac{\partial}{\partial s} \Omega_\varphi(\tilde{F}_1, \dots, \tilde{F}_n) = \sum_{i,j} \mu^{ij} \left(\langle \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i, \tilde{F}_j \rangle + \sqrt{-1} \langle \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i, J(\tilde{F}_j) \rangle \right) \Omega_\varphi(\tilde{F}_1, \dots, \tilde{F}_n) \quad (4.5)$$

With (4.5), the real part of (4.3) recovers the standard formula for $\frac{\partial}{\partial s} \log \det \mu$, and the imaginary part becomes

$$\frac{\partial \theta}{\partial s} = \sum_{i,j} \mu^{ij} \langle \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i, J(\tilde{F}_j) \rangle = \sum_{i,j} \mu^{ij} \omega \left(\tilde{F}_j, \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i \right) . \quad (4.6)$$

Since $s \equiv 0$ corresponds to L and the restriction of Ω_φ on the tangent space of L is the volume form of L , θ vanishes when $s = 0$. Therefore,

$$\theta|_{s=1} = \frac{\partial \theta}{\partial s} \Big|_{s=0} + \int_0^1 (1-s) \frac{\partial^2 \theta}{\partial s^2} ds . \quad (4.7)$$

1st term on the right hand side of (4.7). By (4.1) and (4.2),

$$\bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i = \sum_k u_{;ik} \frac{\partial}{\partial y_k} + \sum_k u_{;k} \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i + s \sum_{k,\ell} u_{;k} u_{;i\ell} \bar{\nabla}_{\frac{\partial}{\partial y_k}} \frac{\partial}{\partial y_\ell} . \quad (4.8)$$

Together with (4.6), it gives

$$\frac{\partial \theta}{\partial s} \Big|_{s=0} = \sum_{i,j,k} \sigma^{ij} u_{;ik} \omega \left(X_j, \frac{\partial}{\partial y_k} \right) + \sum_{i,j,k} u_{;k} \sigma^{ij} \omega \left(X_j, \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i \right) .$$

The first term is $\Delta_\sigma u$. We *claim* that $\omega(\bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i, X_j)$ vanishes when $s = 0$, and thus the second term is zero. Note that

$$\bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i = \bar{\nabla}_{X_i} \frac{\partial}{\partial y_k} + \left[\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x^i} + \sum_{j,\ell} \Lambda_{i\ell}^j y_j \frac{\partial}{\partial y_\ell} \right] = \bar{\nabla}_{X_i} \frac{\partial}{\partial y_k} + \sum_\ell \Lambda_{\ell i}^k \frac{\partial}{\partial y_\ell} .$$

It follows that

$$\begin{aligned} \omega \left(X_j, \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i \right) &= \omega \left(X_j, \bar{\nabla}_{X_i} \frac{\partial}{\partial y_k} \right) + \sum_\ell \Lambda_{\ell i}^k \omega \left(X_j, \frac{\partial}{\partial y_\ell} \right) \\ &= -\omega \left(\bar{\nabla}_{X_i} X_j, \frac{\partial}{\partial y_k} \right) + \Lambda_{ji}^k , \end{aligned}$$

where the last equality uses the fact that $\bar{\nabla} \omega = 0$ and $\omega \left(X_j, \frac{\partial}{\partial y_k} \right) = \delta_{jk}$. When $s = 0$, $y = 0$,

$$\bar{\nabla}_{X_i} X_j = \sum_\ell \Lambda_{ij}^\ell \frac{\partial}{\partial x^\ell} + \bar{\nabla}_{\frac{\partial}{\partial x^i}}^\perp \frac{\partial}{\partial x^j} ,$$

and hence

$$\omega(X_j, \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i) = -\Lambda_{ij}^k - \omega(\bar{\nabla}_{\frac{\partial}{\partial x^t}} \bar{\nabla}_{\frac{\partial}{\partial x^t}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y_k}) + \Lambda_{ji}^k = 0 .$$

This finishes the proof of the claim.

2nd term on the right hand side of (4.7). Since $\bar{\nabla}\omega = 0$, the derivative of (4.6) in s gives

$$\begin{aligned} \frac{\partial^2 \theta}{\partial s^2} &= - \sum_{i,j,k\ell} \mu^{ik} \frac{\partial \mu_{k\ell}}{\partial s} \mu^{j\ell} \omega \left(\tilde{F}_j, \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i \right) \\ &\quad + \sum_{i,j} \mu^{ij} \left[\omega \left(\bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_j, \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i \right) + \omega \left(\tilde{F}_j, \bar{\nabla}_{\frac{\partial}{\partial s}} \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i \right) \right] . \end{aligned} \quad (4.9)$$

We compute the covariant derivative of (4.8) in s :

$$\begin{aligned} \bar{\nabla}_{\frac{\partial}{\partial s}} \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i &= \sum_{k,\ell} u_{;k} u_{;i\ell} \bar{\nabla}_{\frac{\partial}{\partial y_k}} \frac{\partial}{\partial y_\ell} + \sum_{k,\ell} u_{;\ell} u_{;ik} \bar{\nabla}_{\frac{\partial}{\partial y_\ell}} \frac{\partial}{\partial y_k} \\ &\quad + \sum_{k,\ell} u_{;\ell} u_{;k} \bar{\nabla}_{\frac{\partial}{\partial y_\ell}} \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_i + s \sum_{k,\ell,m} u_{;m} u_{;k} u_{;i\ell} \bar{\nabla}_{\frac{\partial}{\partial y_m}} \bar{\nabla}_{\frac{\partial}{\partial y_k}} \frac{\partial}{\partial y_\ell} . \end{aligned} \quad (4.10)$$

With $|du|_\sigma \leq \varepsilon_0$ and $|D^2u|_\sigma \leq \varepsilon_0$, it follows from (4.1), (4.8) and (4.10) that

$$|\tilde{F}_i| \leq c , \quad |\bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i| \leq c(|du|_\sigma + |D^2u|_\sigma) , \quad |\bar{\nabla}_{\frac{\partial}{\partial s}} \bar{\nabla}_{\frac{\partial}{\partial s}} \tilde{F}_i| \leq c(|du|_\sigma^2 + |D^2u|_\sigma^2)$$

for some constant $c > 0$. The norm is computed by φ^*g at $\tilde{F}(x, s)$.

According to Lemma 3.4, the derivative of μ_{ij} in s is

$$\frac{\partial \mu_{ij}}{\partial s}(x, sdu, sD^2u) = \frac{\partial}{\partial s} \left(\sum_k su_{;k} \mathfrak{A}_{ij}^k(x, sdu) + \sum_{k,\ell} su_{;k\ell} \mathfrak{B}_{ij}^{k\ell}(x, sdu, sD^2u) \right) .$$

Since $\mathfrak{A}_{ij}^k(x, y)$ and $\mathfrak{B}_{ij}^{k\ell}(x, y, Q)$ are bounded in (∞, loc) , $|\frac{\partial \mu_{ij}}{\partial s}|$ is no greater than $c(|du|_\sigma + |D^2u|_\sigma)$ for some constant $c > 0$.

By applying these estimates to (4.9), one finds that the absolute value of the last term in (4.7) is bounded from above by $c(|du|_\sigma^2 + |D^2u|_\sigma^2)$. Note that $|du|_\sigma$ and $|D^2u|_\sigma$ are assumed to be no greater than ε_0 , and only their lowest order terms matter. \square

The next step is to study the (spatial) derivative of $\theta(F_u)$.

Lemma 4.2. *There exist $\mathfrak{E}_k^i(x, y, Q), \mathfrak{F}_k^{ij}(x, y, Q)$ which are bounded in (∞, loc) such that*

$$(\theta(F_u) + \kappa u)_{;k} = \sum_{i,j} \mu^{ij} u_{;jki} + \sum_i u_{;i} \mathfrak{E}_k^i(x, du, D^2u) + \sum_{i,j} u_{;ij} \mathfrak{F}_k^{ij}(x, du, D^2u)$$

for any smooth $u : L \rightarrow \mathbb{R}$ with $|du|_\sigma \leq \varepsilon_0$ and $|D^2u|_\sigma \leq \varepsilon_0$.

Proof. As (4.1), denote $\frac{\partial F_u}{\partial x^i}$ by F_i . By Lemma 2.6, $-\mathrm{d}(\theta + \kappa u)$ is $(F_u)^*(\omega(H(F_u), \cdot))$. Since $F_u(L)$ is Lagrangian,

$$(\theta + \kappa u)_{;k} = \sum_{i,j} \omega(\mu^{ij} \bar{\nabla}_{F_i}^\perp F_j, -F_k) = \sum_{i,j} \omega(F_k, \mu^{ij} \bar{\nabla}_{F_i} F_j). \quad (4.11)$$

We compute

$$\begin{aligned} \bar{\nabla}_{F_i} F_j &= \bar{\nabla}_{F_i} \left(X_j + \sum_k u_{;jk} \frac{\partial}{\partial y_k} \right) \\ &= \sum_k \frac{\partial u_{;jk}}{\partial x^i} \frac{\partial}{\partial y_k} + \bar{\nabla}_{F_i} X_j + \sum_k u_{;jk} \bar{\nabla}_{F_i} \frac{\partial}{\partial y_k} \\ &= \sum_k u_{;jki} \frac{\partial}{\partial y_k} + \sum_{k,\ell} (\Lambda_{ji}^\ell u_{;\ell k} + \Lambda_{ki}^\ell u_{;j\ell}) \frac{\partial}{\partial y_k} + \bar{\nabla}_{X_i} X_j \\ &\quad + \sum_k \left(u_{;ik} \bar{\nabla}_{\frac{\partial}{\partial y_k}} X_j + u_{;jk} \bar{\nabla}_{X_i} \frac{\partial}{\partial y_k} \right) + \sum_{k,\ell} u_{;jk} u_{;i\ell} \bar{\nabla}_{\frac{\partial}{\partial y_\ell}} \frac{\partial}{\partial y_k}. \end{aligned}$$

It together with (4.11) and Lemma 3.5 finishes the proof of this lemma. \square

By taking covariant derivatives, D and D^2 , on the expression in Lemma 4.2, it is not hard to prove the following corollary, and we omit its proof.

Corollary 4.3. *There exists $c > 0$ such that if $u : L \rightarrow \mathbb{R}$ is a smooth function with $|\mathrm{d}u|_\sigma \leq \varepsilon_0$ and $|D^2 u|_\sigma \leq \varepsilon_0$, then*

- (i) $\left| (\theta + \kappa u)_{;k} - \sum_{i,j} \mu^{ij} u_{;jki} \right| \leq c(|D^2 u|_\sigma + |\mathrm{d}u|_\sigma);$
- (ii) $\left| (\theta + \kappa u)_{;k\ell} - \sum_{i,j} \mu^{ij} u_{;jkil} \right| \leq c(|D^3 u|_\sigma^2 + |D^3 u|_\sigma + |D^2 u|_\sigma + |\mathrm{d}u|_\sigma);$
- (iii) $\left| (\theta + \kappa u)_{;k\ell m} - \sum_{i,j} \mu^{ij} u_{;jkilm} \right| \leq c(|D^4 u|_\sigma \cdot |D^3 u|_\sigma + |D^4 u|_\sigma + |D^3 u|_\sigma^3 + |D^3 u|_\sigma + |D^2 u|_\sigma + |\mathrm{d}u|_\sigma);$

for any k, ℓ, m .

For a function $u : L \rightarrow \mathbb{R}$, let $\Gamma_{ij}^k(F_u)$ be the Christoffel symbols of its Levi-Civita connection of $\mu_{ij}(x, \mathrm{d}u, D^2 u)$. It is the restriction of $\bar{\nabla}$ on the tangent space of F_u . Recall that the difference between two connections is a tensor:

$$\begin{aligned} \Gamma_{ij}^k(F_u) - \Lambda_{ij}^k &= \frac{1}{2} \sum_\ell \mu^{k\ell} [(\mu - \sigma)_{j\ell;i} + (\mu - \sigma)_{i\ell;j} - (\mu - \sigma)_{ij;\ell}] \\ &= \frac{1}{2} \sum_\ell \mu^{k\ell} [\mu_{j\ell;i} + \mu_{i\ell;j} - \mu_{ij;\ell}], \end{aligned} \quad (4.12)$$

where the equality uses the fact that $D\sigma = 0$.

Lemma 4.4. *There exists a constant $c > 0$ with the following property. Let $u : L \rightarrow \mathbb{R}$ be a smooth function with $|du|_\sigma \leq \varepsilon_0$ and $|D^2u|_\sigma \leq \varepsilon_0$. Then*

$$|\Delta_\mu f - \text{tr}_\mu(\text{Hess}_\sigma f)| \leq c |df|_\sigma \cdot (|D^3u|_\sigma + |D^2u|_\sigma + |du|_\sigma)$$

for any smooth function f on L , where Δ_μ is the Laplace operator of the metric $\mu_{ij}(x, du, D^2u)$.

Proof. We compute

$$\Delta_\mu f - \text{tr}_\mu(\text{Hess}_\sigma f) = \sum_{i,j} \mu^{ij} \sum_k \left(\Gamma_{ij}^k(F_u) - \Lambda_{ij}^k \right) \frac{\partial f}{\partial x^k}.$$

By applying (4.12) and Lemma 3.4, this lemma follows. \square

Remark 4.5. For the maximal principle argument of this paper, both $\Delta_\mu f$ and $\text{tr}_\mu(\text{Hess}_\sigma f)$ will work. We will go with the more geometric one, $\Delta_\mu f$. If one uses $\text{tr}_\mu(\text{Hess}_\sigma f)$, Lemma 4.4 will not be needed. It seems that Δ_σ is not a good choice. This kind of issue also appears implicitly in [18].

5. EVOLUTION EQUATIONS

Now, suppose that $u : L \times [0, T) \rightarrow \mathbb{R}$ is a smooth function such that the graph of $du(\cdot t)$ evolves by the mean curvature flow. As discussed in section 2.5, we may assume that u satisfies (2.8) with $C(t) \equiv 0$. As before, semicolon means the covariant derivative in D .

Lemma 5.1. *There exists a constant $c > 0$ with the following property. Suppose that $u : L \times [0, T) \rightarrow \mathbb{R}$ solves (2.8) with $C(t) \equiv 0$, and $|du|_\sigma \leq \varepsilon_0$, $|D^2u|_\sigma \leq \varepsilon_0$ for all $t \in [0, T)$. Then,*

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) u^2 \leq -|du|_\sigma^2 + 2\kappa u^2 + c|u|(|D^3u|_\sigma^2 + |D^2u|_\sigma^2 + |du|_\sigma^2). \quad (5.1)$$

Proof. By (2.8),

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) u^2 = -2 \sum_{i,j} \mu^{ij} u_{;i} u_{;j} + 2u \cdot (\theta(F_u) - \Delta_\mu u) + 2\kappa u^2.$$

Due to Lemma 3.4, $2 \sum_{i,j} \mu^{ij} u_{;i} u_{;j} \geq |du|_\sigma^2$. According to Proposition 4.1, Lemma 3.4 and Lemma 4.4,

$$\begin{aligned} |\theta(F_u) - \Delta_\mu u| &\leq |\theta(F_u) - \Delta_\sigma u| + |\Delta_\sigma u - \text{tr}_\mu(\text{Hess}_\sigma u)| + |\text{tr}_\mu(\text{Hess}_\sigma u) - \Delta_\mu u| \\ &\leq c'(|du|_\sigma^2 + |D^2u|_\sigma^2) + c''(|D^2u|_\sigma + |du|_\sigma)|D^2u|_\sigma \\ &\quad + c'''|du|_\sigma(|D^3u|_\sigma + |D^2u|_\sigma + |du|_\sigma). \end{aligned}$$

With the assumption that $|du|_\sigma \leq \varepsilon_0$, $|D^2u|_\sigma \leq \varepsilon_0$ and some simple manipulation, it finishes the proof of this lemma. \square

Lemma 5.2. *There exists a constant c which has the following significance. Suppose that $u : L \times [0, T] \rightarrow \mathbb{R}$ solves (2.8), and $|du|_\sigma \leq \varepsilon_0$, $|D^2u|_\sigma \leq \varepsilon_0$ for all $t \in [0, T]$. Then,*

$$\left(\frac{\partial}{\partial t} - \Delta_\mu\right)|du|_\sigma^2 \leq -\frac{1}{2}|D^2u|_\sigma^2 + c(|D^3u|_\sigma \cdot |du|_\sigma + |du|_\sigma^2) \quad \text{and} \quad (5.2)$$

$$\left(\frac{\partial}{\partial t} - \Delta_\mu\right)|D^2u|_\sigma^2 \leq -\frac{1}{2}|D^3u|_\sigma^2 + c(|D^3u|_\sigma^2 \cdot |D^2u|_\sigma + |D^2u|_\sigma^2 + |du|_\sigma^2). \quad (5.3)$$

Proof. The equation (5.2). Differentiating (2.8) gives

$$\frac{\partial}{\partial t}u_{;k} = (\theta + \kappa u)_{;k}. \quad (5.4)$$

We compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_\mu\right)|du|_\sigma^2 &= 2 \sum_{k,\ell} \sigma^{k\ell} u_{;\ell} \frac{\partial}{\partial t} u_{;k} - \text{tr}_\mu(\text{Hess}_\sigma |du|_\sigma^2) \\ &\quad + (\text{tr}_\mu(\text{Hess}_\sigma |du|_\sigma^2) - \Delta_\mu |du|_\sigma^2). \end{aligned} \quad (5.5)$$

Due to (5.4) and Corollary 4.3 (i),

$$\begin{aligned} 2 \sum_{k,\ell} \sigma^{k\ell} u_{;\ell} \frac{\partial}{\partial t} u_{;k} &= 2 \sum_{k,\ell} \sigma^{k\ell} u_{;\ell} (\theta + \kappa u)_{;k} \\ &\leq 2 \sum_{i,j,k,\ell} \mu^{ij} \sigma^{k\ell} u_{;\ell} u_{;jki} + c' |du|_\sigma (|D^2u|_\sigma + |du|_\sigma). \end{aligned}$$

By Lemma 3.4 and $u_{;jki} - u_{;kij} = u_{;kji} - u_{;kij} = (\text{curvature of } \sigma) * du$,

$$\begin{aligned} -\text{tr}_\mu(\text{Hess}_\sigma |du|_\sigma^2) &= -2 \sum_{i,j,k,\ell} \mu^{ij} \sigma^{k\ell} (u_{;ki} u_{\ell j} + u_{;kij} u_{;\ell}) \\ &\leq -|D^2u|_\sigma^2 - 2 \sum_{i,j,k,\ell} \mu^{ij} \sigma^{k\ell} u_{;\ell} u_{;jki} + c'' |du|_\sigma^2. \end{aligned}$$

According to Lemma 4.4 and $|D^2u|_\sigma \leq \varepsilon_0$,

$$\text{tr}_\mu(\text{Hess}_\sigma |du|_\sigma^2) - \Delta_\mu |du|_\sigma^2 \leq c''' |du|_\sigma (|D^3u|_\sigma + |D^2u|_\sigma + |du|_\sigma).$$

Since $|du|_\sigma |D^2u|_\sigma \leq \delta |D^2u|_\sigma^2 + |du|_\sigma^2 / \delta$ for any $\delta > 0$, putting these estimates into (5.5) finishes the proof of (5.2).

The equation (5.3). Differentiating (5.4) gives

$$\frac{\partial}{\partial t}u_{;kp} = (\theta + \kappa u)_{;kp}. \quad (5.6)$$

Similar to (5.5),

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_\mu\right)|D^2u|_\sigma^2 &= 2 \sum_{k,\ell,p,q} \sigma^{k\ell} \sigma^{pq} u_{;\ell q} \frac{\partial}{\partial t} u_{;kp} - \text{tr}_\mu(\text{Hess}_\sigma |D^2u|_\sigma^2) \\ &\quad + (\text{tr}_\mu(\text{Hess}_\sigma |D^2u|_\sigma^2) - \Delta_\mu |D^2u|_\sigma^2). \end{aligned} \quad (5.7)$$

Due to (5.6) and Corollary 4.3 (ii) ,

$$\begin{aligned} & 2 \sum_{k,\ell,p,q} \sigma^{k\ell} \sigma^{pq} u_{;\ell q} \frac{\partial}{\partial t} u_{;kp} \\ & \leq 2 \sum_{i,j,k,\ell} \mu^{ij} \sigma^{k\ell} \sigma^{pq} u_{;\ell q} u_{;jkip} + c' |D^2 u|_\sigma (|D^3 u|_\sigma^2 + |D^3 u|_\sigma + |D^2 u|_\sigma + |du|_\sigma) . \end{aligned}$$

By Lemma 3.4 and commuting covariant derivatives,

$$\begin{aligned} -\operatorname{tr}_\mu(\operatorname{Hess}_\sigma |D^2 u|_\sigma^2) &= -2 \sum_{i,j,k,\ell,p,q} \mu^{ij} \sigma^{k\ell} \sigma^{pq} (u_{;kpi} u_{\ell qj} + u_{;kpij} u_{;\ell q}) \\ &\leq -|D^3 u|_\sigma^2 - 2 \sum_{i,j,k,\ell} \mu^{ij} \sigma^{k\ell} \sigma^{pq} u_{;\ell q} u_{;jkip} + c'' |D^2 u|_\sigma (|D^2 u|_\sigma + |du|_\sigma) . \end{aligned}$$

According to Lemma 4.4,

$$\operatorname{tr}_\mu(\operatorname{Hess}_\sigma |D^2 u|_\sigma^2) - \Delta_\mu |D^2 u|_\sigma^2 \leq c''' |D^3 u|_\sigma |D^2 u|_\sigma (|D^3 u|_\sigma + |D^2 u|_\sigma + |du|_\sigma) .$$

It is not hard to see that (5.3) follows from these estimates and (5.7). \square

By combining these quantities, one can produce a monotone quantity along the flow.

Proposition 5.3. *When the Kähler–Einstein constant $\kappa \leq 0$, there exist positive constants $C_0, C_1 \geq 1$ and $\varepsilon_1 \leq \varepsilon_0$ such that the following holds. Suppose that $u : L \times [0, T) \rightarrow \mathbb{R}$ solves (2.8) with $C(t) \equiv 0$. Let*

$$\psi = C_0 u^2 + C_1 |du|_\sigma^2 + |D^2 u|_\sigma^2 . \quad (5.8)$$

If $\psi < (\varepsilon_1)^2$ at $t = 0$, then $\max_{L \times \{t\}} \psi$ is non-increasing in t . Moreover, ψ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) \psi \leq 2C_0 \kappa u^2 - \frac{1}{4} (C_0 |du|_\sigma^2 + C_1 |D^2 u|_\sigma^2 + |D^3 u|_\sigma^2) . \quad (5.9)$$

Proof. The key step is to show that the constants C_0, C_1, ε_1 can be chosen so that (5.9) holds true on $L \times [0, T')$ as long as $\psi < (\varepsilon_1)^2$ on $L \times [0, T')$. For $C_1 \geq 1$ and $\varepsilon_1 \leq \varepsilon_0$, $\psi < (\varepsilon_1)^2$ implies that $|du|_\sigma < \varepsilon_0$ and $|D^2 u|_\sigma < \varepsilon_0$.

If $\psi < (\varepsilon_1)^2$, $|u| < \varepsilon_1 / \sqrt{C_0}$, and (5.1) implies that

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) C_0 u^2 \leq -C_0 |du|_\sigma^2 + 2C_0 \kappa u^2 + c \sqrt{C_0} \varepsilon_1 (|D^3 u|_\sigma^2 + |D^2 u|_\sigma^2 + |du|_\sigma^2) .$$

Since $|D^3 u|_\sigma \cdot |du|_\sigma \leq \delta |D^3 u|_\sigma^2 + \delta^{-1} |du|_\sigma^2$ for any $\delta > 0$, (5.2) leads to

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) C_1 |du|_\sigma^2 \leq -\frac{1}{2} C_1 |D^2 u|_\sigma^2 + c C_1 (\delta |D^3 u|_\sigma^2 + \delta^{-1} |du|_\sigma^2 + |du|_\sigma^2) .$$

If $\psi < (\varepsilon_1)^2$, $|D^2 u|_\sigma < \varepsilon_1$, and (5.3) implies that

$$\left(\frac{\partial}{\partial t} - \Delta_\mu \right) |D^2 u|_\sigma^2 \leq -\frac{1}{2} |D^3 u|_\sigma^2 + c (\varepsilon_1 |D^3 u|_\sigma^2 + |D^2 u|_\sigma^2 + |du|_\sigma^2) .$$

It follows that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_\mu\right)\psi &\leq 2C_0\kappa u^2 - \left[C_0 - c\left(\sqrt{C_0}\varepsilon_1 + C_1(1 + \delta^{-1}) + 1\right)\right] |du|_\sigma^2 \\ &\quad - \left[\frac{1}{2}C_1 - c\left(\sqrt{C_0}\varepsilon_1 + 1\right)\right] |D^2u|_\sigma^2 - \left[\frac{1}{2} - c\left(\sqrt{C_0}\varepsilon_1 + C_1\delta + \varepsilon_1\right)\right] |D^3u|_\sigma^2. \end{aligned}$$

First, choose $C_1 > 10c$; second, choose $\delta < 1/(10cC_1)$; next, choose $C_0 > 10c(C_1(1 + \delta^{-1}) + 1)$. Finally, (5.9) can be obtained by choosing sufficiently small ε_1 .

To finish the proof of this proposition, consider

$$T^* = \sup\{T' \in (0, T) : \psi < (\varepsilon_1)^2 \text{ on } L \times [0, T']\}.$$

By a maximal principle argument, T^* must be equal to T . \square

The last gadget needed in the proof of the main theorem is the evolution equation of the third order derivative of u , which is more or less equivalent to the second fundamental form of the graph of du . The evolution equation of the second fundamental form along the mean curvature flow was derived in [22, section 7]. With the help of Corollary 4.3 (iii), the proof of the following lemma is basically the same as that of Lemma 5.2, and is left as an exercise for the reader.

Lemma 5.4. *There exists a constant $c > 0$ with the following property. Suppose that $u : L \times [0, T) \rightarrow \mathbb{R}$ solves (2.8), and $|du|_\sigma \leq \varepsilon_0$, $|D^2u|_\sigma \leq \varepsilon_0$ for all $t \in [0, T)$. Then,*

$$\left(\frac{\partial}{\partial t} - \Delta_\mu\right)|D^3u|_\sigma^2 \leq -\frac{1}{2}|D^4u|_\sigma^2 + c(|D^3u|_\sigma^4 + |D^3u|_\sigma^2 + |D^2u|_\sigma^2 + |du|_\sigma^2). \quad (5.10)$$

6. PROOF OF THE MAIN THEOREM

We are now ready to prove the dynamical stability theorem.

Theorem 6.1. *Under Assumption 3.1, when the Kähler–Einstein constant $\kappa \leq 0$, there exists a positive constant $\varepsilon > 0$ with the following significance. Suppose that $u_0 : L \rightarrow \mathbb{R}$ is a smooth function satisfying $(u_0)^2 + |du_0|_\sigma^2 + |D^2u_0|_\sigma^2 < \varepsilon^2$. Then, the Lagrangian mean curvature flow starting from $\varphi \circ du_0 : L \rightarrow M$ exists for all time. Moreover, as $t \rightarrow \infty$, the flow converges smoothly to $L \subset M$.*

Proof. Let C_0, C_1 and ε_1 be the constants given by Proposition 5.3. Suppose that the initial condition satisfies

$$C_0(u_0)^2 + C_1|du_0|_\sigma^2 + |D^2u_0|_\sigma^2 < (\varepsilon_2)^2$$

for some $\varepsilon_2 \leq \varepsilon_1$; the precise value of ε_2 will be determined later.

Let T be the maximal existence time of the smooth solution u to (2.8) (with $C(t) \equiv 0$) with initial condition u_0 . According to Proposition 5.3, ψ defined by (5.8) is always less than $(\varepsilon_2)^2$.

Uniform boundedness of D^3u . With the equation of ψ , we employ the trick in [23, section 4] to bound D^3u . For any $K > 0$, consider the function $\log(1 + |D^3u|_\sigma^2) + K\psi$. It follows from (5.10) that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_\mu\right) \log(1 + |D^3u|_\sigma^2) \\ & \leq \frac{1}{1 + |D^3u|_\sigma^2} \left[-\frac{1}{2}|D^4u|_\sigma^2 + c_1(|D^3u|_\sigma^4 + |D^3u|_\sigma^2 + |D^2u|_\sigma^2 + |du|_\sigma^2) \right] \\ & \quad + \langle d \log(1 + |D^3u|_\sigma^2), d(\log(1 + |D^3u|_\sigma^2) + K\psi) \rangle_\mu - K \langle d \log(1 + |D^3u|_\sigma^2), d\psi \rangle_\mu \end{aligned}$$

where $\langle df, d\tilde{f} \rangle_\mu = \sum_{i,j} \mu^{ij} f_{;i} \tilde{f}_{;j}$. By Lemma 3.3 and the fact that $\psi < (\varepsilon_2)^2$,

$$\begin{aligned} & K |\langle d \log(1 + |D^3u|_\sigma^2), d\psi \rangle_\mu| \\ & \leq \frac{Kc_2}{1 + |D^3u|_\sigma^2} |D^3u|_\sigma |D^4u|_\sigma (|D^3u|_\sigma |D^2u|_\sigma + |D^2u|_\sigma |du|_\sigma + |du|_\sigma |u|) \\ & \leq \frac{Kc_3}{1 + |D^3u|_\sigma^2} (\varepsilon_2 |D^3u|_\sigma^2 |D^4u|_\sigma + (\varepsilon_2)^2 |D^3u|_\sigma |D^4u|_\sigma) \\ & \leq \frac{1}{1 + |D^3u|_\sigma^2} \left(\frac{1}{4} |D^4u|_\sigma^2 + c_4 K^2 (\varepsilon_2)^2 (|D^3u|_\sigma^4 + |D^3u|_\sigma^2) \right). \end{aligned}$$

These estimates with (5.9) imply that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta_\mu\right) (\log(1 + |D^3u|_\sigma^2) + K\psi) \\ & \leq (c_1 + c_4(K\varepsilon_2)^2) |D^3u|_\sigma^2 + c_1(|D^2u|_\sigma^2 + |du|_\sigma^2) - \frac{1}{4} \frac{|D^4u|_\sigma^2}{1 + |D^3u|_\sigma^2} \\ & \quad - \frac{K}{4} (|D^3u|_\sigma^2 + |D^2u|_\sigma^2 + |du|_\sigma^2) \\ & \quad + \langle d \log(1 + |D^3u|_\sigma^2), d(\log(1 + |D^3u|_\sigma^2) + K\psi) \rangle_\mu. \end{aligned}$$

By picking $K > 8c_1$ and $\varepsilon_2 < \sqrt{\frac{c_1}{c_4}} \frac{1}{K}$, one finds that $\max_{L \times \{t\}} (\log(1 + |D^3u|_\sigma^2) + K\psi)$ is non-increasing in t . It follows that $|D^3u|_\sigma$ is uniformly bounded on $L \times [0, T)$.

Since $|du|_\sigma$, $|D^2u|_\sigma$ and $|D^3u|_\sigma$ are all uniformly bounded, the second fundamental form of the graph of du is uniformly bounded. It follows from the work of Huisken [4] that the mean curvature flow exists for all time, $T = \infty$.

Moreover, the standard theory of mean curvature flow implies that all the higher order derivatives of the second fundamental form are uniformly bounded; see for instance [1, Proposition 4.8]. In the current setting, it is equivalent to the boundedness of $\sup_{L \times [0, \infty)} |D^\ell u|_\sigma$ for any $\ell \geq 0$.

Convergence. With the boundedness of $|D^\ell u|_\sigma$, there exists a time sequence $t_j \rightarrow \infty$ such that $u(\cdot, t_j)$ converges smoothly to u_∞ as $j \rightarrow \infty$. Thus, the graph of $du(\cdot, t_j)$ converges smoothly to the graph of du_∞ .

It is known that a Kähler–Einstein metric is analytic; see [10, 2]. Note that the mean curvature flow is the negative gradient flow of the volume functional. With the analyticity of the metric, the general theorem of Simon [16, Corollary 2] implies that the graph of $du(\cdot t)$ converges smoothly to the graph of du_∞ as $t \rightarrow \infty$.

The limit. When $\kappa < 0$, it follows from (5.9) that ψ decays to 0 exponentially, and thus $u_\infty \equiv 0$.

When $\kappa = 0$, it follows from (2.5) that Ω_φ is holomorphic volume form of constant magnitude, and $U \subset T^*L$ is a Calabi–Yau manifold. The graph of du_∞ is a minimal Lagrangian. The function u_∞ can be viewed as a function on the ambient space $U \subset T^*L$. It is not hard to see that it gives a time-independent Hamiltonian isotopy between $(du_\infty)(L)$ and L . By [20, Lemma 4.2], $(du_\infty)(L)$ coincides with L , and u_∞ is a constant function. It finishes the proof of this theorem. \square

We remark that the Thomas–Yau uniqueness theorem invoked in the last step has further generalizations; especially by Imagi [5] and Li [7].

Remark 6.2. When $\kappa = 0$, one may also apply Proposition 5.3 to conclude the limit is L . Replace u by $\tilde{u} = u - \int_L u dV_L / \int_L 1 dV_L$, where dV_L is the volume form of (L, σ) . It follows that $\int_L \tilde{u} \Delta_\sigma \tilde{u} dV_L \leq -\lambda_1 \int_L \tilde{u}^2 dV_L$. Note that $\Delta_\sigma \tilde{u} - \Delta_\mu \tilde{u} = \Delta_\sigma u - \Delta_\mu u$ can be bounded by Lemma 3.4. By integrating (5.9) over L , one can show that $\int_L \tilde{u}^2 dV_L$ decays to zero exponentially, provided that ε_2 is sufficiently small.

6.1. Positive Kähler–Einstein constant. We finish this paper by mentioning the case when $\kappa > 0$. In this case, the formula (1.1) of Oh [13, 14] implies that a minimal Lagrangian is Hamiltonian stable if and only if the first eigenvalue of Δ_L is no less than κ . Oh also showed that $\mathbb{R}\mathbb{P}^n$ and the Clifford torus are Hamiltonian stable in $\mathbb{C}\mathbb{P}^n$, and their first eigenfunctions do give *integrable* variations. That is to say, for any first eigenfunction ψ , there exists a one-parameter family of Hamiltonian isotopic minimal Lagrangians $\{L_s\}_{-\epsilon < s < \epsilon}$ with $L_0 = L$, and $\omega(\frac{\partial L_s}{\partial s}|_{s=0}, \cdot)|_L$ is $d\psi$.

In the situation of Hamiltonian stable minimal Lagrangian whose first eigenfunctions are integrable, one can apply the result of Simon [16, Theorem 2] to conclude the C^3 (in potential u_0) dynamical stability. Note that the limit may be a different minimal Lagrangian from the original one. For instance, if one perturbs the equator of the round 2-sphere so that it divides the sphere into two components of the same area, the curve will remain so along the mean curvature flow. Thus, it will not shrink to a point, but it may converge to a different equator as $t \rightarrow \infty$.

It would be interesting to see whether the dynamical stability can be improved to C^2 -closeness in potential when $\kappa > 0$.

REFERENCES

- [1] C. Baker, *The mean curvature flow of submanifolds of high codimension*, 2010. Thesis (Ph.D.)—Australian National University.
- [2] Ph. Delanoë and A. Hirschowitz, *About the proofs of Calabi’s conjectures on compact Kähler manifolds*, Enseign. Math. (2) **34** (1988), no. 1-2, 107–122.
- [3] R. Harvey and H. B. Lawson Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [4] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299.
- [5] Y. Imagi, *Generalized Thomas-Yau uniqueness theorems*, Birational geometry, Kähler-Einstein metrics and degenerations, Springer Proc. Math. Stat., vol. 409, Springer, 2023, pp. 323–342.
- [6] X. Jin and J. Liu, *Stability of the generalized Lagrangian mean curvature flow in the cotangent bundle*, preprint, available at [arXiv:2406.04591](https://arxiv.org/abs/2406.04591).
- [7] Y. Li, *Quantitative Thomas-Yau uniqueness*, preprint, available at [arXiv:2207.08047](https://arxiv.org/abs/2207.08047).
- [8] J. D. Lotay and F. Schulze, *Consequences of strong stability of minimal submanifolds*, Int. Math. Res. Not. IMRN **8** (2020), 2352–2360.
- [9] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd ed., Oxford Graduate Texts in Mathematics, Oxford University Press, 2017.
- [10] C. B. Morrey Jr., *On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior*, Amer. J. Math. **80** (1958), 198–218.
- [11] J. Morrow and K. Kodaira, *Complex manifolds*, AMS Chelsea Publishing, 2006. Reprint of the 1971 edition with errata.
- [12] A. Neves, *Finite time singularities for Lagrangian mean curvature flow*, Ann. of Math. (2) **177** (2013), no. 3, 1029–1076.
- [13] Y.-G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), no. 2, 501–519.
- [14] ———, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. **212** (1993), no. 2, 175–192.
- [15] ———, *Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kähler manifolds*, Math. Z. **216** (1994), no. 3, 471–482.
- [16] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.
- [17] K. Smoczyk, *A canonical way to deform a Lagrangian submanifold*, preprint, available at [arXiv:dg-ga/9605005](https://arxiv.org/abs/dg-ga/9605005).
- [18] K. Smoczyk, M.-P. Tsui, and M.-T. Wang, *Generalized Lagrangian mean curvature flows: the cotangent bundle case*, J. Reine Angew. Math. **750** (2019), 97–121.
- [19] K. Smoczyk and M.-T. Wang, *Generalized Lagrangian mean curvature flows in symplectic manifolds*, Asian J. Math. **15** (2011), no. 1, 129–140.
- [20] R. P. Thomas and S.-T. Yau, *Special Lagrangians, stable bundles and mean curvature flow*, Comm. Anal. Geom. **10** (2002), no. 5, 1075–1113.
- [21] C.-J. Tsai and M.-T. Wang, *A strong stability condition on minimal submanifolds and its implications*, J. Reine Angew. Math. **764** (2020), 111–156.
- [22] M.-T. Wang, *Mean curvature flow of surfaces in Einstein four-manifolds*, J. Differential Geom. **57** (2001), no. 2, 301–338.

- [23] ———, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, Invent. Math. **148** (2002), no. 3, 525–543.
- [24] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982), no. 1, 11–29.

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