Categorical aspects of Campana orbifolds

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vorgelegt von

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To my mother.

Abstract (English)

This thesis reports on two projects related to adapted differential forms on Campana's geometric orbifolds.

The first project concerns the definition of adapted differential forms. They are usually defined only over suitably ramified covers of the geometric orbifold, and we show that they can be regarded in a natural way as a qfh-sheaf on the category of schemes over the geometric orbifold. In particular, adapted differentials form an étale sheaf with transfers.

In the second project we construct the Albanese variety of a mildly singular geometric orbifold with respect to a given suitably ramified cover. This generalizes current work in progress by Stefan Kebekus and Erwan Rousseau to geometric orbifolds with quotient singularities. A key ingredient is the following, which works more generally for klt geometric orbifolds: any adapted (reflexive) differential 1-form on a suitably ramified cover of the geometric orbifold extends to a regular differential form on a resolution of singularities of the cover.

Abstract (German)

Diese Dissertation berichtet über zwei Projekte, die sich mit angepassten Differentialformen auf Campanas geometrischen Orbifaltigkeiten befassen.

Das erste Projekt betrifft die Definition angepasster Differentialformen. Sie sind normalerweise nur über genug verzweigte Überlagerungen der geometrischen Orbifaltigkeit definiert, und wir zeigen, dass sie auf natürliche Weise als eine qfh-Garbe auf der Kategorie der Schemata über die geometrische Orbifaltigkeit betrachtet werden können. Insbesondere bilden angepasste Differentialformen eine étale Garbe mit Transferabbildungen.

Im zweiten Projekt konstruieren wir die Albanesevarietät einer mildsingulären geometrischen Orbifaltigkeit bezüglich einer genug verzweigten Überlagerung. Dies verallgemeinert unfertige Erzeugnisse von Stefan Kebekus und Erwan Rousseau zu geometrischen Orbifaltigkeiten mit Quotientensingularitäten. Eine wichtige Zutat der Konstruktion, die allgemeiner für klt geometrische Orbifaltigkeiten gilt, ist folgende: jede angepasste (reflexive) differentielle 1-Form auf einer genug verzweigten Überlagerung kann zu einer regulären Differentialform auf einer Auflösung der Singularitäten der Überlagerung erweitert werden.

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Contents

CHAPTER 0

Introduction

1. Motivation for Campana orbifolds

Campana's program for the birational classification of algebraic varieties features objects that he calls *geometric orbifolds* [Cam04], which are pairs (X, Δ) consisting of a normal (irreducible) variety X over $\mathbb{C}$ and a Weil $\mathbb{Q}$-divisor of the form $\sum_i \frac{m_i-1}{m_i} D_i$ for $m_i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, with the convention that $\frac{\infty-1}{\infty} = 1$. Therefore, strictly speaking, geometric orbifolds are particular cases of the log varieties considered in the minimal model program (MMP), cf [KM98, Example 2.20]. But Campana's point of view is to regard these pairs as geometric objects, and in this sense Campana's program differs from the MMP.

A first motivation for these geometric orbifolds is that they appear naturally in many geometric situations:

Example 0.1 (Ramified covers)**.** Let $\pi\colon Y \to X$ be a finite morphism between smooth varieties of the same dimension. Say, π is a degree m morphism with an irreducible ramification locus $R \subseteq Y$ and with branching locus $D \subseteq X$, such that $\pi^* D = mR$. The Riemann–Hurwitz adjunction formula implies that

$$K_Y = \pi^*(K_X) + (m-1)R = \pi^*\left(K_X + \frac{m-1}{m} D\right),$$

and this suggests to consider the orbifold boundary $\Delta := \frac{m-1}{m} D$ on X.

Example 0.2 (Fiber spaces)**.** Let $f\colon S \to C$ be a minimal elliptic surface, i.e., a morphism from a smooth surface onto a smooth curve whose general fiber is a smooth connected curve of genus one and which does not contain any (-1)-curve in its fibers. Let $F_1, \dots, F_l$ be the multiple fibers, say with $f(F_i) = P_i$ and $f^*(P_i) = m_i F_i$ for all $i \in \{1, \dots, l\}$. Kodaira's canonical bundle formula for elliptic surfaces [BPV84, Corollary V.12.3] implies that there exists some divisor L on C of degree $\chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_C)$ such that

$$K_S = f^*(L) + \sum_{i=1}^{l} (m_i - 1) F_i = f^*\left(L + \sum_{i=1}^{l} \frac{m_i - 1}{m_i} P_i\right),$$

and this suggests to consider the orbifold boundary $\Delta := \sum_{i=1}^{l} \frac{m_i-1}{m_i} P_i$ on C.

The previous two examples show that orbifold boundary divisors appear quite naturally in frequent geometric situations, but they do not explain why we would be interested in considering such orbifold pairs as geometric objects. After all, Example 0.2 is also featured in [KM98, Example 2.20] as a motivation for the logarithmic MMP, and such log pairs are not necessarily regarded as geometric objects in [KM98]. The following example of Campana is meant to illustrate why we would like to consider such pairs as geometric objects:

Example 0.3 ([Cam11b, Example 2.12]). Let C be the smooth projective curve corresponding to $\{(x,y) \in \mathbb{A}^2 \mid y^2 = x^6 - 1\}$ and let $\pi\colon C \longrightarrow \mathbb{P}^1$ be the degree 2 morphism corresponding to the projection $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x,y) \mapsto x$. Hence C is a hyperelliptic curve of genus 2, ramified over $\mathbb{P}^1$ at the 6 points $P_0, \dots, P_5$, with

$$P_i := \zeta_6^i \in \mathbb{A}^1 \subseteq \mathbb{P}^1$$

for all $i \in \{0, \dots, 5\}$. Let now E be the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$, and consider the product surface $S_0 := C \times E$. Let $\Phi_{S_0}\colon S_0 \longrightarrow C$ be the projection from the product. The canonical bundle on S_0 is nef, so S_0 is a minimal surface, and its Kodaira dimension is the sum of the Kodaira dimensions of the factors, i.e., $\kappa(S_0) = 1$. In the case of curves we have a nice trichotomy, in which every smooth projective curve falls into one of three "pure geometries", namely:

- Kodaira dimension $-\infty$,
- Kodaira dimension 0, which we refer to as "Calabi–Yau", and
- maximal Kodaira dimension, i.e., of general type.

In the case of surfaces the situation gets a bit more complicated. Our surface S_0 already provides an example of a geometry that does not fall into any of those three "pure geometries". But the Iitaka fibration [Mat02, Definition-Proposition 1-6-1] allows us to fiber our minimal surface with Kodaira dimension 1 into "Calabi–Yau" curves over a smooth projective curve. In our example, the projection Φ_{S_0} is the Iitaka fibration, and we see moreover that the curve on the base is of general type. That is, we have managed to decompose the intermediate geometry $\kappa(S_0) = 1$ into the two "pure geometries" of "Calabi–Yau" curves and curves of general type, as shown in Figure 1.

Since the dimension of the base of the Iitaka fibration is always the Kodaira dimension of the minimal variety on top, we can rephrase this by saying that the Kodaira dimension of the total space equals the Kodaira dimension of the general fiber plus the Kodaira dimension of the base:

$$\kappa(S_0) = \kappa(E) + \kappa(C).$$

Let us now twist the picture a bit. We consider the $\mathbb{Z}/2\mathbb{Z}$-action on S_0 given as follows. On C we act with the hyperelliptic involution, swapping the two branches of the ramified cover over $\mathbb{P}^1$. On E we act via a translation of order 2, say $1 + 2\mathbb{Z}$ acts as $\lambda + \mathbb{Z}[i] \mapsto \lambda + \frac{1}{2} + \mathbb{Z}[i]$. This gives us a fixed-point free action on the smooth projective surface S_0, so we can consider the geometric quotient $q\colon S_0 \longrightarrow S$ and this is then a finite étale morphism of degree 2. The smooth projective surface S is again

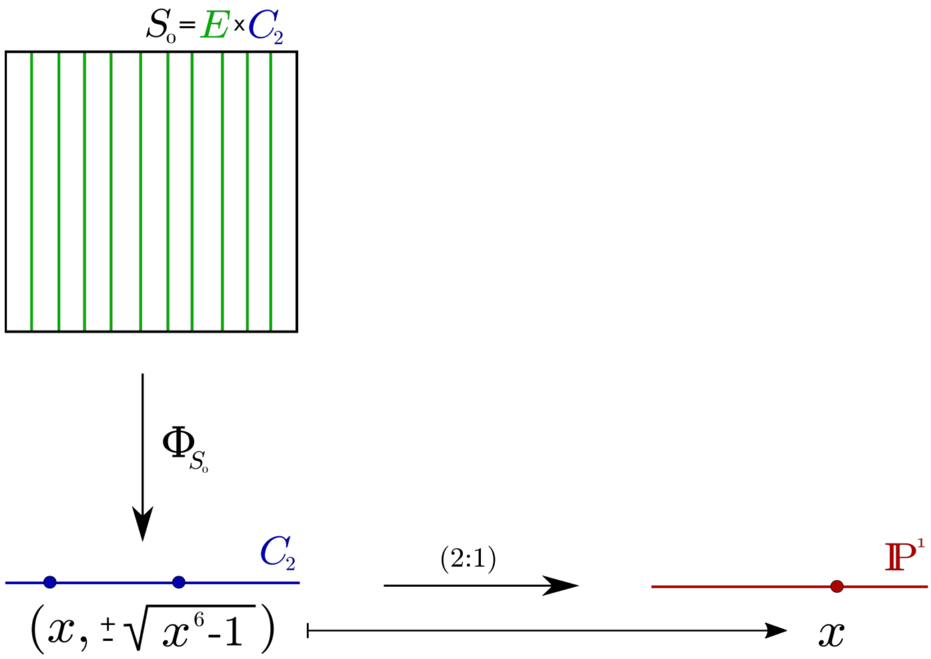

FIGURE 1. Nice decomposition of the surface S_0.

minimal with $\kappa(S) = 1$, and the Iitaka fibration is precisely the morphism $\Phi_S \colon S \to \mathbb{P}^1$ which makes the diagram in Figure 2 commute.

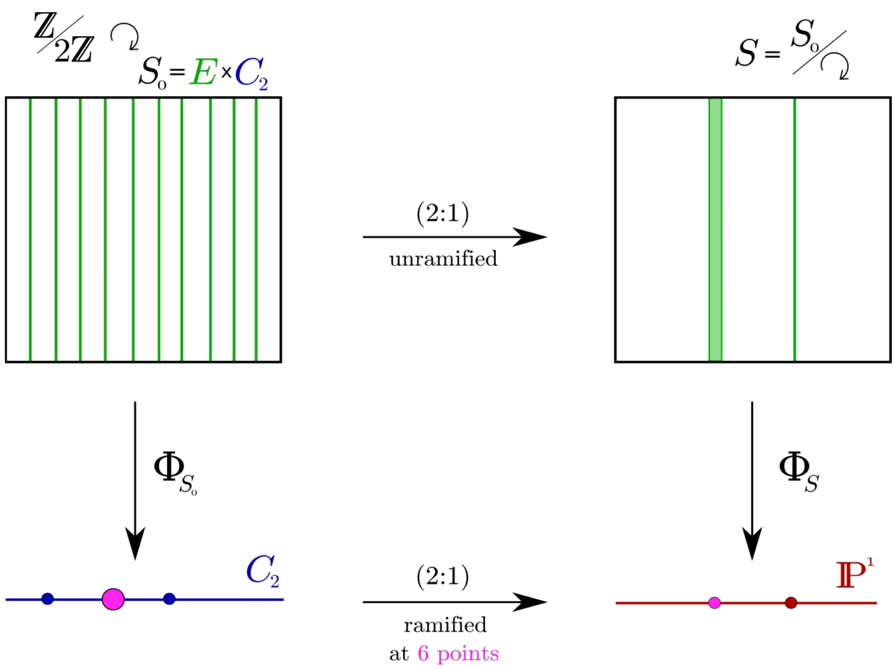

FIGURE 2. Failed decomposition of the surface S.

In this case, the fibers of Φ_S are still "Calabi–Yau" varieties, by definition of the Iitaka fibration. But the base is far from being of general type, so now we have

$$\kappa(S) = 1 \neq -\infty = \kappa(E) + \kappa(\mathbb{P}^1).$$

Campana points out that one can blame this "failed decomposition" on the multiple fibers that appear over the branching points P_i, and the solution that he proposes is to take orbifold structures into account. Namely, if we consider the orbifold boundary

$$\Delta := \sum_{i=0}^{5} \frac{1}{2} P_i$$

on $\mathbb{P}^1$, as suggested both by Example 0.1 and by Example 0.2, then we have $\deg(K_{\mathbb{P}^1} + \Delta) = 1$. So we can say that $\mathbb{P}^1$ equipped with this orbifold structure is of general type, and we have:

$$\kappa(S) = \kappa(E) + \kappa(\mathbb{P}^1, \Delta).$$

Therefore, after taking the natural orbifold structures into account, we do obtain the pleasant decomposition that we were hoping for with the Iitaka fibration.

Remark 0.4 (Orbifold base)**.** The common theme in all previous examples is the presence of multiple fibers. Orbifold structures can be used to keep track of such multiple fibers, as we did in Example 0.3. The general construction is called the *orbifold base* of a fibration [Cam04, §1].

The upshot of Campana's classification program is, roughly speaking, the following. A smooth projective variety X is called *special* if it does not admit a fibration $f : X \longrightarrow Y$ whose orbifold base is of general type, see [Cam04, Definition 2.1] for a precise definition. In the case of smooth projective curves, this class of varieties groups together curves with Kodaira dimension $-\infty$ and "Calabi–Yau" curves in the trichotomy mentioned in Example 0.3, so we now obtain a dichotomy between being special and being of general type. In higher dimensions the situation gets again more complicated, and we have "intermediate geometries" which are neither special nor of general type. As in the MMP, the idea now is to use fibrations to decompose "intermediate geometries" into these two basic kinds of geometries, and this is precisely what Campana's *core fibration* successfully does [Cam11b, Theorem 7.1].

2. Main results

As suggested in Section 1, we would like to think of geometric orbifolds as geometric objects. In particular, one should also be able to talk about differential forms on them. Geometric orbifolds interpolate between projective varieties and logarithmic pairs in the sense of Iitaka [Iit82, §11], so their differential forms should also interpolate between regular differentials on a projective variety and logarithmic differentials on a logarithmic pair. For example, in the case of $X = \mathrm{Spec}(\mathbb{C}[x,y])$ with $\Delta = \frac{1}{2}\{x=0\}+\frac{2}{3}\{y=0\}$, we would like to consider something of the form $x^{-1/2}dx$ and $y^{-2/3}dy$. There are two common approaches to make this idea precise in the literature:

(1) Working with symmetric differential forms and allowing logarithmic poles to appear only on high enough symmetric powers [Cam11a, §2.5]. For instance, in our example we could look at $x^{-1}dxdx$ and think of it as the symmetric square of $x^{-1/2}dx$.
(2) Passing to a suitable ramified cover of X on which the "multi-valued differential forms" that we want to consider pull back to well-defined differential forms [CP19, §5]. For instance, in our example we could look at the differential form $3dt$ on $Y = \mathrm{Spec}(\mathbb{C}[s,t])$ and think of it as the pull-back of $y^{-2/3}dy$ along the cover $Y \to X$ given by $x \mapsto s^2$ and $y \mapsto t^3$.

The objects of study in this thesis are *adapted differentials*, which correspond to the approach outlined in Item 2. They are a well-established tool in Campana's classification program, cf. [CP19]. If $\gamma\colon \hat{X} \to X$ is such a suitably ramified cover and $p \in \mathbb{N}$, then we denote by $\Omega^{[p]}_{(X,\Delta,\gamma)}$ the sheaf of adapted differential p-forms on $\hat{X}$. Since we allow mild singularities on $\hat{X}$, these are really reflexive differentials, and this is the reason to use the $[-]$ in the notation, cf. Item 10 in Notation 0.7. We refer to Chapter 1 for the precise definition.

2.1. Adapted differentials as a qfh-sheaf. Adapted differentials are defined only on suitably ramified covers, so we can think of them as being defined locally with respect to the qfh-topology [Voe96, §3]. The main result of Chapter 2 makes this intuition precise, showing that they are indeed the sections of a qfh-sheaf:

Theorem (cf. Theorem 2.1)**.** *Let (X,Δ) be a geometric orbifold and let $p \in \mathbb{N}$. Then there exists a qfh-sheaf $\Omega^p_{(X,\Delta)}$ on $\mathbf{Sch}/X$, characterized as a presheaf by a certain universal property, such that for every adapted morphism $\gamma\colon \hat{X} \to X$, the sections $\Omega^p_{(X,\Delta)}(\hat{X})$ are the adapted differential p-forms on $\hat{X}$.*

In particular, $\Omega^p_{(X,\Delta)}$ is an étale sheaf with transfers, i.e., enjoying certain functoriality with respect to finite correspondences [CD19, Proposition 10.5.9]. Therefore, this result provides a well-behaved and globally defined object that encodes all adapted differential forms on the geometric orbifold.

2.2. Adapted Albanese. One of the leading conjectures in Campana's theory of geometric orbifolds is the following counterpart to the Bombieri–Lang–Vojta conjecture:

Conjecture 0.5 ([Cam11b, Conjecture 11.2])**.** *Let X be a smooth projective variety defined over a number field k. Then X is special if and only if rational* *points are potentially dense in X, i.e., if and only if there exists a finite field extension K/k such that $X(K)$ is Zariski dense in X.*

For example, abelian varieties are special and their rational points are potentially dense [Has03, Proposition 4.2]. This conjecture has also a hyperbolic counterpart, and Albanese varieties are a useful tool in the study of these conjectures, see [JR22]. Therefore, it is desirable to extend the machinery of Albanese varieties to the orbifold setting. Stefan Kebekus and Erwan Rousseau construct the Albanese of a normal crossing geometric orbifold with respect to a given adapted cover in the analytic setting [KR] (work in progress). The main result of Chapter 3 generalizes this construction to singular pairs in the algebraic setting:

Theorem (cf. Theorem 3.4)**.** *Let (X, Δ) be a $\mathcal{C}$-pair with quotient singularities in which X is a projective variety. Let $\gamma\colon \hat{X} \to X$ be an adapted cover, let $\Delta_{\hat{X}} := \gamma^*(\lfloor\Delta\rfloor)_{\mathrm{red}}$, let $\hat{U} := \hat{X} \setminus \mathrm{Supp}(\Delta_{\hat{X}})$ and let $y_0 \in \hat{U}$ be a base point. Then there exists a semi-abelian variety $\mathrm{Alb}(X, \Delta, \gamma)$ and a morphism*

$$\mathrm{alb}_{(X,\Delta,\gamma)}\colon \hat{U} \to \mathrm{Alb}(X, \Delta, \gamma)$$

with the following universal property:

(1) *The morphism $\mathrm{alb}_{(X,\Delta,\gamma)}$ sends y_0 to 0.*
(2) *Let $T_1(\mathrm{Alb}(X, \Delta, \gamma))$ denote the space of logarithmic differentials with respect to a smooth compactification, cf. Item 9 in Notation 0.7. Then, the pull-back of logarithmic differential 1-forms*

$$\mathrm{alb}^*_{(X,\Delta,\gamma)}\colon T_1(\mathrm{Alb}(X, \Delta, \gamma)) \to \Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$$

has image contained in $\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X}) \subseteq \Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$. See also Items 9 and 13 in Notation 0.7 and Remark 1.70 for further clarifications of the notation.
(3) *The variety $\mathrm{Alb}(X, \Delta, \gamma)$ is initial among semi-abelian varieties under $\hat{U}$ with the previous two properties.*

Moreover, we have

$$\dim(\mathrm{Alb}(X, \Delta, \gamma)) \leq \dim_{\mathbb{C}}\left(\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})\right),$$

and the image of $\mathrm{alb}_{(X,\Delta,\gamma)}$ generates $\mathrm{Alb}(X, \Delta, \gamma)$ in the sense of Item 15 in Notation 0.7.

Remark 0.6. If $\lfloor\Delta\rfloor = 0$, then the adapted Albanese is an abelian variety. In that case, if we can moreover find a resolution of singularities of the cover whose Albanese morphism is surjective with connected fibers, then the adapted Albanese morphism is surjective and has connected fibers as well. See Remark 3.59.

An ingredient that needed to be generalized is the following extension property of adapted differential 1-forms over a klt geometric orbifold:

Theorem (cf. Theorem 3.10)**.** *Let (X, Δ) be a C-pair with klt singularities [KM98, Definition 2.34]. Let $\gamma \colon \hat{X} \to X$ be an adapted morphism and let $\pi \colon \tilde{X} \to \hat{X}$ be a log resolution of singularities of $\hat{X}$. Then the pull-back of rational differentials induces an $\mathcal{O}_{\tilde{X}}$-module morphism*

$$\pi^* \Omega^{[1]}_{(X,\Delta,\gamma)} \to \Omega^1_{\tilde{X}}.$$

3. Conventions and notation

We will mostly follow notation and conventions from [Har77] and [KM98]. For the most part, we will be interested in the algebraic setting. However, some definitions and results need to be stated in the analytic setting, to allow for the use of analytic-local arguments. Unless explicitly stated otherwise, open subsets will refer to Zariski-open subsets.

When citing The Stacks Project, we will use the format [SP, 0000]. The four characters on the right represent the tag, so the corresponding URL would be https://stacks.math.columbia.edu/tag/0000.

Notation 0.7. Some more specific notation that we will use:

(1) We will frequently use bold font to denote categories: $\mathbf{C}$, $\mathbf{D}$, ... For instance, we use **Set** to denote the category of sets. We denote by $\mathbf{C}^{\mathrm{op}}$ the *opposite category* of a category $\mathbf{C}$.

(2) An open subset U of a scheme X is called *big* if the closed subset $X \setminus U$ has codimension at least 2 in X.

(3) If X is a normal scheme, then we will denote by $\mathrm{Div}(X)$ the group of Weil divisors on X and by $W(X)$ the set of prime Weil divisors on X. We will frequently work with pairs (X, Δ) where X is a normal scheme and Δ is a Weil $\mathbb{Q}$-divisor on X. If we need to be specific about it we will write (X, Δ_X) instead. If we write

$$\Delta = \sum_{i \in I} a_i D_i,$$

then it is understood that $a_i \neq 0$ for all $i \in I$ and the D_i are distinct irreducible components of Δ. In this case, for $D \in W(X)$, we define

$$\mathrm{mult}_D(\Delta) := \begin{cases} a_i & \text{if } D = D_i \text{ for some } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

A *reduced divisor* on X is a Weil $\mathbb{Q}$-divisor whose non-zero coefficients are all 1. If $\Delta = \sum_{i \in I} a_i D_i$ is any Weil $\mathbb{Q}$-divisor, then we denote by $\Delta_{\mathrm{red}} = \sum_{i \in I} D_i$ the underlying reduced divisor. See [KM98, Notation 0.4] for more conventions regarding Weil $\mathbb{Q}$-divisors. If D is a Cartier divisor on a scheme X, then we denote by $\mathcal{O}_X(D)$ the corresponding line bundle. Note that this differs from the notation $\mathcal{L}(D)$ in [Har77].

(4) If k is a field and X and Y are k-schemes, then a *morphism* $f : X \to Y$ will mean a k-scheme morphism unless we explicitly say otherwise. We will usually work over the field $\mathbb{C}$ of complex numbers, so by a *variety* we will mean a separated integral scheme of finite type over $\mathbb{C}$. In Chapter 2, all schemes are assumed to be of finite type over $\mathbb{C}$. We denote by $\mathbf{Sch}/X$ the category of finite type X-schemes, i.e., schemes over X such that the corresponding morphism is of finite type.

(5) Let $f : X \to Y$ be a morphism of normal varieties along which we can pull back rational functions [SP, 02OT] and let Δ be a $\mathbb{Q}$-divisor on Y. If Δ is $\mathbb{Q}$-Cartier, then we define

$$f^*(\Delta) := \frac{1}{m} f^*(m\Delta)$$

for any $m \in \mathbb{N}_{>0}$ such that $m\Delta$ is Cartier, where $f^*(m\Delta)$ denotes the pull-back of $m\Delta$ as a Cartier divisor [GW10, Definition 11.47]. If f is quasi-finite and $\dim(X) = \dim(Y)$, then we can pull back Δ as in Lemma B.7. These two notions of pull-back of Δ along f are compatible by Lemma B.7.

(6) By a *Galois cover* we mean a finite morphism $f : X \to Y$ between normal varieties of the same dimension such that there exists a finite group G acting algebraically on X with geometric quotient $f : X \to Y$, see Definition B.18. In particular, a Galois cover does not need to be an étale morphism. In this setting, the group G is determined by the morphism, cf. Lemma B.22; we refer to G as the *Galois group* of the cover. By a *cyclic cover* we mean a Galois cover with cyclic Galois group. By an *abelian cover* we mean a Galois cover with abelian Galois group. See Appendix B for more on Galois covers.

(7) Let X be a normal variety. A *log pair* is a pair (X, Δ) consisting of a normal variety X and a reduced divisor Δ. We say that the log pair (X, Δ) is *projective* if X is projective, and *smooth* if X is smooth and Δ is simple normal crossing (snc) or normal crossing (nc) in the analytic setting. More generally, if (X, Δ) is a pair consisting of a normal variety X and a Weil $\mathbb{Q}$-divisor Δ, we say that (X, Δ) is *projective* (resp. *smooth*) if the log pair $(X, \Delta_{\mathrm{red}})$ is projective (resp. smooth).

(8) If X is a (complex) variety and $i \in \mathbb{N}$, then we denote by $H_i(X, \mathbb{Z})$ the i-th singular homology group of its analytification. Similarly for singular cohomology and for log pairs.

(9) Let U be a smooth variety and let (X, Δ) be a smooth compactification[1] of U, i.e., a smooth projective log pair (X, Δ) such that

[1]Such compactifications always exist thanks to Nagata's compactification theorem and Hironaka's resolution of singularities, cf. [Iit82, §7.10.d].

$X \setminus \operatorname{Supp}(\Delta) = U$. We will use the shorthand notation

$$T_1(U) := \Omega^1_X(\log \Delta)(X),$$

where $\Omega^1_X(\log \Delta)$ is the sheaf of logarithmic differentials, see [Del70, II.§3]. By [Iit82, Theorem 11.1], this $\mathbb{C}$-vector space does not depend on the choice of such a compactification, so there is not much ambiguity in the notation.

(10) We follow [GKKP11, §2.E] when it comes to reflexive sheaves. If X is a normal variety, then we denote by $\Omega^{[p]}_X$ the sheaf of *reflexive differential p-forms* on X. It can be described as the pushforward of the sheaf of differential p-forms along the inclusion of the smooth locus. If $f\colon X \to Y$ is a morphism of normal varieties and $\mathcal{F}$ is a coherent sheaf on Y, then we denote by $f^{[*]}\mathcal{F} := (f^*\mathcal{F})^{\vee\vee}$ the reflexive hull of $f^*\mathcal{F}$. Note that this differs from the notation $(f^*\mathcal{F})^{[1]}$ in [KM98]. If (X, Δ) is a log pair, and if $i\colon U \to X$ is the inclusion of the largest open subset such that $(U, \Delta|_U)$ is smooth, then we define the sheaf of *reflexive logarithmic differential p-forms* as $\Omega^{[p]}_X(\log \Delta) := i_*\Omega^p_U(\log \Delta|_U)$ for every $p \in \mathbb{N}$. See Appendix A for more on reflexive sheaves.

(11) Let X be a noetherian scheme. Then we denote by $\mathcal{K}_X$ the sheaf of meromorphic functions on X, as defined in [SP, 01X2]. Since we will also work in the analytic setting sometimes, we refer to this sheaf as the *sheaf of rational functions* on X in order to avoid potential confusion with (analytic) meromorphic functions. Since we will work only with noetherian schemes, the notions of meromorphic and rational functions coincide, cf. [SP, 0EMF]. Similarly, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then we will refer to $\mathcal{K}_X(\mathcal{F}) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ as the *sheaf of rational sections* of $\mathcal{F}$, cf. [SP, 01X4]. Moreover, X will frequently be an integral scheme, in which case $\mathcal{K}_X$ is isomorphic to the constant sheaf with value the function field of X and $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value the stalk of $\mathcal{F}$ at the generic point of X [SP, 01X5]. In particular, if $U \subseteq X$ is a dense open subset, then any section in $\mathcal{F}(U)$ defines a rational section.[2] See also [Iit82, §2.19.a].

(12) Let X be a normal algebraic variety and let $p \in \mathbb{N}$. Then we refer to sections of $\mathcal{K}_X(\Omega^p_X)$ as *rational differential p-forms*, or simply rational differentials if no confusion is likely to happen. Let now Δ be a reduced Weil divisor, possibly zero. Then $\Omega^{[p]}_X(\log \Delta)$ is reflexive, hence torsion-free. So we have natural injective[3] morphisms such that the diagram

[2]This is not true anymore in the analytic setting. For example, $\sin\colon \mathbb{A}^1 \to \mathbb{A}^1$ does not define a meromorphic function on the Riemann sphere $\mathbb{P}^1$.

[3]However, the natural morphism $\Omega^p_X \to \mathcal{K}_X(\Omega^p_X)$ need not be injective, because the sheaf of Kähler differentials may contain torsion. See [GR11].

$$\begin{tikzcd} \Omega_X^{[p]}(\log \Delta) \arrow[r, hook] \arrow[dr, hook] & \mathcal{K}_X(\Omega_X^{[p]}(\log \Delta)) \arrow[d, "\cong"] \\ & \mathcal{K}_X(\Omega_X^p) \end{tikzcd}$$

commutes. We will sometimes identify $\Omega_X^{[p]}(\log \Delta)$ with the corresponding subsheaf of $\mathcal{K}_X(\Omega_X^p)$ under the injective morphism above, so let us describe the image of this morphism explicitly. Since it is an injective sheaf morphism, the image sheaf agrees with the image presheaf, so it suffices to compute the image of the morphism on sections over an open subset $V \subseteq X$, which we may assume to be non-empty. So let $\sigma \in \Omega_X^{[p]}(\log \Delta)(V)$ and let $i \colon U \to X$ be the inclusion of the largest open subset such that $(U, \Delta|_U)$ is smooth. By definition we have $\sigma \in \Omega_U^p(\log \Delta|_U)(V \cap U)$. Let $W := (V \cap U) \setminus \mathrm{Supp}(\Delta|_{V \cap U})$. Then we have $\sigma|_W \in \Omega_X^p(W)$, and the image of σ in $\mathcal{K}_X(\Omega_X^p)(V) \cong (\Omega_X^p)_\eta$ is given by the germ of $\sigma|_W$ at the generic point η of X. Conversely, a section $\xi \in \mathcal{K}_X(\Omega_X^p)(V)$ corresponding to the germ of some $\tau \in \Omega_X^p(V')$ is in the image of $\Omega_X^{[p]}(\log \Delta)(V)$ if there exists some $\sigma \in \Omega_U^p(\log \Delta|_U)(V \cap U)$ such that $(\sigma|_W)_\eta = \tau_\eta$, where $W = (V \cap U) \setminus \mathrm{Supp}(\Delta|_{V \cap U})$ is as above. In plain words, a germ $\tau_\eta \in (\Omega_X^p)_\eta$ is in the image of $\Omega_X^{[p]}(\log \Delta)(V)$ if it extends to a reflexive logarithmic form over V.

(13) Let $f \colon X \to Y$ be a morphism of normal algebraic varieties and let $p \in \mathbb{N}$. We will be interested in pulling back differential p-forms along f, and there are several ways we could go about this:
- Usual pull-back of Kähler differentials.
- If defined, pull-back of rational differentials.
- If defined, pull-back of reflexive (logarithmic) differentials.

Since we are working with normal varieties, we are only interested in rational and reflexive differentials, and not so much in usual Kähler differentials. Let us summarize here the usual setting in which we can pull back rational (resp. reflexive) differentials, and how the two notions of pull-back agree. If f is dominant, then we can always pull-back rational differentials. Indeed, we can pull back rational functions [SP, 02OU] and we can pull back Kähler differentials, so we obtain a pull-back morphism

$$f^* \colon (\Omega_Y^p)_{\eta_Y} \to (\Omega_X^p)_{\eta_X}$$

given as the morphism on global sections of the composition

$$\mathcal{K}_Y(\Omega_Y^p) \to f_*(\mathcal{K}_X(f^*\Omega_Y^p)) \to f_*(\mathcal{K}_X(\Omega_X^p)),$$

in which both morphisms are induced by the canonical ones, cf. [SP, 02OY]. See also [Iit82, §2.20.d]. Since f is dominant and the sheaves of Kähler differentials are locally free over some dense open subsets, f^* is injective [Iit82, Lemma 2.35]. Therefore, we have the following situation:

$$\begin{tikzcd}
\Omega_Y^{[p]}(Y) \arrow[r, dashed, "\exists ?"] \arrow[d, hook, "(-)_{\eta_Y}"'] & \Omega_X^{[p]}(X) \arrow[d, hook, "(-)_{\eta_X}"] \\
(\Omega_Y^p)_{\eta_Y} \arrow[r, hook, "f^*"] & (\Omega_X^p)_{\eta_X}.
\end{tikzcd}$$

The diagram shows that, if at all possible, there is at most one way to define a pull-back of reflexive (logarithmic) differentials in a way which is compatible with the pull-back of rational differentials. Some example situations when this is possible are when f is a quasi-finite morphism between normal varieties of the same dimension, or when the singularities of Y are mild enough.

Sometimes we will also be interested in the existence of pull-backs at the level of sheaves, rather than just at the level of global sections. In this case, we consider the morphism $f^*(\mathcal{K}_Y(\Omega_Y^p)) \to \mathcal{K}_X(\Omega_X^p)$ adjoint to the composition $\mathcal{K}_Y(\Omega_Y^p) \to f_*(\mathcal{K}_X(\Omega_X^p))$ discussed above. Then, we have the following situation:

$$\begin{tikzcd}
f^*\Omega_Y^{[p]} \arrow[r, dashed, "\exists ?"] \arrow[d, "f^*((-)_{\eta_Y})"'] & \Omega_X^{[p]} \arrow[d, hook, "(-)_{\eta_X}"] \\
f^*(\mathcal{K}_Y(\Omega_Y^p)) \arrow[r, "f^*"] & \mathcal{K}_X(\Omega_X^p).
\end{tikzcd}$$

As before, since $(-)_{\eta_X}$ is a monomorphism, there is at most one way to complete the diagram into a commutative square. If it is possible to complete it, we will say that the *pull-back of rational differentials induces a morphism*

$$f^*\Omega_Y^{[p]} \to \Omega_X^{[p]}.$$

The uniqueness ensures that there is no ambiguity in saying it like that. Note that, by construction, if such a morphism exists, then the morphism on global sections induced by the composition

$$\Omega_Y^{[p]} \to f_*f^*\Omega_Y^{[p]} \to f_*\Omega_X^{[p]}$$

is the pull-back of reflexive differentials discussed before. In relation to Item 12, the existence of such a morphism is equivalent to the image of the sheaf morphism $f^*\Omega_Y^{[p]} \to \mathcal{K}_X(\Omega_X^p)$ being contained in the image of $\Omega_X^{[p]}$ in $\mathcal{K}_X(\Omega_X^p)$, and similarly for logarithmic differentials if any boundaries are being considered.

More generally, suppose $\mathcal{G} \subseteq \mathcal{K}_Y(\Omega_Y^p)$ and $\mathcal{F} \subseteq \mathcal{K}_X(\Omega_X^p)$ are subsheaves of quasi-coherent modules. Then, we will say that the *pull-back of rational differentials induces a morphism*

$$f^*\mathcal{G} \to \mathcal{F}$$

if the analogous statement holds.

Suppose moreover that $\mathcal{G}$ and $\mathcal{F}$ are coherent and that $\mathcal{F}$ is reflexive. If the pull-back of rational differentials induces a morphism $f^*\mathcal{G} \to \mathcal{F}$, then the universal property of the reflexive hull yields a morphism $f^{[*]}\mathcal{G} \to \mathcal{F}$. In this situation, we will also say that *the pull-back of rational differentials induces a morphism*

$$f^{[*]}\mathcal{G} \to \mathcal{F}.$$

By definition, this morphism fits into a commutative diagram

$$\begin{tikzcd} f^*\mathcal{G} \arrow[r] \arrow[dr] & f^{[*]}\mathcal{G} \arrow[d] & \\ & \mathcal{F} \arrow[r, hook] & \mathcal{K}_X(\Omega_X^p) \end{tikzcd}$$

in which the morphism $f^*\mathcal{G} \to f^{[*]}\mathcal{G}$ is the natural morphism into the reflexive hull. This means, for instance, that if the pull-back of rational differentials induces a morphism

$$f^*\Omega_Y^{[p]} \to \Omega_X^{[p]},$$

then the induced morphism

$$f^{[*]}\Omega_Y^{[p]} \to \Omega_X^{[p]}$$

has the property that it restricts to the usual pull-back of Kähler differentials over the appropriate smooth loci.

(14) We denote by $\mathbb{G}_a$ the algebraic group corresponding to the additive group $(\mathbb{C}, +)$, and by $\mathbb{G}_m$ the algebraic group corresponding to the multiplicative group $(\mathbb{C}^\times, \cdot)$.

(15) Let $f \colon X \to G$ be a morphism from a variety to a commutative algebraic group. The subgroup

$$G' := \{f(x) - f(y) \mid x, y \in X\} \subseteq G$$

is an irreducible algebraic subgroup [Ser59, §1]. We say that *the image of f generates G* if $G' = G$ [Ser59, Définition 1].

CHAPTER 1

Campana orbifolds and adapted differentials

This chapter introduces the objects of study: Campana's geometric orbifolds and adapted differentials. In Section 1 we define C-pairs and their morphisms in the sense of Campana. Such morphisms, despite not playing an important role in this thesis, are an important guiding principle to keep in mind when discussing any aspect of the theory of geometric orbifolds. In Section 2 we discuss some constructions with geometric orbifolds. Some of these will be relevant to define the singularities that we are interested in, which we do in Section 3. In Section 4 we study adapted covers in some detail, paying particular attention to the constructions that will be used to prove the results in the following chapters. The main contributions in this section are the notion of a perfectly adapted cover and the existence of suitable such covers in the singular setting, cf. Definition 1.36 and Corollary 1.60 respectively. In Section 5 we study adapted differential forms on adapted covers. A key observation here, which is one of the reasons that make perfectly adapted covers useful, is that the sheaf of adapted differentials on a perfectly adapted cover agrees with the usual sheaf of reflexive (logarithmic) differentials, cf. Lemma 1.73.

1. C-pairs and their category

The word *orbifold* is used already in various different settings, so we will use the term *C-pair* instead of the term *geometric orbifold* used by Campana in [Cam11a, §2.1].

Definition 1.1 (C-pair, cf. [Cam11a, Définition 2.1])**.** A *C-pair* (X, Δ) consists of a normal algebraic or analytic variety X and a $\mathbb{Q}$-divisor Δ of the form

$$\Delta = \sum_{i \in I} \frac{m_i - 1}{m_i} D_i,$$

where the D_i are pairwise distinct prime Weil divisors and where $m_i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $i \in I$, with the convention that $\frac{\infty - 1}{\infty} = 1$.

Remark 1.2. In the analytic case, the indexing set I need not be finite, but since Δ is a divisor the sum will at least be locally finite.

Definition 1.3 (C-multiplicity, cf. [Cam11a, Définition 2.1])**.** Let (X, Δ) be a C-pair and let $D \in W(X)$ be a prime Weil divisor on X. Then we define

$$\operatorname{mult}_{C,D}(\Delta) := \begin{cases} m_i & \text{if } D \text{ appears in } \Delta \text{ with non-zero coefficient } \frac{m_i - 1}{m_i}, \\ 1 & \text{otherwise.} \end{cases}$$

We call $\mathrm{mult}_{C,D}(\Delta)$ the *C-multiplicity of Δ along D*.

Notation 1.4. Let (X, Δ) be a C-pair and let I be the corresponding indexing set. We define the following subsets of indices:

- The *logarithmic indices* $I^{\log} := \{i \in I \mid m_i = \infty\}$.
- The *orbifold indices* $I^{\mathrm{orb}} := I \setminus I^{\log}$.

We also define the following $\mathbb{Q}$-divisors:

- The *logarithmic boundary* $\Delta^{\log} := \lfloor \Delta \rfloor = \sum_{i \in I^{\log}} D_i$.
- The *orbifold boundary* $\Delta^{\mathrm{orb}} := \lceil \Delta \rceil - \Delta = \sum_{i \in I^{\mathrm{orb}}} \frac{1}{m_i} D_i$.

Definition 1.5 (C-morphism, cf. [Cam11a, Définition 2.3])**.** Let (X, Δ_X) and (Y, Δ_Y) be two C-pairs. Assume that Y is $\mathbb{Q}$-factorial[1] and let $f\colon X \longrightarrow Y$ be a morphism such that $f(X) \not\subseteq \mathrm{Supp}(\Delta_Y)$. We say that f *induces an orbifold morphism* if for every prime Weil divisor $D' \in W(Y)$ and every prime Weil divisor $D \in W(X)$ with $\mathrm{mult}_D(f^*D') \neq 0$ we have

$$\mathrm{mult}_D(f^*D')\,\mathrm{mult}_{C,D}(\Delta_X) \geq \mathrm{mult}_{C,D'}(\Delta_Y).$$

We say that f is a *morphism of C-pairs*, or a *C-morphism*, if the following additional conditions are satisfied:

(1) we are in the algebraic setting,
(2) the variety X is also $\mathbb{Q}$-factorial,
(3) the morphism f is dominant, and
(4) the morphism f induces an orbifold morphism.

In this case, we denote it by $f\colon (X, \Delta_X) \longrightarrow (Y, \Delta_Y)$.

Remark 1.6. Let (X, Δ_X) and (Y, Δ_Y) be two algebraic C-pairs. Then, a C-morphism is a morphism of the underlying varieties satisfying the properties in Definition 1.5. So being a C-morphism is a property of morphisms of the underlying varieties, and not an extra structure on them.

Lemma 1.7. *The property of being a C-morphism is (Zariski-)local on the target [GW10, Appendix C].*

Proof. Being dominant is local on the target for morphisms between integral schemes [SP, 0CC1], and $\mathbb{Q}$-factoriality is local in the Zariski topology, cf. Lemma C.1. Let now (X, Δ_X) and (Y, Δ_Y) be C-pairs with X and Y $\mathbb{Q}$-factorial, and let $f\colon X \longrightarrow Y$ be a dominant morphism. If $\{V_i\}_{i \in I}$ is an open cover of Y, say with $V_i \neq \varnothing$ for all $i \in I$, then $f|_{f^{-1}(V_i)}\colon (f^{-1}(V_i), \Delta) \longrightarrow (V_i, \Delta)$ is a C-morphism for all $i \in I$ as a consequence of Lemma 1.13 and Remark 1.17 below. Conversely, if $f|_{f^{-1}(V_i)}\colon (f^{-1}(V_i), \Delta) \longrightarrow (V_i, \Delta)$ is a C-morphism for all $i \in I$, then $f\colon (X, \Delta_X) \longrightarrow (Y, \Delta_Y)$ is a C-morphism because we can compute $\mathrm{mult}_D(f^*D')$ locally for all $D \in W(X)$ and all $D' \in W(Y)$. □

[1] If Y is an analytic variety, this means globally analytically $\mathbb{Q}$-factorial, cf. Appendix C.

Remark 1.8. Lemma 1.7 is no longer true in the analytic setting, because being $\mathbb{Q}$-factorial is not local in the analytic topology, cf. Example C.2. For this and other reasons, in [KR], the term C-morphism is used with a different meaning.

Remark 1.9. In the setting of Definition 1.5, for $D \in W(X)$ and $D' \in W(Y)$, the following are equivalent:

(1) We have $\mathrm{mult}_D(f^*D') \neq 0$.
(2) We have $f(D) \subseteq D'$.
(3) We have $D \subseteq \mathrm{Supp}(f^*D')$.

Indeed, since the pull-back is defined by $\mathbb{Q}$-linearity, we may assume that D' is a Cartier divisor. Then we have $\mathrm{Supp}(f^*D') = f^{-1}(D')$ at the level of topological spaces [GW10, Corollary 11.49]. If $f(D) \subseteq D'$, then $D \subseteq f^{-1}(f(D)) \subseteq f^{-1}(D')$, so Item 3 follows from Item 2. Conversely, if $D \subseteq f^{-1}(D')$, then $f(D) \subseteq D'$, so Item 2 and Item 3 are equivalent. And Item 3 is equivalent to Item 1 by definition of the support.

Example 1.10. A dominant morphism between $\mathbb{Q}$-factorial varieties $f\colon X \to Y$ is not always a C-morphism $f\colon (X,0) \to (Y,0)$. Indeed, let $q\colon \mathbb{A}^2 \to X$ be the geometric quotient of $\mathbb{A}^2$ by the $\mathbb{Z}/2\mathbb{Z}$-action $(x,y) \mapsto (-x,-y)$, i.e., q is induced by the injective $\mathbb{C}$-algebra morphism

$$\begin{aligned} q^\sharp\colon \mathbb{C}[u,v,w]/(uw-v^2) &\to \mathbb{C}[x,y] \\ u + (uw-v^2) &\mapsto x^2, \\ v + (uw-v^2) &\mapsto xy, \\ w + (uw-v^2) &\mapsto y^2. \end{aligned}$$

Denoting by $i\colon X \hookrightarrow \mathbb{A}^3$ the closed immersion, we see that the map $q^\sharp$ is the map induced by the universal property of the quotient $i^\sharp$ and the $\mathbb{C}$-algebra morphism

$$\begin{aligned} (i \circ q)^\sharp\colon \mathbb{C}[u,v,w] &\to \mathbb{C}[x,y] \\ u &\mapsto x^2, \\ v &\mapsto xy, \\ w &\mapsto y^2. \end{aligned}$$

Note that X is the cone over the plane conic $\{(u:v:w) \in \mathbb{P}^2 \mid uw-v^2 = 0\}$, so we can consider its standard resolution of singularities $\pi_X\colon \tilde{X} \to X$, which can also be described as the quotient of the blow-up of $\mathbb{A}^2$ at the origin by the induced $\mathbb{Z}/2\mathbb{Z}$-action. Let us make this more explicit.

Let $\nu\colon \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ denote the blow-up at the origin. We consider coordinates $(t_0 : t_1)$ on $\mathbb{P}^1$, so that denoting $t_{i/j} = t_i/t_j$ we can cover $\tilde{\mathbb{A}}^2$ by the two affine charts

$$\tilde{\mathbb{A}}^2_0 = \{(x,y,t_{1/0}) \in \mathbb{A}^3 \mid y = xt_{1/0}\} \text{ and } \tilde{\mathbb{A}}^2_1 = \{(x,y,t_{0/1}) \in \mathbb{A}^3 \mid x = yt_{0/1}\}.$$

The $\mathbb{Z}/2\mathbb{Z}$-action on $\tilde{\mathbb{A}}^2$ is given by

$$(x, y; t_0 : t_1) \mapsto (-x, -y; t_0 : t_1),$$

which is well-defined because $xt_1 = yt_0$ implies that $-xt_1 = -yt_0$.

On the other hand, in order to resolve X, we first blow-up the ambient space $\mathbb{A}^3$ at the origin. This gives us a morphism $\pi\colon \tilde{\mathbb{A}}^3 \to \mathbb{A}^3$. Consider coordinates $(s_0 : s_1 : s_2)$ on $\mathbb{P}^2$ and let $s_{i/j}$ denote s_i/s_j again. Then we can cover $\tilde{\mathbb{A}}^3$ by the three affine charts

$$\tilde{\mathbb{A}}^3_0 = \{(u, v, w, s_{1/0}, s_{2/0}) \in \mathbb{A}^5 \mid v = us_{1/0}, w = us_{2/0}\},$$

$$\tilde{\mathbb{A}}^3_1 = \{(u, v, w, s_{0/1}, s_{2/1}) \in \mathbb{A}^5 \mid u = vs_{0/1}, w = vs_{2/1}\}, \text{ and}$$

$$\tilde{\mathbb{A}}^3_2 = \{(u, v, w, s_{0/2}, s_{1/2}) \in \mathbb{A}^5 \mid u = ws_{0/2}, v = ws_{1/2}\}.$$

With respect to these charts, the strict transform $\tilde{X} \subseteq \tilde{\mathbb{A}}^3$ of $X \subseteq \mathbb{A}^3$ is given by

$$\tilde{X}_0 = \{(u, v, w, s_{1/0}, s_{2/0}) \mid v = us_{1/0}, w = us_{2/0}, s_{2/0} = s_{1/0}^2\},$$

$$\tilde{X}_1 = \{(u, v, w, s_{0/1}, s_{2/1}) \mid u = vs_{0/1}, w = vs_{2/1}, s_{0/1}s_{2/1} = 1\}, \text{ and}$$

$$\tilde{X}_2 = \{(u, v, w, s_{0/2}, s_{1/2}) \mid u = ws_{0/2}, v = ws_{1/2}, s_{0/2} = s_{1/2}^2\}.$$

The Jacobian criterion shows that $\tilde{X}$ is smooth, so $\pi_X\colon \tilde{X} \to X$ is a resolution of singularities.

We now look at the situation on the affine charts $\tilde{\mathbb{A}}^2_0$ and $\tilde{\mathbb{A}}^3_0$. We want to find a dashed arrow making the following diagram commutative:

$$\begin{tikzcd} \tilde{\mathbb{A}}^2_0 \arrow[r, "\nu_0"] \arrow[d, dashed, "\varphi_0"'] & \mathbb{A}^2 \arrow[d, "i\circ q"] \\ \tilde{\mathbb{A}}^3_0 \arrow[r, "\pi_0"] & \mathbb{A}^3. \end{tikzcd}$$

This arrow would correspond to a $\mathbb{C}$-algebra morphism

$$\mathbb{C}[u, v, w, s_{1/0}, s_{2/0}]/J \to \mathbb{C}[x, y, t_{1/0}]/I,$$

where $I = (y - xt_{1/0})$ and $J = (v - us_{1/0}, w - us_{2/0})$, so we only need to determine the image of the generators and check that the corresponding morphism is well-defined. Since we want the diagram to commute, we need

$$u + J \mapsto x^2 + I, \quad v + J \mapsto xy + I \quad \text{and} \quad w + J \mapsto y^2.$$

In order for the corresponding morphism to be well-defined, we need that

$$v - us_{1/0}, w - us_{2/0} \mapsto 0 + I,$$

so we can define $s_{1/0} \mapsto t_{1/0}$ and $s_{2/0} \mapsto t_{1/0}^2$. This defines a morphism $\varphi_0\colon \tilde{\mathbb{A}}^2_0 \to \tilde{\mathbb{A}}^3_0$, and since $\tilde{X}_0$ is cut out by the equation $s_{2/0} = s_{1/0}^2$, we can factor this morphism as

$$\begin{tikzcd} \tilde{\mathbb{A}}^2_0 \arrow[r, "\nu_0"] \arrow[d, "p_0"] \arrow[dd, bend right, "\varphi_0"'] & \mathbb{A}^2 \arrow[d, "q"] \\ \tilde{X}_0 \arrow[r, "\pi_{X,0}"] \arrow[d, "j_0"] & X \arrow[d, "i"] \\ \tilde{\mathbb{A}}^3_0 \arrow[r, "\pi_0"] & \mathbb{A}^3, \end{tikzcd}$$

where p_0 is the geometric quotient of $\tilde{\mathbb{A}}^2_0$ by the induced $\mathbb{Z}/2\mathbb{Z}$-action and j_0 is the closed immersion.

Doing analogous computations over the remaining affine charts shows that we have a commutative diagram

$$\begin{tikzcd} \tilde{\mathbb{A}}^2 \arrow[r, "\nu"] \arrow[d, "p"'] & \mathbb{A}^2 \arrow[d, "q"] \\ \tilde{X} \arrow[r, "\pi_X"] & X \end{tikzcd}$$

in which p and q are the respective geometric quotients. Moreover, π_X is surjective and $\tilde{X}$ and X are $\mathbb{Q}$-factorial. We claim that π_X is not a C-morphism $\pi_X\colon (\tilde{X}, 0) \to (X, 0)$. Indeed, we consider the closed subscheme $D \subseteq X$ defined by the ideal

$$(u + (uw - v^2), v + (uw - v^2)) \subseteq \mathbb{C}[u, v, w]/(uw - v^2).$$

This subscheme is a prime Weil divisor such that $2D$ is a Cartier divisor defined by the ideal $(u+(uw-v^2))$, see [Har77, Example II.6.5.2]. Therefore we can write

$$\pi_X^*(D) = \frac{1}{2}\pi_X^*(2D).$$

We compute the pull-back of the Cartier divisor $2D$ along π_X as an effective Cartier divisor [GW10, Definition 11.47]. The restriction $\pi_X|_{\tilde{X}_0}\colon \tilde{X}_0 \to X$ is given by the $\mathbb{C}$-algebra morphism

$$\mathbb{C}[u, v, w]/(uw - v^2) \to \mathbb{C}[u, v, w, s_{1/0}, s_{2/0}]/(v - us_{1/0}, w - us_{2/0}, s_{2/0} - s_{1/0}^2)$$

induced by the identity on $\mathbb{C}[u, v, w]$. Therefore, the closed subscheme $\pi_X^*(2D) \cap \tilde{X}_0 \subseteq \tilde{X}_0$ is defined by the ideal $(u+(v-us_{1/0}, w-us_{2/0}, s_{2/0}-s_{1/0}^2))$. This subscheme is the intersection $E \cap \tilde{X}_0$, where $E \subseteq \tilde{X}$ is the exceptional divisor. On the other hand, the restriction $\pi_X|_{\tilde{X}_1}\colon \tilde{X}_1 \to X$ corresponds to the $\mathbb{C}$-algebra morphism

$$\mathbb{C}[u, v, w]/(uw - v^2) \to \mathbb{C}[u, v, w, s_{1/0}, s_{2/0}]/(u - vs_{0/1}, w - vs_{2/1}, s_{0/1}s_{2/1} - 1)$$

induced again by the identity on $\mathbb{C}[u, v, w]$. Therefore, the closed subscheme $\pi_X^*(2D) \cap \tilde{X}_1 \subseteq \tilde{X}_1$ is again the exceptional divisor $E \cap \tilde{X}_1$. And, similarly, the closed subscheme $\pi_X^*(2D) \cap \tilde{X}_2 \subseteq \tilde{X}_2$ is the union of the exceptional divisor and the strict transform $\widetilde{(2D)} \cap \tilde{X}_2$, defined as a closed subscheme by the ideal $(s_{1/2} + (u - ws_{0/2}, v - ws_{1/2}, s_{0/2} - s_{1/2}^2))^2$. It follows that

$$\pi_X^*(D) = \tilde{D} + \frac{1}{2}E,$$

where $\tilde{D} \subseteq \tilde{X}$ is the strict transform of D and $E \subseteq \tilde{X}$ is the exceptional divisor. So π_X is not a C-morphism, because

$$\operatorname{mult}_E(\pi_X^* D)\operatorname{mult}_{C,E}(0) = \operatorname{mult}_E(\pi_X^* D) = \frac{1}{2} \ngeq 1 = \operatorname{mult}_{C,D}(0).$$

However, we have the following:

Remark 1.11. If $f: X \to Y$ is a quasi-finite morphism between $\mathbb{Q}$-factorial varieties of the same dimension, Lemma B.8 shows that $f: (X,0) \to (Y,0)$ is always a C-morphism.

Remark 1.12. Even if both f and g induce orbifold morphisms, the composition $(g \circ f)$ may fail to induce an orbifold morphism, because the image of the composition may be contained in the support of the boundary divisor of the target. For example, let $f: D \to \mathbb{A}^1$ be the inclusion of the origin and let $g: \mathbb{A}^1 \to \mathbb{A}^1$ be the morphism given by $x \mapsto x^2$. Then f induces an orbifold morphism $(D,0) \to (\mathbb{A}^1, 0)$ and g induces an orbifold morphism $(\mathbb{A}^1,0) \to (\mathbb{A}^1, \frac{1}{2}D)$, but $g \circ f$ does not induce an orbifold morphism because $(g \circ f)(D) \subseteq D$.

However, we have the following:

Lemma 1.13 ([Cam11a, Remarque 2.5])**.** *Let $f: (X, \Delta_X) \to (Y, \Delta_Y)$ and $g: (Y, \Delta_Y) \to (Z, \Delta_Z)$ be C-morphisms. Then $g \circ f: (X, \Delta_X) \to (Z, \Delta_Z)$ is a C-morphism as well.*

Proof. Let $D'' \in W(Z)$ be prime Weil divisor. We have

$$g^*D'' = \sum_{D' \in W(Y)} \operatorname{mult}_{D'}(g^*D'')D'$$

and for each $D' \in W(Y)$ we also have

$$f^*D' = \sum_{D \in W(X)} \operatorname{mult}_D(f^*D')D,$$

hence

$$(g \circ f)^*D'' = \sum_{D \in W(X)} \left(\sum_{D' \in W(Y)} \operatorname{mult}_{D'}(g^*D'')\operatorname{mult}_D(f^*D') \right) D.$$

Therefore, for every $D \in W(X)$ we have

$$\operatorname{mult}_D((g \circ f)^*D'')\operatorname{mult}_{C,D}(\Delta_X) = \\ = \sum_{D' \in W(Y)} \operatorname{mult}_{D'}(g^*D'')\operatorname{mult}_D(f^*D')\operatorname{mult}_{C,D}(\Delta_X).$$

Assume now that $g(f(D)) \subseteq D''$, so that $\operatorname{mult}_D((g\circ f)^*D'') \neq 0$, cf. Remark 1.9. We want to show that

$$\operatorname{mult}_D((g\circ f)^*D'')\operatorname{mult}_{C,D}(\Delta_X) \geq \operatorname{mult}_{C,D''}(\Delta_Z),$$

so by the previous discussion it suffices to show that

$$\sum_{D'\in W(Y)} \operatorname{mult}_{D'}(g^*D'')\operatorname{mult}_D(f^*D')\operatorname{mult}_{C,D}(\Delta_X) \geq \operatorname{mult}_{C,D''}(\Delta_Z).$$

Using Remark 1.9 again, this is equivalent to showing that

$$\sum_{\substack{D'\in W(Y)\\ f(D)\subseteq D'\\ g(D')\subseteq D''}} \operatorname{mult}_{D'}(g^*D'')\operatorname{mult}_D(f^*D')\operatorname{mult}_{C,D}(\Delta_X) \geq \operatorname{mult}_{C,D''}(\Delta_Z).$$

Since f induces a C-morphism, it suffices to show that

$$\sum_{\substack{D'\in W(Y)\\ f(D)\subseteq D'\\ g(D')\subseteq D''}} \operatorname{mult}_{D'}(g^*D'')\operatorname{mult}_{C,D'}(\Delta_Y) \geq \operatorname{mult}_{C,D''}(\Delta_Z).$$

Since $f(D) \subseteq \operatorname{Supp}(g^*D'')$, there exists some $D'_0 \in W(Y)$ such that $f(D) \subseteq D'_0$ and $g(D'_0) \subseteq D''$. Therefore, it suffices to show that

$$\operatorname{mult}_{D'_0}(g^*D'')\operatorname{mult}_{C,D'_0}(\Delta_Y) \geq \operatorname{mult}_{C,D''}(\Delta_Z),$$

which follows from the choice of D'_0 and the assumption that g induces a C-morphism. □

Definition 1.14 ($\mathbb{Q}$-factorial C-pair)**.** We will say that a C-pair (X,Δ) is $\mathbb{Q}$-factorial if X is $\mathbb{Q}$-factorial.

By Lemma 1.13 we can compose C-morphisms, and Remark 1.6 ensures that this composition is associative. Therefore, we can consider the following:

Definition 1.15 (Category of C-pairs)**.** The *category of C-pairs* is defined as follows:

(1) its objects are $\mathbb{Q}$-factorial algebraic C-pairs, and
(2) its morphisms are (dominant) C-morphisms.

2. Constructions with C-pairs

Definition 1.16 (Restriction C-pair)**.** Let (X,Δ) be a C-pair and let $U \subseteq X$ be a non-empty open subset. Then there is an induced C-pair $(U, \Delta|_U)$, which we will call the *restriction* of (X,Δ) to U, denoted (U,Δ_U) or simply (U,Δ) if no confusion is likely to happen.

Remark 1.17. Let (X,Δ) be a $\mathbb{Q}$-factorial C-pair and let $U \subseteq X$ be a non-empty open subset. Then the inclusion induces an orbifold morphism. In the algebraic case, U is $\mathbb{Q}$-factorial as well, cf. Appendix C, and the inclusion is a C-morphism $(U, \Delta_U) \to (X, \Delta)$. Moreover, this C-morphism satisfies the following universal property. For every C-morphism $(T, \Delta_T) \to (X, \Delta)$ such that the image of T is contained in U, there exists a unique factorization as shown in the following diagram:

$$\begin{tikzcd} (T, \Delta_T) \arrow[r, "\exists!"] \arrow[dr] & (U, \Delta_U) \arrow[d, hook] \\ & (X, \Delta_X). \end{tikzcd}$$

If X and Y are normal varieties, then $X \times Y$ is also a normal variety [EGA IV$_2$, (6.14.1)]. Moreover, we can pull-back Weil divisors along the projections of a product in a meaningful way. This allows us to define:

Definition 1.18 (Product of C-pairs, cf. [Cam11a, Remarque 2.2]). Let (X, Δ_X) and (Y, Δ_Y) be two C-pairs. Define

$$\Delta_{X \times Y} := p_X^*(\Delta_X) + p_Y^*(\Delta_Y),$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. We call the resulting C-pair $(X \times Y, \Delta_{X \times Y})$ the *product* of the C-pairs (X, Δ_X) and (Y, Δ_Y).

Lemma 1.19. *Let (X, Δ_X) and (Y, Δ_Y) be $\mathbb{Q}$-factorial algebraic C-pairs and let $(X \times Y, \Delta_{X \times Y})$ be their product. Then $X \times Y$ is $\mathbb{Q}$-factorial and the two projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are C-morphisms. Moreover, if X and Y are locally factorial, then $(X \times Y, \Delta_{X \times Y})$ is a product of (X, Δ_X) and (Y, Δ_Y) in the category of C-pairs.*

Proof. $\mathbb{Q}$-factoriality of the product is shown in [BGS11]. We check that p_X induces a morphism of C-pairs, hence also p_Y by symmetry. Let $D \in W(X)$ be a prime Weil divisor and let $D' \in W(X \times Y)$ be a prime Weil divisor such that $D' \subseteq \operatorname{Supp}(p_X^* D) = D \times Y$. Since $D \times Y$ is already irreducible, we must have $D' = D \times Y$, so the claim follows by construction of $\Delta_{X \times Y}$.

Suppose now that X and Y are locally factorial. Then $X \times Y$ is also locally factorial [BGS11, Corollaire 5.2]. We check that $(X \times Y, \Delta_{X \times Y})$ is indeed a product in the category of C-pairs. Let (T, Δ_T) be a $\mathbb{Q}$-factorial C-pair and let $f : (T, \Delta_T) \to (X, \Delta_X)$ and $g : (T, \Delta_T) \to (Y, \Delta_Y)$ be morphisms of C-pairs. We want to show that there exists a unique morphism of C-pairs $h : (T, \Delta_T) \to (X \times Y, \Delta_{X \times Y})$ such that $f = p_X \circ h$ and $g = p_Y \circ h$. Uniqueness follows from the universal property of the product $X \times Y$ as varieties. We check that the morphism of varieties $h : T \to X \times Y$ induced by this universal property also induces an orbifold morphism $h : (T, \Delta_T) \to (X \times Y, \Delta_{X \times Y})$. Let $D \in W(T)$ and $D' \in W(X \times Y)$ be prime Weil divisors such that $h(D) \subseteq D'$. Since $X \times Y$ is locally factorial, $h^*(D')$ is a divisor with integer coefficients. Therefore, we may assume that D' is contained in the support of $\Delta_{X \times Y}$. So assume that $D' = D_0 \times Y = p_X^*(D_0)$ for some $D_0 \in$

$W(X)$, and assume that $\mathrm{mult}_{C,D_0}(\Delta_X) \geq 2$. Since f induces an orbifold morphism and $D \subseteq h^*(D') = h^* p_X^*(D_0)$, we have

$$\mathrm{mult}_D(h^*D')\,\mathrm{mult}_{C,D}(\Delta_T) \geq \mathrm{mult}_{C,D_0}(\Delta_X) = \mathrm{mult}_{C,D'}(\Delta_{X\times Y}).$$

Similarly for divisors of the form $p_Y^*(D_0)$ for $D_0 \in W(Y)$. So h induces an orbifold morphism. Since T is $\mathbb{Q}$-factorial and h is dominant, it is also a morphism of C-pairs. □

Definition 1.20 (Action on a C-pair)**.** Let (X, Δ) be an algebraic (resp. analytic) C-pair and let G be a finite group. We say that an action of G on X is an *action of* G *on* (X, Δ) if it is algebraic (resp. analytic) and $\phi_g(\Delta) = \Delta$ for all $g \in G$, where ϕ_g denotes the automorphism of X corresponding to $g \in G$. In this case we also say that G *acts on* (X, Δ).

Remark 1.21. In the algebraic setting we always mean an algebraic action unless we explicitly say otherwise.

Remark 1.22. We are not asking that each $g \in G$ fixes Δ pointwise, but rather that it leaves the $\mathbb{Q}$-divisor Δ invariant. In particular, it can happen that some $g \in G$ maps an irreducible component of Δ onto another irreducible component of Δ, but in this case the two irreducible components need to have the same coefficient.

Definition-Lemma 1.23 (Quotient C-pair)**.** Let (X', Δ') be a C-pair and let G be a finite group acting on (X', Δ'). Assume[2] that we can form the quotient of X' by G and let $q\colon X' \to X$ be the quotient morphism. We define a C-pair (X, Δ) as follows. For each prime Weil divisor D on X, we let D' be a prime Weil divisor contained in the support of q^*D. We set then

$$\mathrm{mult}_{C,D}(\Delta) := \mathrm{mult}_{D'}(q^*D) \cdot \mathrm{mult}_{C,D'}(\Delta') \tag{1.1}$$

with the usual convention that $m \cdot \infty = \infty$ for all $m \in \mathbb{N}_{>0}$. This construction yields a well-defined C-pair.

Proof. The analytic case is treated in [KR], so we discuss only the algebraic case here. Since X' is a normal variety and G is a finite group, X is a normal variety as well. We check that $\mathrm{mult}_{C,D}(\Delta)$ does not depend on the choice of D'.

It suffices to show this over the big open subsets on which X' and X are smooth, the question is local on X and the morphism q is finite. So we may assume that $X' = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A^G)$ and $q = \mathrm{Spec}(\iota)$, where $\iota\colon A^G \to A$ is the inclusion. We may also assume that $D = \{t = 0\}$ is a principal effective Cartier divisor, so that $q^*D = \{\iota(t) = 0\}$ is given by a G-invariant equation. Denoting by $\phi_g \in \mathrm{Aut}(X')$ the automorphism corresponding to $g \in G$, we have

$$q^{-1}D = \bigcup_{g\in G} \phi_g(D')$$

[2] This is for instance the case if X' is quasi-projective.

as topological spaces, and each $\phi_g(D')$ is an irreducible component of q^*D. So for any other choice of prime Weil divisor $D'' \subseteq q^*D$ we can find a $g \in G$ such that $\phi_g(D'') = D'$. We have

$$\mathrm{mult}_{D''}(q^*D) = \nu_{D''}(\iota(t)),$$

where $\nu_{D''}$ denotes the valuation of the discrete valuation ring $\mathcal{O}_{X',\eta''}$ for η'' the generic point of D''. So $\nu_{D''}(\iota(t))$ is the vanishing order of the regular function $\iota(t)$ along $D'' = \phi_{g^{-1}}(\phi_g(D''))$, which is in turn the vanishing order of the regular function $\iota(t) \circ \phi_{g^{-1}}$ along $\phi_g(D'')$, hence $\nu_{D''}(\iota(t)) = \nu_{\phi_g(D'')}(\iota(t) \circ \phi_{g^{-1}})$. And since $\iota(t)$ is G-invariant we deduce that $\nu_{D''}(\iota(t)) = \nu_{\phi_g(D'')}(\iota(t))$. Therefore

$$\mathrm{mult}_{D''}(q^*D) = \nu_{D''}(\iota(t)) = \nu_{\phi_g(D'')}(\iota(t)) = \nu_{D'}(\iota(t)) = \mathrm{mult}_{D'}(q^*D).$$

This shows that the left factor in the definition of $\mathrm{mult}_{C,D}(\Delta)$ is independent of the choice of D'. The assumption that the G-action leaves Δ' invariant ensures that the same is true for the right factor, cf. Remark 1.22. □

Remark 1.24. Quotient C-pairs are a particular case of Campana's *orbifold base* of a proper surjective morphism from a C-pair to a normal variety, cf. [Cam11a, Définition 4.2].

Lemma 1.25. *Let (X', Δ') be a C-pair and let G be a finite group acting on (X', Δ'). Assume that we can form the quotient of X' by G and let (X, Δ) be the quotient C-pair. Let $q\colon X' \longrightarrow X$ be the quotient morphism and let* $\mathrm{Ram}(q)$ *denote its ramification divisor.*

(1) *We have* $q^*(\Delta^{\mathrm{orb}}) = (\Delta')^{\mathrm{orb}}$.
(2) *If* $\Delta^{\log} = 0$*, then we have* $\mathrm{Ram}(q) = q^*(\Delta) - \Delta'$.

Proof. We start by proving Item 1. Let $D \in W(X)$ such that $D \subseteq \mathrm{Supp}(\Delta^{\mathrm{orb}})$. We have

$$\begin{aligned} q^*\left(\frac{1}{\mathrm{mult}_{C,D}(\Delta)}D\right) &= \sum_{\substack{D' \in W(X') \\ D' \subseteq q^{-1}D}} \frac{\mathrm{mult}_{D'}(q^*D)}{\mathrm{mult}_{D'}(q^*D)\,\mathrm{mult}_{C,D'}(\Delta')} D' \\ &= \sum_{\substack{D' \in W(X') \\ D' \subseteq q^{-1}D}} \frac{1}{\mathrm{mult}_{C,D'}(\Delta')} D', \end{aligned}$$

where the D' are prime Weil divisors on X'. Hence $q^*(\Delta^{\mathrm{orb}}) = (\Delta')^{\mathrm{orb}}$, because different prime Weil divisors on X cannot have a common prime Weil divisor on X' above them.

To prove Item 2 we use Equation (B.1). This allows us to write

$$\begin{aligned}\mathrm{Ram}(q) &= \sum_{D\in W(X)} (q^*(D) - (q^*D)_{\mathrm{red}}) \\ &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} (\mathrm{mult}_{D'}(q^*D) - 1)D'.\end{aligned}$$

On the other hand, note that by construction we have $\Delta^{\log} = 0$ if and only if $(\Delta')^{\log} = 0$, so can write

$$\begin{aligned}q^*(\Delta) &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} \frac{\mathrm{mult}_{C,D}(\Delta) - 1}{\mathrm{mult}_{C,D}(\Delta)} \mathrm{mult}_{D'}(q^*D)D' \\ &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} \frac{\mathrm{mult}_{C,D}(\Delta) - 1}{\mathrm{mult}_{C,D'}(\Delta')} D'.\end{aligned}$$

Therefore,

$$\begin{aligned}q^*(\Delta) - \Delta' &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} \frac{\mathrm{mult}_{C,D}(\Delta) - 1 - (\mathrm{mult}_{C,D'}(\Delta') - 1)}{\mathrm{mult}_{C,D'}(\Delta')} D' \\ &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} \frac{\mathrm{mult}_{C,D}(\Delta) - \mathrm{mult}_{C,D'}(\Delta')}{\mathrm{mult}_{C,D'}(\Delta')} D' \\ &= \sum_{D\in W(X)} \sum_{\substack{D'\in W(X') \\ D'\subseteq q^{-1}D}} (\mathrm{mult}_{D'}(q^*D) - 1)D' \\ &= \mathrm{Ram}(q).\end{aligned}$$

□

In [KR], Kebekus and Rousseau show that the quotient morphism $q\colon (X',\Delta') \to (X,\Delta)$ is a C-morphism with the expected universal property. We show the analogous statement using our definition of C-morphism:

Lemma 1.26. *Let (X',Δ') be a $\mathbb{Q}$-factorial algebraic C-pair and let G be a finite group acting on (X',Δ'). Assume that we can form the quotient of X' by G and let (X,Δ) be the quotient C-pair. Then the quotient morphism is a C-morphism $q\colon (X',\Delta') \to (X,\Delta)$. Moreover, this C-morphism satisfies the following universal property: for every C-morphism $(X',\Delta') \to (T,\Delta_T)$ which is constant on G-orbits there exists a unique factorization*

$$\begin{tikzcd} (X', \Delta') \arrow[r, "q"] \arrow[dr] & (X, \Delta) \arrow[d, "\exists!"] \\ & (T, \Delta_T) \end{tikzcd}$$

in the category of C-pairs.

Proof. $\mathbb{Q}$-factoriality of X follows from [KM98, Lemma 5.16], and $q\colon (X', \Delta') \to (X, \Delta)$ being a C-morphism follows immediately from Equation (1.1). We show the desired universal property. Let (T, Δ_T) be a $\mathbb{Q}$-factorial C-pair and let $f\colon (X', \Delta') \to (T, \Delta_T)$ be a C-morphism that is constant on G-orbits. We want to show that there exists a unique C-morphism $\bar{f}\colon (X, \Delta) \to (T, \Delta_T)$ such that $f = \bar{f} \circ q$. Uniqueness follows from the universal property of geometric quotients [MFK94, Proposition 0.1] and Remark 1.6. For the existence, we need to show that the induced (dominant) morphism $\bar{f}\colon X \to T$ induces an orbifold morphism. So let $D \in W(X)$ and $D'' \in W(T)$ be prime Weil divisors such that $D \subseteq \mathrm{Supp}(\bar{f}^* D'')$. For any $D' \subseteq \mathrm{Supp}(q^* D)$ we have then $D' \subseteq \mathrm{Supp}(f^* D'')$ and

$$f^* D'' = \mathrm{mult}_D(\bar{f}^* D'')\, \mathrm{mult}_{D'}(q^* D) D' + \dots,$$

where D' does not appear in the omitted part of the sum. Therefore, since $f\colon (X', \Delta') \to (T, \Delta)$ is a C-morphism, we have

$$\begin{aligned} \mathrm{mult}_D(\bar{f}^* D'')\, \mathrm{mult}_{C,D}(\Delta) &= \mathrm{mult}_D(\bar{f}^* D'')\, \mathrm{mult}_{D'}(q^* D)\, \mathrm{mult}_{C,D'}(\Delta') \\ &= \mathrm{mult}_{D'}(f^* D'')\, \mathrm{mult}_{C,D'}(\Delta') \\ &\geq \mathrm{mult}_{D''}(\Delta_T). \end{aligned}$$

□

Example 1.27. Let $X = \mathbb{A}^2$ and let $q\colon X \to Y$ be the quotient of X by the $\mathbb{Z}/2\mathbb{Z}$-action given by $(x, y) \mapsto (-x, -y)$. Then the quotient C-pair of $(X, 0)$ is $(Y, 0)$, because the quotient morphism $q\colon X \to Y$ is quasi-étale. Therefore $q\colon (X, 0) \to (Y, 0)$ is a C-morphism, confirming what we already knew from Remark 1.11.

3. Singularities of C-pairs

In this section we define various notions of singularities for C-pairs.

Definition 1.28 (Smooth C-pair)**.** A C-pair (X, Δ) is called *smooth* if it is smooth in the sense of Item 7 in Notation 0.7, i.e., if X is smooth and $\mathrm{Supp}(\Delta)$ is simple normal crossing (normal crossing in the analytic setting), cf. [KM98, Notation 0.4.(8)].

Definition 1.29 (C-pairs with quotient singularities)**.** We will say that a C-pair (X, Δ) has *quotient singularities* if it is analytic-locally the quotient C-pair of a smooth C-pair.

Example 1.30. Let (X, Δ) be a C-pair.

(1) If (X, Δ) is smooth, then it has quotient singularities, because it is the quotient C-pair of itself by the trivial group.
(2) If X has quotient singularities [KM98, Definition 5.14], then $(X, 0)$ has quotient singularities. Indeed, by the Chevalley–Shephard–Todd theorem [Ben93, §7] we may assume that the quotient morphism is quasi-étale.

Remark 1.31. If (X, Δ) is a C-pair with quotient singularities, then X has quotient singularities in the usual sense. In particular, X has rational singularities and is $\mathbb{Q}$-factorial [KM98, Proposition 5.15].

Definition 1.32 (klt C-pair)**.** We will say that a C-pair (X, Δ) is klt if the pair (X, Δ) is klt in the usual sense [KM98, Definition 2.34].

Lemma 1.33. *Let (X', Δ') be a C-pair and let G be a finite group acting on (X', Δ'). Assume that we can form the quotient of X' by G and let (X, Δ) be the quotient C-pair. Then (X', Δ') is klt if and only if (X, Δ) is klt.*

Proof. By definition, klt implies that the boundary has coefficients strictly less than 1. Therefore, we may apply Item 2 in Lemma 1.25 to deduce that

$$\begin{aligned} K_{X'} + \Delta' &= q^*(K_X) + \mathrm{Ram}(q) + \Delta' \\ &= q^*(K_X) + q^*(\Delta) - \Delta' + \Delta' \\ &= q^*(K_X + \Delta). \end{aligned}$$

Hence the equivalence follows from [KM98, Proposition 5.20]. □

Corollary 1.34. *Let (X, Δ) be a C-pair with quotient singularities such that $\Delta^{\log} = 0$. Then (X, Δ) is klt.*

The analogous of Corollary 1.34 for pairs with $\Delta^{\log} \neq 0$ replacing klt by dlt [KM98, Definition 2.37] is not true:

Example 1.35. Let $X' = \mathbb{A}^2$ and $\Delta' = \{x = 0\} + \{y = 0\}$. Consider the $\mathbb{Z}/2\mathbb{Z}$-action given by $(x, y) \mapsto (-x, -y)$ and let (X, Δ) be the quotient C-pair. Then (X, Δ) has quotient singularities but it is not dlt. Indeed, X is the A_1 singularity and Δ consists of two lines intersecting at the singular point, so [KM98, Corollary 5.55] implies that (X, Δ) is not dlt.

4. Adapted covers

In order to study C-pairs, we will consider suitable covers of the underlying variety.

Definition 1.36 (Adapted morphism)**.** Let (X, Δ) be a C-pair and let $\gamma\colon \hat{X} \to X$ be a quasi-finite morphism with $\hat{X}$ a normal variety of the same dimension as X.

(1) We say that γ is an *adapted morphism* if $\gamma^*(\Delta^{\mathrm{orb}})$ is a divisor with integer coefficients, cf. Remark 1.37.

(2) We say that γ is *strongly adapted* if $\gamma^*(\Delta^{\mathrm{orb}})$ is moreover a reduced divisor.
(3) We say that γ is *precisely adapted* if γ is adapted and quasi-étale over $X \setminus \mathrm{Supp}(\Delta)$.
(4) We say that γ is *perfectly adapted* if γ is strongly and precisely adapted.

We denote by $\mathbf{Adpt}(X, \Delta)$ the full subcategory of $\mathbf{Sch}/X$ whose objects are adapted morphisms. We refer to adapted morphisms which are finite as *adapted covers* instead, i.e., an adapted morphism is an adapted cover if it is a cover in the sense of Definition B.11.

Remark 1.37. Note that in the situation of Definition 1.36, the morphism γ is dominant and open and we can pull Weil divisors back in a way that respects linear equivalence. See Lemma B.4 and Lemma B.7 respectively.

Remark 1.38. If $\gamma_1 \colon Y_1 \to X$ is an adapted morphism (resp. cover) and $f \colon Y_2 \to Y_1$ is a quasi-finite (resp. finite) morphism between normal varieties of the same dimension, then $\gamma_1 \circ f \colon Y_2 \to X$ is again an adapted morphism (resp. cover).

Remark 1.39. If $\gamma_1 \colon Y_1 \to X$ and $\gamma_2 \colon Y_2 \to X$ are adapted morphisms and $f \colon Y_2 \to Y_1$ is a morphism over X, then f is a quasi-finite morphism between normal varieties of the same dimension [GW10, Proposition 12.17.(3)], hence open and dominant as well. Similarly, any morphism of adapted covers is finite [GW10, Appendix C].

Remark 1.40. If $\gamma \colon \hat{X} \to X$ is a perfectly adapted morphism (resp. cover) for (X, Δ) and $f \colon Y \to \hat{X}$ is a quasi-finite and quasi-étale morphism between normal varieties of the same dimension (resp. quasi-étale cover), then $\gamma \circ f$ is again a perfectly adapted morphism (resp. cover) for (X, Δ).

Remark 1.41. Since adapted morphisms are dominant, an adapted morphism is an adapted cover if and only if it is finite. But there are surjective adapted morphisms which are not adapted covers, cf. Example B.17.

Lemma 1.42. *Let (X, Δ_X) and (Y, Δ_Y) be two C-pairs and let $(X \times Y, \Delta_{X\times Y})$ be their product C-pair. Let $\gamma_X \colon \hat{X} \to X$ and $\gamma_Y \colon \hat{Y} \to Y$ be quasi-finite morphisms of normal varieties of the same dimension.*

(1) *If γ_X and γ_Y are adapted morphisms (resp. adapted covers), then*
$$\gamma_X \times \gamma_Y \colon \hat{X} \times \hat{Y} \to X \times Y$$
is an adapted morphism (resp. adapted cover) for $(X \times Y, \Delta_{X\times Y})$.
(2) *If γ_X and γ_Y are strongly adapted, then $\gamma_X \times \gamma_Y$ is strongly adapted.*
(3) *If γ_X and γ_Y are precisely adapted, then $\gamma_X \times \gamma_Y$ is precisely adapted.*
(4) *If γ_X and γ_Y are perfectly adapted, then $\gamma_X \times \gamma_Y$ is perfectly adapted.*

Proof. If $\hat{X}$ and $\hat{Y}$ are normal varieties of the same dimension as X and Y respectively, then $\hat{X} \times \hat{Y}$ is a normal variety [EGA IV$_2$, (6.14.1)] of the same dimension as $X \times Y$. Moreover, the product of quasi-finite maps (resp. covers) remains quasi-finite (resp. a cover). So $\gamma_X \times \gamma_Y : \hat{X} \times \hat{Y} \to X \times Y$ is a quasi-finite morphism (resp. cover) of normal varieties of the same dimension.

For the remaining assertions, we consider the following commutative diagram:

$$\begin{tikzcd}
 & \hat{X} \times \hat{Y} \arrow[dl, "p_{\hat{X}}"'] \arrow[d, "\gamma_X \times \gamma_Y"] \arrow[dr, "p_{\hat{Y}}"] & \\
\hat{X} \arrow[dd, "\gamma_X"'] & X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \hat{Y} \arrow[dd, "\gamma_Y"] \\
 & & \\
X & & Y.
\end{tikzcd}$$

If $\gamma_X^*(\Delta_X^{\mathrm{orb}})$ and $\gamma_Y^*(\Delta_Y^{\mathrm{orb}})$ are divisors with integer coefficients (resp. reduced), then so is

$$p_{\hat{X}}^* \gamma_X^*(\Delta_X^{\mathrm{orb}}) + p_{\hat{Y}}^* \gamma_Y^*(\Delta_Y^{\mathrm{orb}}) = (\gamma_X \times \gamma_Y)^*(\Delta_{X \times Y}^{\mathrm{orb}}).$$

This shows Items 1 and 2.

Suppose now that γ_X and γ_Y are precisely adapted and let $(x, y) \in X \times Y \setminus \mathrm{Supp}(\Delta_{X \times Y})$ be a closed point. Then $x \in X \setminus \mathrm{Supp}(\Delta_X)$ and $y \in Y \setminus \mathrm{Supp}(\Delta_Y)$, so γ_X and γ_Y are quasi-étale in a neighborhood $U_x \subseteq X$ of x and $U_y \subseteq Y$ of y respectively. This means that there are closed subsets $Z_x \subseteq U_x$ and $Z_y \subseteq U_y$ of codimension at least 2 in U_x and U_y respectively such that γ_X and γ_Y are étale over $U_x \setminus Z_x$ and $U_y \setminus Z_y$ respectively. Product of étale morphisms is étale [SP, 03PC], so this implies that $\gamma_X \times \gamma_Y$ is étale over $U_x \times U_y \setminus (Z_x \times U_y \cup U_x \times Z_y)$. Hence $\gamma_X \times \gamma_Y$ is quasi-étale at (x, y). This shows Item 3.

Item 4 follows from Items 2 and 3. □

Lemma 1.43. *Let (X', Δ') be a C-pair and let G be a finite group acting on (X', Δ'). Assume that we can form the quotient of X' by G. Let $q : X' \to X$ be the quotient morphism and let (X, Δ) be the quotient C-pair. Let $\gamma' : \hat{X} \to X'$ be an quasi-finite morphism of normal varieties of the same dimension and let $\gamma := q \circ \gamma'$.*

(1) *If γ' is an adapted morphism (resp. cover) for (X', Δ'), then γ is an adapted morphism (resp. cover) for (X, Δ).*
(2) *If γ' is strongly adapted, then γ is strongly adapted.*
(3) *If γ' is precisely adapted, then γ is precisely adapted.*
(4) *If γ' is perfectly adapted, then γ is perfectly adapted.*

Proof. We check Item 1 and Item 2 first. Since q is finite, the assertion about adapted covers follows from the assertion about adapted morphisms. The assertion about adapted morphisms follows from Item 1 in Lemma 1.25.

To show Item 3, we note first that q is quasi-étale over $X \setminus \mathrm{Supp}(\Delta)$, because Δ lies under all codimension 1 ramification by construction. Again by construction we have $\mathrm{Supp}(\Delta') \subseteq q^{-1}(\mathrm{Supp}(\Delta))$, so $q^{-1}(X \setminus \mathrm{Supp}(\Delta)) \subseteq X' \setminus \mathrm{Supp}(\Delta')$. If γ' is precisely adapted, then all morphisms in the commutative diagram

$$\begin{tikzcd}
(\gamma')^{-1}q^{-1}(X \setminus \mathrm{Supp}(\Delta)) \arrow[r, "\gamma'"] \arrow[dr, "\gamma"'] & q^{-1}(X \setminus \mathrm{Supp}(\Delta)) \arrow[d, "q"] \\
& X \setminus \mathrm{Supp}(\Delta)
\end{tikzcd}$$

are quasi-étale, so γ is precisely adapted as well.

Finally, Item 4 follows from Item 2 and Item 3. □

Remark 1.44. The property of being a (finite) Galois field extension is not transitive, so in the setting of Lemma 1.43, we cannot conclude that γ is Galois from γ' being Galois.

Corollary 1.45. *Let X' be a normal variety and let G be a finite group acting on X', so that G acts on $(X', 0)$ as well. Assume that we can form the quotient of X' by G. Let (X, Δ) be the quotient C-pair of $(X', 0)$ by this group action and let $q\colon X' \to X$ be the quotient morphism. Then $q\colon X' \to X$ is a perfectly adapted cover.*

Proof. Take $\gamma' = \mathrm{id}_{X'}$ in Lemma 1.43. □

Remark 1.46. We would like the conclusion of Corollary 1.45 to hold true for the quotient morphism $q\colon \mathbb{A}^2 \to Y$ of the A_1-singularity, where $Y = \mathbb{A}^2/\{\pm 1\}$ and the action is given by $(x, y) \mapsto (-x, -y)$. This quotient morphism is quasi-étale, so nothing happens at the level of divisors and we would have $\Delta_Y = 0$. If we had defined precisely adapted covers as being étale away from the support of the boundary divisor, then q would fail to be a perfectly adapted cover, because q is not unramified at $(0, 0)$.

Lemma 1.47. *Let (X, Δ) be a klt C-pair and let $\gamma\colon \hat{X} \to X$ be a perfectly adapted cover. Then $(\hat{X}, 0)$ is klt.*

Proof. Since (X, Δ) is klt, we have $\Delta^{\log} = 0$. Since γ is precisely adapted, we can rewrite Equation (B.1) as

$$\mathrm{Ram}(\gamma) = \sum_{i \in I} \left(\gamma^* D_i - (\gamma^* D_i)_{\mathrm{red}}\right).$$

Since γ is strongly adapted, we can in turn rewrite this as

$$\mathrm{Ram}(\gamma) = \left(\sum_{i \in I} \gamma^* D_i\right) - \gamma^*(\Delta^{\mathrm{orb}}),$$

hence we deduce that $\mathrm{Ram}(\gamma) = \gamma^*(\Delta)$. It follows then from the ramification formula that

$$K_{\hat{X}} = \gamma^*(K_X) + \mathrm{Ram}(\gamma) = \gamma^*(K_X + \Delta),$$

so we can apply [KM98, Proposition 5.20] to conclude that $(\hat{X}, 0)$ is klt. □

Example 1.48. Let (X, Δ) be the C-pair with $X = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$ and $\Delta = \frac{1}{2}\{x = 0\}$. Let $n \in \mathbb{N}_{>0}$ be an even number. Then the morphism $\gamma\colon \mathbb{A}^1 \to \mathbb{A}^1$ corresponding to the $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \mathbb{C}[x] &\to \mathbb{C}[z] \\ x &\mapsto z^n \end{aligned}$$

is a precisely adapted cover, and it is perfectly adapted if and only if $n = 2$.

Perfectly adapted covers will be particularly useful, cf. Corollary 1.45 and Lemmas 1.47 and 1.73. But the following example shows that they do not always exist globally:

Example 1.49. Let $X := \mathbb{P}^1$ with homogeneous coordinates x_0 and x_1 and let $\Delta := \frac{1}{2}P$ for $P := \{x_1 = 0\}$. Then there isn't any perfectly adapted cover $\gamma\colon \hat{X} \to \mathbb{P}^1$. Indeed, suppose $\gamma\colon \hat{X} \to \mathbb{P}^1$ was such a cover. Normal curves are smooth, so γ is a finite and surjective morphism between smooth projective curves. Since γ is perfectly adapted we have $\deg(\gamma) \geq 2$ and $\deg(R) = 1$, where $R = P$ is the ramification divisor. It follows from Hurwitz's Theorem [Har77, Corollary IV.2.4] that

$$-2 \leq 2g(\hat{X}) - 2 = -2\deg(\gamma) + 1 \leq -3,$$

a contradiction.

However, for smooth C-pairs it is possible to construct perfectly adapted abelian covers analytic-locally:

Lemma 1.50. *Let (X, Δ) be a smooth C-pair. Then, for every point $x \in X$, there exists an analytic-open neighborhood U of x in X and a perfectly adapted abelian cover $\gamma\colon \hat{U} \to U$ such that $(\hat{U}, \gamma^*\Delta|_U)$ is smooth.*

Proof. Let $x \in X$ be a closed point. Since (X, Δ) is smooth, we can find an analytic-open neighborhood $x \in U \subseteq X$ with analytic-local coordinates $z_1, \dots, z_n$ such that $(\Delta|_U)_{\mathrm{red}}$ is either zero or given by the equation $\{z_1 \cdots z_r = 0\}$ for some $r \in \{1, \dots, n\}$. If $(\Delta^{\mathrm{orb}})|_U = 0$, then we can take $\gamma = \mathrm{id}_U$. So we may assume that there exists an $r \in \{1, \dots, n\}$, there exists an $l \in \{1, \dots, r\}$ and there exist some $m_1, \dots, m_l \in \mathbb{N}_{\geq 2}$ such that $(\Delta|_U)_{\mathrm{red}}$ is given by the equation $\{z_1 \cdots z_r = 0\}$ and

$$\Delta^{\mathrm{orb}}|_U = \frac{1}{m_1}\{z_1 = 0\} + \dots + \frac{1}{m_l}\{z_l = 0\}.$$

Consider $\mathbb{A}^n$ with coordinates $x_1, \dots, x_n$, and consider the C-pair $(\mathbb{A}^n, \Delta_0)$ with

$$\Delta_0 := \frac{m_1 - 1}{m_1}\{x_1 = 0\} + \dots + \frac{m_l - 1}{m_l}\{x_l = 0\} + \{x_{l+1} = 0\} + \dots + \{x_r = 0\}.$$

By construction, there is an analytic-open neighborhood B of the origin in $\mathbb{A}^n$ such that $(U, \Delta|_U) \cong (B, (\Delta_0)|_B)$. Since the statement that we want to show is analytic-local on X, we may assume that $(X, \Delta) = (\mathbb{A}^n, \Delta_0)$ to begin with. Under this assumption, we construct a perfectly adapted abelian cover $\gamma\colon \hat{X} \to X$ such that $(\hat{X}, \gamma^*\Delta)$ is smooth.

Let $\hat{X} := \mathbb{A}^n$, with coordinates $y_1, \dots, y_n$, and consider the morphism $\gamma\colon \hat{X} \to X$ given with respect to these coordinates by

$$(y_1, \dots, y_l, y_{l+1}, \dots, y_n) \mapsto (y_1^{m_1}, \dots, y_l^{m_l}, y_{l+1}, \dots, y_n).$$

Algebraically, this morphism corresponds to the $\mathbb{C}$-algebra morphism

$$\begin{aligned} \mathbb{C}[x_1, \dots, x_n] &\to \mathbb{C}[y_1, \dots, y_n] \\ x_i &\mapsto \begin{cases} y_i^{m_i} & \text{if } i \in \{1, \dots, l\}, \\ y_i & \text{if } i \in \{l+1, \dots, n\}. \end{cases} \end{aligned}$$

From the algebraic description we see that γ is a finite morphism, because it makes $\mathbb{C}[y_1, \dots, y_n]$ a finite $\mathbb{C}[x_1, \dots, x_n]$-module. By construction, $\gamma\colon \hat{X} \to X$ is strongly and precisely adapted such that $(\hat{X}, \gamma^*\Delta)$ is smooth, so it remains to show that it is an abelian cover. We argue algebraically to show this using Lemma B.22, so let us assume that $X = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n])$ and $\hat{X} = \operatorname{Spec}(\mathbb{C}[y_1, \dots, y_n])$. We can regard $\hat{X}$ as the subspace of

$$\mathbb{A}^{n+l} = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n, t_1, \dots, t_l])$$

defined by the ideal

$$I := (t_1^{m_1} - x_1, \dots, t_l^{m_l} - x_l).$$

Indeed, an isomorphism between the two affine schemes is given by the $\mathbb{C}$-algebra isomorphism

$$y_i \mapsto \begin{cases} t_i + I & \text{if } i \in \{1, \dots, l\}, \\ x_i + I & \text{if } i \in \{l+1, \dots, n\}. \end{cases}$$

Its inverse is the (well-defined) $\mathbb{C}$-algebra morphism given by

$$t_i + I \mapsto y_i$$

for all $i \in \{1, \dots, l\}$ and

$$x_i + I \mapsto \begin{cases} y_i^{m_i} & \text{if } i \in \{1, \dots, l\}, \\ y_i & \text{if } i \in \{l+1, \dots, n\}. \end{cases}$$

Under this identification, and denoting $A := \mathbb{C}[x_1, \dots, x_n]$, the cover $\gamma\colon \hat{X} \to X$ is given by the natural A-algebra morphism

$$A \to \frac{A[t_1, \dots, t_l]}{(t_1^{m_1} - x_1, \dots, t_l^{m_l} - x_l)} =: B.$$

Let $K := A_{(0)}$ be the field of fractions of the integral domain A, and let

$$L := B_{(0)} = \frac{K[t_1, \dots, t_l]}{(t_1^{m_1} - x_1, \dots, t_l^{m_l} - x_l)}.$$

Then we can characterize B as the integral closure of A in the field extension $K \subseteq L$, cf. Lemma B.15. Moreover, L is a finite composite of cyclic extensions by [Bos18, Proposition 4.8/3], so it is an abelian extension by [Lan02, Theorem VI.1.17]. More precisely, applying [Bos18, Proposition 4.8/3] and [Lan02, Theorem VI.1.14], we deduce that

$$G := \mathrm{Gal}(L/K) \cong (\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_l\mathbb{Z}).$$

By Lemma B.22, $\gamma\colon \hat{X} \to X$ is an abelian cover with Galois group G. □

Remark 1.51. One can also show that the cover constructed in Lemma 1.50 is abelian by explicitly writing down a group action in coordinates, namely, multiplying each coordinate with an appropriate root of unity.

Remark 1.52. The covers constructed in Lemma 1.50, which are essentially determined analytic-locally by Δ, are not necessarily cyclic. If Δ^{orb} can be written around a point as

$$\frac{1}{m_1}\{x_1 = 0\} + \cdots + \frac{1}{m_l}\{x_l = 0\},$$

then the Galois cover constructed in Lemma 1.50 has Galois group

$$(\mathbb{Z}/m_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_l\mathbb{Z}),$$

which by the Chinese remainder theorem is cyclic if and only if $\gcd(m_1, \dots, m_l) = 1$. See also Examples 1.53 and 1.54.

Example 1.53. In the setting of Lemma 1.50, suppose $X = \mathrm{Spec}(\mathbb{C}[x_1, x_2])$ and $\Delta = \frac{1}{2}\{x_1 = 0\} + \frac{2}{3}\{x_2 = 0\}$. Then we would consider $\hat{X} = \mathrm{Spec}(\mathbb{C}[y_1, y_2])$ and $\gamma\colon \hat{X} \to X$ given by $(x_1, x_2) \mapsto (y_1^2, y_2^3)$. In the proof of Lemma 1.50 we have seen that the Galois group is

$$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z},$$

so in this case the cover $\gamma\colon \hat{X} \to X$ is cyclic. By [Bos18, Proposition 4.8/3], we should be able to write the corresponding field extension as

$$K \subseteq K[t]/(t^6 - a) =: F$$

for some $a \in K$, where K denotes the function field of X. In the proof of Lemma 1.50 we have seen that the function field of $\hat{X}$ is given by

$$L = \frac{K[t_1, t_2]}{(t_1^2 - x_1, t_2^3 - x_2)}.$$

Taking $a = x_1^3 x_2^2$, we get a well-defined K-algebra morphism $F \to L$ induced by $t \mapsto t_1 t_2$. So we can regard $K \subseteq F \subseteq L$, and since $[L : K] = [F : K] = 6$, the previous K-algebra morphism is necessarily an isomorphism.

Example 1.54. In the setting of Lemma 1.50, suppose again that $X = \mathrm{Spec}(\mathbb{C}[x_1, x_2])$, but let now $\Delta = \frac{1}{2}\{x_1 = 0\} + \frac{1}{2}\{x_2 = 0\}$. Then we would consider $\hat{X} = \mathrm{Spec}(\mathbb{C}[y_1, y_2])$ and $\gamma\colon \hat{X} \to X$ given by $(x_1, x_2) \mapsto (y_1^2, y_2^2)$. In the proof of Lemma 1.50 we have seen that the Galois group is

$$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

so in this case the cover $\gamma\colon \hat{X} \to X$ is not cyclic.

Corollary 1.55. *Let (X, Δ) be a $\mathcal{C}$-pair with quotient singularities. Then, for every point $x \in X$, there exists an analytic-open neighborhood U of x in X and a perfectly adapted cover $\gamma\colon \hat{U} \to U$ such that $(\hat{U}, \gamma^*(\lfloor \Delta|_U \rfloor))$ is smooth.*

Proof. This follows from Lemma 1.50 and Lemma 1.43. □

We will also need perfectly adapted covers in a more singular setting. For this we will use the cyclic cover construction [KM98, Definition 2.52].

Lemma 1.56. *Let (X, Δ) be a $\mathcal{C}$-pair in which every irreducible component of Δ^{orb} is $\mathbb{Q}$-Cartier. Then, for every point $x \in X$, there exists a Zariski-open neighborhood U of x in X and a perfectly adapted cyclic cover $\gamma\colon \hat{U} \to U$.*

Proof. Let $N \in \mathbb{N}_{>0}$ be a natural number such that ND_i is Cartier for every prime Weil divisor D_i appearing in Δ^{orb}. Let $U \subseteq X$ be an open neighborhood of x in X such that every such ND_i is a principal (effective, Cartier) divisor on U. If $\mathrm{Supp}(\Delta^{\mathrm{orb}}) \cap U = \emptyset$, then we can take $\gamma_U = \mathrm{id}_U$, so assume this is not the case and write

$$\Delta^{\mathrm{orb}}|_U = \sum_{i=1}^{l} \frac{1}{m_i}(D_i)|_U$$

with $l \geq 1$. Let $m := \mathrm{lcm}\{m_1, \ldots, m_l\} > 1$ and consider

$$D := mN\left(\Delta^{\mathrm{orb}}|_U\right) = \sum_{i=1}^{l} \frac{m}{m_i}(ND_i)|_U.$$

Since each ND_i is a principal Cartier divisor on U, so is D. Say, $D = (f)$ for some $f \in \mathcal{O}_U(U)$ non-zero, notation as in [Har77, §II.6]. Then we have an isomorphism $\mathcal{O}_U \cong \mathcal{O}_U(D)$ given by multiplication with f^{-1}. So under this isomorphism the section $1 \in \mathcal{O}_U(D)(U)$ corresponds to the regular function $f \in \mathcal{O}_U(U)$ and its zero locus is the divisor D. We also have an isomorphism $\mathcal{O}_U \cong \mathcal{O}_U^{\otimes m}$, hence an isomorphism $\mathcal{O}_U(D) \cong \mathcal{O}_U^{\otimes m}$. Under this isomorphism, the section $1 \in \mathcal{O}_U(D)(U)$ corresponds to a section $s \in \mathcal{O}_U^{\otimes m}(U)$ whose zero locus is D, so we can apply the cyclic cover construction from [KM98, Definition 2.52] to obtain the desired cover. To ensure that it satisfies the properties that we want, we want to apply the results in [EV92, §3.5], so let $W \subseteq U$ be a big open subset such that $(W, D|_W)$ is smooth. The cyclic cover construction in [EV92, §3.5] applied to $(W, D|_W)$ yields a cyclic (Galois) cover $\gamma_W\colon \hat{W} \to W$. If $\hat{W}$ is not irreducible, then there exists an $m' > 1$ dividing mN and a section $s' \in \mathcal{O}_W^{\otimes m'}(W)$ such that

$s = (s')^{m'}$ [EV92, Lemma 3.15]. Therefore, there exists another effective Cartier divisor $D'|_W$ on W such that $m'(D'|_W) = D|_W$, cf. proof of [EH16, Proposition 1.30]. Note that m' divides

$$\gcd\left(\frac{mN}{m_1}, \dots, \frac{mN}{m_l}\right) = N,$$

so D' is still of the form $mN'(\Delta^{\mathrm{orb}}|_U)$ for some $N' \in \mathbb{N}_{>0}$; and $(W, D'|_W)$ is still smooth because $\mathrm{Supp}(D') = \mathrm{Supp}(D)$. Therefore, choosing m' maximal and replacing $D|_W$ by $D'|_W$, we may assume that $\hat{W}$ is irreducible. By Lemma B.22, γ_W induces a finite Galois field extension of function fields $K(U) = K(W) \subseteq K(\hat{W})$. Let now $\gamma\colon \hat{U} \to U$ be the normalization of U inside $K(\hat{W})$. Again by Lemma B.22, γ is a cyclic cover. In particular it is a finite surjective morphism between normal varieties of the same dimension, and $\gamma^{-1}(W)$ is a big open subset in $\hat{U}$ by Lemma B.5. Moreover, compatibility of normalization with open immersions and the fact that $\hat{W}$ is the normalization of W in $K(\hat{W})$ implies that $\gamma^{-1}(W) = \hat{W}$ and $\gamma|_{\hat{W}} = \gamma_W$. So in order to check that $\gamma\colon \hat{U} \to U$ is perfectly adapted for (U, Δ_U), it suffices to show that $\gamma_W\colon \hat{W} \to W$ is perfectly adapted for (W, Δ_W). For every $i \in \{1, \dots, l\}$ we have

$$\frac{mN\frac{mN}{m_i}}{\gcd\left(mN, \frac{mN}{m_i}\right)} = \mathrm{lcm}\left(mN, \frac{mN}{m_i}\right) = mN,$$

so [EV92, Lemma 3.15.c)] implies that

$$\gamma_W^*(\Delta^{\mathrm{orb}}|_W) = \frac{1}{mN}\gamma_W^*(D|_W)$$

is a reduced divisor, i.e., that γ_W is a strongly adapted cover. Furthermore, [EV92, Lemma 3.15.b)] implies that γ_W is also precisely adapted, so γ_W is a perfectly adapted cover. Therefore, $\gamma\colon \hat{U} \to U$ is a cyclic cover which is perfectly adapted for (U, Δ_U). □

Remark 1.57. The covers constructed in Lemma 1.50 do not need to be cyclic, cf. Remark 1.52. So they do not need to agree with the covers constructed in Lemma 1.56. See also Examples 1.58 and 1.59

Example 1.58. Let $X = \mathrm{Spec}\, A$, where $A = \mathbb{C}[x_1, x_2]$, and let $\Delta = \frac{1}{2}\{x_1 = 0\} + \frac{2}{3}\{x_2 = 0\}$. It follows from the computations in Example 1.53 that in this case the covers produced by the proofs of Lemma 1.50 and Lemma 1.56 are isomorphic, because both of them are the normalization of X in the field extension

$$K \subseteq K[t]/(t^6 - x_1^3 x_2^2)$$

of its function field $K = A_{(0)}$. It follows from the Jacobian criterion that the singular locus in $\mathrm{Spec}(A[t]/(t^6 - x_1^3 x_2^2))$ has codimension 1, so this affine scheme is not normal. Following the recipe in Lemma 1.56, we normalize it to obtain the desired cover, and in particular we deduce that the integral closure of $A[t]/(t^6 - x_1^3 x_2^2)$ in its function field is isomorphic to $A[t_1, t_2]/(t_1^2 - x_1, t_2^3 - x_2) \cong \mathbb{C}[y_1, y_2]$.

Example 1.59. We consider again $X = \operatorname{Spec} A$ for $A = \mathbb{C}[x_1, x_2]$, but now we consider the divisor $\Delta = \frac{1}{2}\{x_1 = 0\} + \frac{1}{2}\{x_2 = 0\}$ as in Example 1.54. In this case, the cover produced by the proof of Lemma 1.50 has degree 4, whereas the one produced by the proof of Lemma 1.56 has degree 2. Moreover, as discussed in Example 1.54, the cover produced by the proof of Lemma 1.50 is not cyclic, whereas the one produced by the proof of Lemma 1.56 is. So the two covers are not isomorphic. In fact, the domains of the two covers are not isomorphic either. Indeed, the cover produced by the proof of Lemma 1.56 is given by the normalization of X in the field extension

$$K \subseteq K[t]/(t^2 - x_1 x_2) =: L$$

of its function field $K = A_{(0)}$. Consider the integral domain $B := A[t]/(t^2 - x_1 x_2)$. Then $Y := \operatorname{Spec}(B)$ has function field L. It follows from the Jacobian criterion that the singular locus of $Y \subseteq \mathbb{A}^3$ is just the origin. Since Y is a hypersurface with singularities in codimension at least 2, it is a normal variety. Therefore, B is integrally closed in L and Y is the normalization of X in the field extension $K \subseteq L$. But $Y \not\cong \mathbb{A}^2$, because Y is singular and $\mathbb{A}^2$ is smooth. So the domains of the two covers are not isomorphic.

Corollary 1.60. *Let (X, Δ) be a klt $\mathcal{C}$-pair. Then there exists a closed subset $Z \subseteq X$ of codimension at least 3 in X with the following property. For every point $x \in X \setminus Z$ there exists an analytic-open neighborhood U of x in X and a perfectly adapted cover $\gamma \colon \hat{U} \to U$ in which $\hat{U}$ is smooth.*

Proof. Since we may remove closed subsets of codimension at least 3, we may assume that X is $\mathbb{Q}$-factorial, cf. [GKKP11, Proposition 9.1]. By Lemma 1.56 we may assume that there exists a perfectly adapted cyclic cover $\gamma \colon \hat{X} \to X$. It follows from Lemma 1.47 that $(\hat{X}, 0)$ is klt. By [GKKP11, Proposition 9.3], there exists a closed subset $\hat{Z} \subseteq \hat{X}$ of codimension at least 3 such that every point $y \in \hat{X} \setminus \hat{Z}$ has an analytic-open neighborhood that is a quasi-étale quotient of affine space by a finite group action. Since γ is finite and surjective, $Z := \gamma(\hat{Z})$ is also a closed subset of codimension at least 3 in X, and the same is true about $\gamma^{-1}(Z) \supseteq \hat{Z}$. So, after removing Z, we may assume that there exists a perfectly adapted cover $\gamma \colon \hat{X} \to X$ such that every point $y \in \hat{X}$ has an analytic-open neighborhood U_y that is a quasi-étale quotient of affine space by a finite group action. By the local description of finite maps [GR84, Theorem in p. 48], we may assume that $\gamma|_{U_y} \colon U_y \to \gamma(U_y)$ is also a (perfectly adapted) cover. The claim follows now from Remark 1.40. □

Remark 1.61. There are other useful covering constructions in the literature, cf. [Laz04, §4.1.B]. A notable example are Kawamata covers [Laz04, Proposition 4.1.12], which have already been used in the literature to produce adapted covers, see [CP19, §5.1]. They have the advantage of being globally defined, but the price to pay is to allow some extra ramification, cf. Example 1.49. The cyclic cover construction [KM98, Definition 2.52] used in Lemma 1.56 is more suitable for our purposes.

5. Adapted differentials

Given a C-pair (X, Δ) with $\lfloor \Delta \rfloor = 0$ and an adapted morphism $\gamma\colon \hat{X} \to X$, adapted differentials will be usual reflexive[3] differentials on the normal variety $\hat{X}$. If $\lfloor \Delta \rfloor \neq 0$, then they will be (reflexive) logarithmic differentials on $(\hat{X}, \lfloor \Delta \rfloor)$. We refer to [Del70, II.§3] for background on logarithmic differentials.

Definition-Lemma 1.62 (Adapted differentials in the smooth case, [CP19, Definition 5.3])**.** Let (X, Δ) be a smooth C-pair and let $\gamma\colon \hat{X} \to X$ be an adapted morphism such that $(\hat{X}, \gamma^*\Delta)$ is smooth as well. Then $\gamma^*\mathcal{O}_{D_i} = \mathcal{O}_{\gamma^* D_i}$ for all $i \in I$, and since $\gamma^*\lceil \Delta \rceil \geq \gamma^*\Delta^{\mathrm{orb}}$ we also obtain a well-defined quotient morphism $q_i\colon \gamma^*\mathcal{O}_{D_i} \to \mathcal{O}_{\gamma^*\left(\frac{1}{m_i} D_i\right)}$ for all $i \in I^{\mathrm{orb}}$. This yields in turn a quotient morphism

$$q\colon \bigoplus_{i \in I} \gamma^*\mathcal{O}_{D_i} \to \bigoplus_{i \in I^{\mathrm{orb}}} \mathcal{O}_{\gamma^*\left(\frac{1}{m_i} D_i\right)}$$

in which we send each logarithmic summand on the left to zero on the right. We consider also the residue map

$$\mathrm{res}\colon \Omega^1_X(\log \lceil \Delta \rceil) \to \bigoplus_{i \in I} \mathcal{O}_{D_i}$$

and define the $\mathcal{O}_{\hat{X}}$-module morphism φ as the composition

$$\gamma^*\Omega^1_X(\log \lceil \Delta \rceil) \xrightarrow{\gamma^*(\mathrm{res})} \bigoplus_{i \in I} \gamma^*\mathcal{O}_{D_i} \xrightarrow{q} \bigoplus_{i \in I^{\mathrm{orb}}} \mathcal{O}_{\gamma^*\left(\frac{1}{m_i} D_i\right)}.$$

We define the $\mathcal{O}_{\hat{X}}$-module of *adapted differentials* as

$$\Omega^1_{(X,\Delta,\gamma)} := \ker(\varphi).$$

The sheaf of adapted differentials fits then into the following short exact sequence of $\mathcal{O}_{\hat{X}}$-modules:

$$0 \to \Omega^1_{(X,\Delta,\gamma)} \to \gamma^*\Omega^1_X(\log \lceil \Delta \rceil) \xrightarrow{\varphi} \bigoplus_{i \in I^{\mathrm{orb}}} \mathcal{O}_{\gamma^*\left(\frac{1}{m_i} D_i\right)} \to 0.$$

We also set $\Omega^p_{(X,\Delta,\gamma)} := \bigwedge^p \Omega^1_{(X,\Delta,\gamma)}$ for all $p \in \mathbb{N}$, and call this $\mathcal{O}_{\hat{X}}$-module the sheaf of *adapted differential p-forms.*

[3]Recall from Item 10 in Notation 0.7 that reflexive differential forms can be regarded as differential forms defined only over the smooth locus.

Proof. We need to show that $\gamma^*\mathcal{O}_{D_i} = \mathcal{O}_{\gamma^* D_i}$ for all $i \in I$ and that φ is surjective. For the first assertion, note that the morphism γ is flat, because it is a quasi-finite morphism between smooth varieties of the same dimension. Therefore the pull-back functor γ^* is exact, so for each $i \in I$ we have a short exact sequence

$$0 \to \gamma^*\mathcal{O}_X(-D_i) \to \gamma^*\mathcal{O}_X \to \gamma^*\mathcal{O}_{D_i} \to 0.$$

We have $\gamma^*\mathcal{O}_X = \mathcal{O}_{\hat{X}}$, and since D_i is an effective Cartier divisor we also have $\gamma^*\mathcal{O}_X(-D_i) = \mathcal{O}_{\hat{X}}(-\gamma^* D_i)$ [GW10, (11.16)]. This proves the first claim. The second assertion follows from right-exactness of γ^* and surjectivity of res and of q. □

Remark 1.63. In the situation of Definition-Lemma 1.62, let $(y_1, \dots, y_n)$ be a system of analytic-local coordinates around a point $y \in \hat{X}$ and let $(x_1, \dots, x_n)$ be a system of analytic-local coordinates around $x := \gamma(y)$. Then, up to shrinking the coordinate domains and choosing coordinates appropriately, we may assume the following:

(1) The morphism γ is given by

$$\gamma(y_1, \dots, y_n) = (y_1^{a_1}, \dots, y_l^{a_l}, y_{l+1}, \dots, y_p, y_{p+1}^{b_{p+1}}, \dots, y_n^{b_n})$$

with respect to these coordinates, with $a_1, \dots, a_l, b_{p+1}, \dots, b_n \in \mathbb{N}_{>0}$.

(2) The divisor $\lceil \Delta \rceil$ is given by $x_1 \cdots x_l = 0$ around x with $D_i = \{x_i = 0\}$ for each $i \in I$.

(3) For each $i \in I^{\mathrm{orb}}$ we have $a_i = k_i m_i$ for some $k_i \in \mathbb{N}_{>0}$.

(4) The divisors given analytic-locally around x by the equations $\{x_{p+1} = 0\}, \dots, \{x_n = 0\}$ correspond to the extra ramification of γ, which is not necessary to make $\gamma^*\Delta^{\mathrm{orb}}$ a divisor with integer coefficients.

Let $\Delta_{\hat{X}} := (\gamma^*\lfloor \Delta \rfloor)_{\mathrm{red}}$ and assume in the following discussion that $I^{\mathrm{orb}} = \{1, \dots, l'\}$ and $I^{\log} = \{l'+1, \dots, l\}$ for some $1 \le l' < l$. The idea behind adapted differentials is to consider those logarithmic differential forms on $(\hat{X}, \Delta_{\hat{X}})$ which would correspond to the pull-back of the following "multi-valued differential forms" on X:

$$\frac{dx_1}{x_1^{1-\frac{1}{m_1}}}, \dots, \frac{dx_{l'}}{x_{l'}^{1-\frac{1}{m_{l'}}}}, \frac{dx_{l'+1}}{x_{l'+1}}, \dots, \frac{dx_l}{x_l}, dx_{l+1}, \dots, dx_n.$$

With respect to our current analytic-local coordinate system, the pull-back of the first such "differential form" would be

$$\frac{d(y_1^{a_1})}{y_1^{a_1-k_1}} = \frac{a_1 y_1^{a_1-1} dy_1}{y_1^{a_1-1+1-k_1}} = a_1 y_1^{k_1} \frac{dy_1}{y_1},$$

which is a regular differential form on $\hat{X}$. Allowing ourselves a bit of sloppiness in the notation, we may write

$$\bigoplus_{i\in I} \mathcal{O}_{\gamma^* D_i} = \bigoplus_{i\in I} \mathcal{O}/(y_i^{a_i})$$

and also

$$\bigoplus_{i\in I^{\mathrm{orb}}} \mathcal{O}_{\gamma^*\left(\frac{1}{m_i}D_i\right)} = \bigoplus_{i\in I^{\mathrm{orb}}} \mathcal{O}/(y_i^{k_i}).$$

We check that the logarithmic form $y_1^{k_1}\frac{dy_1}{y_1}$ is indeed in $\Omega^1_{(X,\Delta,\gamma)}$. We have

$$y_1^{k_1}\frac{dy_1}{y_1} \xmapsto{\gamma^*(\mathrm{res})} (y_1^{k_1} + (y_1^{a_1}), 0 + (y_2^{a_2}), \ldots, 0 + (y_l^{a_l}))$$

and

$$(y_1^{k_1} + (y_1^{a_1}), 0 + (y_2^{a_2}), \ldots, 0 + (y_l^{a_l})) \overset{q}{\mapsto} (0 + (y_1^{k_1}), \ldots, 0 + (y_{l'}^{a_{l'}})),$$

hence the claim.

On the other hand, the pull-back of the l-th differential form above would be

$$\frac{d(y_l^{a_l})}{y_l^{a_l}} = \frac{a_l y_l^{a_l-1} dy_l}{y_l^{a_l}} = \frac{a_l dy_l}{y_l},$$

and the logarithmic differential $\frac{dy_l}{y_l}$ lies in $\Omega^1_{(X,\Delta,\gamma)}$ by construction of q. The same is true for the pull-backs of the remaining differential forms listed above.

Therefore, with respect to our current analytic-local coordinate system around y, the $\mathcal{O}_{\hat{X}}$-module $\Omega^1_{(X,\Delta,\gamma)}$ is generated by the logarithmic differential forms

$$y_1^{k_1-1}dy_1, \ldots, \frac{dy_l}{y_l}, dy_{l+1}, \ldots, dy_p, y_{p+1}^{b_{p+1}-1}dy_{p+1}, \ldots, y_n^{b_n-1}dy_n.$$

See also [CP19, §5.2].

Example 1.64. In the setting of Example 1.48, let $n = 2n'$ for some $n' \in \mathbb{N}_{>0}$. Then we have

$$\Omega^1_{(\mathbb{A}^1,\Delta,\gamma)}(\mathbb{A}^1) = \mathbb{C}[z] \cdot z^{n'-1}dz.$$

In particular, for $n = 2$ we get $\Omega^1_{(\mathbb{A}^1,\Delta,\gamma)} = \Omega^1_{\mathbb{A}^1}$.

Example 1.65. Let $X = \mathbb{A}^1$ with affine coordinate x and consider the prime Weil divisor $D_0 = \{x = 0\}$. Consider the divisors $\Delta = \frac{1}{2}D_0$ and $\Delta' = \frac{2}{3}D_0$. We have then $\Delta' \geq \Delta$. Let now $\hat{X} = \mathbb{A}^1$ with affine coordinate z and consider the cover $\gamma\colon \hat{X} \to X$ induced by the coordinate ring map $x \mapsto z^6$. Then γ is adapted both for (X, Δ) and for (X, Δ') and we are in the situation of Definition-Lemma 1.62 with respect to both (X, Δ) and (X, Δ'), so we have two sheaves of modules $\Omega^1_{(X,\Delta,\gamma)}$ and $\Omega^1_{(X,\Delta',\gamma)}$ on $\hat{X}$. It follows from the local computation in Remark 1.63 that

$$\Omega^1_{(X,\Delta,\gamma)}(\hat{X}) = \mathbb{C}[z] \cdot z^2 dz,$$

because γ is given by $z \mapsto z^6$ and $6 = 3\,\mathrm{mult}_{C,D_0}(\Delta)$. Similarly, we have

$$\Omega^1_{(X,\Delta',\gamma)}(\hat{X}) = \mathbb{C}[z] \cdot z dz.$$

Since $\hat{X}$ is affine, this implies that we have an inclusion of $\mathcal{O}_{\hat{X}}$-modules

$$\Omega^1_{(X,\Delta,\gamma)} \subseteq \Omega^1_{(X,\Delta',\gamma)}.$$

Lemma 1.66 ([CP19, §5])**.** *In the setting of Definition-Lemma 1.62, the sheaves $\Omega^p_{(X,\Delta,\gamma)}$ are locally free for all $p \in \mathbb{N}$.*

Proof. For $p = 0$ we have $\Omega^0_{(X,\Delta,\gamma)} = \mathcal{O}_{\hat{X}}$, which is locally free. For $p = 1$, the assertion follows from the local computation in Remark 1.63. The assertion for $p > 1$ follows from the assertion for $p = 1$, because the corresponding sheaf is defined as an exterior power of the sheaf for $p = 1$. □

In the general case, we will follow [KR] and use reflexive differentials:

Definition 1.67 (Adapted reflexive differentials)**.** Let (X, Δ) be a C-pair and let $\gamma\colon \hat{X} \to X$ be an adapted morphism. Let $U \subseteq X$ be the largest open subset of X such that the restriction (U, Δ_U) is smooth, which is a big open subset in X. By Lemma B.5, $\gamma^{-1}(U)$ is also a big open subset in $\hat{X}$. If we intersect $\gamma^{-1}(U)$ with the largest open subset in $\hat{X}$ over which $(\hat{X}, \gamma^*\Delta)$ is smooth, which is again big, we obtain a new big open subset $\hat{U} \subseteq \hat{X}$ such that the restriction $\gamma|_{\hat{U}}\colon \hat{U} \to U$ satisfies the assumptions in Definition-Lemma 1.62. Consider the inclusion morphism $\iota\colon \hat{U} \to \hat{X}$. We define the $\mathcal{O}_{\hat{X}}$-module of *adapted reflexive differentials* as

$$\Omega^{[1]}_{(X,\Delta,\gamma)} := \iota_*\Omega^1_{(U,\Delta_U,\gamma|_{\hat{U}})}.$$

For each $p \in \mathbb{N}$ we define the $\mathcal{O}_{\hat{X}}$-module of *adapted reflexive differential p-forms* as

$$\Omega^{[p]}_{(X,\Delta,\gamma)} := \iota_*\Omega^p_{(U,\Delta_U,\gamma|_{\hat{U}})}.$$

Corollary 1.68. *In the setting of Definition 1.67, the sheaves $\Omega^{[p]}_{(X,\Delta,\gamma)}$ are reflexive for all $p \in \mathbb{N}$.*

Proof. By Lemma 1.66, this sheaf is the push-forward of a locally free sheaf defined over a big open subset, hence reflexive by Fact A.2. □

Remark 1.69. By Remark A.5, shrinking the big open subsets in Definition 1.67 to a smaller big open subset will produce the same $\mathcal{O}_{\hat{X}}$-module.

Remark 1.70 ([KR])**.** In the setting of Definition 1.67, for each $p \in \mathbb{N}$, identifying certain sheaves with their images under the natural morphisms and with the notation of Item 12 in Notation 0.7, we have the following diagram of inclusions of $\mathcal{O}_{\hat{X}}$-modules:

$$\begin{tikzcd} \gamma^{[*]}\Omega^p_X \arrow[r, hook] & \Omega^{[p]}_{(X,\Delta,\gamma)} \arrow[r, hook] \arrow[d, hook] & \Omega^{[p]}_{\hat{X}}(\log(\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}) \arrow[d, hook] & \\ & \gamma^{[*]}\Omega^{[p]}_X(\log\lceil\Delta\rceil) \arrow[r, hook] & \Omega^{[p]}_{\hat{X}}(\log(\gamma^*\lceil\Delta\rceil)_{\mathrm{red}}) \arrow[r, hook] & \mathcal{K}_{\hat{X}}(\Omega^p_{\hat{X}}). \end{tikzcd}$$

Indeed, except for $\mathcal{K}_{\hat{X}}(\Omega^p_{\hat{X}})$, all sheaves involved are coherent and reflexive by definition. So it suffices to check the corresponding inclusions in the setting of Definition-Lemma 1.62. In this setting, the inclusions follow from the local computation in Remark 1.63. The remaining inclusion was discussed in Item 12 in Notation 0.7.

Remark 1.71. By definition, a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ is reflexive if and only if the canonical map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ to its reflexive hull is an isomorphism, cf. Appendix A. So $f^*\mathcal{F} = f^{[*]}\mathcal{F}$ if and only if $f^*\mathcal{F}$ is reflexive. This is the case, for instance, if $\mathcal{F}$ is locally free or if f is flat [Har80, Proposition 1.8].

Generalizing Example 1.65 we have the following:

Lemma 1.72. *Let (X, Δ) and (X, Δ') be two C-pairs with $\Delta' \geq \Delta$ and let $\gamma\colon \hat{X} \to X$ be a morphism which is adapted for (X, Δ) and (X, Δ') simultaneously. Then we have for each $p \in \mathbb{N}$ an inclusion of $\mathcal{O}_{\hat{X}}$-modules*

$$\Omega^{[p]}_{(X,\Delta,\gamma)} \subseteq \Omega^{[p]}_{(X,\Delta',\gamma)}.$$

Proof. By Corollary 1.68, both sheaves are reflexive. So it suffices to show the inclusion over an appropriate big open subset and we may assume that we are in the situation of Definition-Lemma 1.62 with respect to both Δ and Δ'. It suffices also to show the result for $p = 1$. The claim follows then from the local computation in Remark 1.63 as in Example 1.65. ◻

And generalizing Example 1.64 we have the following:

Lemma 1.73. *Let (X, Δ) be a C-pair and let $\gamma\colon \hat{X} \to X$ be a perfectly adapted morphism. Let $\Delta_{\hat{X}} := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}$. Then, for all $p \in \mathbb{N}$, we have*

$$\Omega^{[p]}_{(X,\Delta,\gamma)} = \Omega^{[p]}_{\hat{X}}(\log \Delta_{\hat{X}}).$$

Proof. Since both sheaves involved are reflexive, it suffices to show that they agree over a big open subset. Moreover, over a big open subset, for each $p \in \mathbb{N}$, both sides are by definition obtained as the p-th exterior power of the corresponding sheaves for $p = 1$, cf. [Iit82, §11.1.b]. So we may assume that we are in the situation of Definition-Lemma 1.62 with $p = 1$. We can write γ analytic-locally as

$$\gamma(y_1, \ldots, y_n) = (y_1^{a_1}, \ldots, y_l^{a_l}, y_{l+1}, \ldots, y_p, y_{p+1}^{b_{p+1}}, \ldots, y_n^{b_n})$$

as in Remark 1.63. The assumption that γ is perfectly adapted translates then into $a_i = m_i$ for all $1 \le i \le l'$ and $b_i = 1$ for all $p + 1 \le i \le n$, i.e., we can write γ as

$$\gamma(y_1, \ldots, y_n) = (y_1^{m_1}, \ldots, y_{l'}^{m_{l'}}, y_{l'+1}^{a_{l'+1}}, \ldots, y_l^{a_l}, y_{l+1}, \ldots, y_n)$$

for some $a_{l'+1}, \ldots, a_l \in \mathbb{N}_{>0}$. It follows then from the analytic-local description in Remark 1.63 that $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is generated in this coordinate domain by

$$dy_1, \ldots, dy_{l'}, \frac{dy_{l'+1}}{y_{l'+1}}, \ldots, \frac{dy_l}{y_l}, dy_{l+1}, \ldots, dy_n,$$

which shows that the two sheaves agree over the big open subset in which the setting of Definition-Lemma 1.62 holds. □

Given two adapted morphisms related by a morphism over the variety underlying the C-pair, we can relate adapted differentials on each of their domains as follows:

Lemma 1.74 (cf. [KR])**.** *Let $\gamma_1 \colon \hat{X}_1 \to X$ and $\gamma_2 \colon \hat{X}_2 \to X$ be two adapted morphisms and let $f \colon \hat{X}_2 \to \hat{X}_1$ be a morphism over X. Let $p \in \mathbb{N}$. The pull-back of rational differentials induces an isomorphism*

$$f^{[*]}\Omega^{[p]}_{(X,\Delta,\gamma_1)} \cong \Omega^{[p]}_{(X,\Delta,\gamma_2)},$$

using the terminology of Item 13 of Notation 0.7.

Proof. Both sheaves are reflexive, so we may restrict to suitable big open subsets. Furthermore, f is a quasi-finite morphism between normal varieties of the same dimension, cf. Remark 1.39. So by Lemma B.5 we may assume that (X, Δ), $(\hat{X}_1, \gamma_1^*\Delta)$ and $(\hat{X}_2, \gamma_2^*\Delta)$ are all smooth. Let $\Delta_i := (\gamma_i^*\lfloor\Delta\rfloor)_{\mathrm{red}}$ for each $i \in \{1, 2\}$. Then $\Omega^p_{(X,\Delta,\gamma_1)}$ is a locally free subsheaf of $\Omega^p_{\hat{Y}_1}(\log \Delta_1)$ and f is dominant, so the pull-back of logarithmic differential forms as in [Iit82, §11.c] induces an injective morphism

$$f^*\Omega^p_{(X,\Delta,\gamma_1)} \to \Omega^p_{\hat{X}_2}(\log \Delta_2).$$

The claim is that the image is precisely $\Omega^p_{(X,\Delta,\gamma_2)}$. It suffices to show this on the stalks and for $p = 1$. The morphism from a noetherian local ring to its completion is faithfully flat [SP, 00MC], and the completion of a finitely generated module over said ring can be expressed as the tensor product with the completion of the ring [AM69, Proposition 10.13]. Therefore, it suffices to show that the image is $\Omega^1_{(X,\Delta,\gamma_2)}$ on the completions of the stalks. The natural map from the algebraic stalks to the analytic stalks induces an isomorphism on completions, cf. [Ser56, Proposition 3]. So we may assume that we are in the analytic setting.

We do this using the local computation in Remark 1.63. We choose analytic-local coordinates on $\hat{X}_1$ such that

$$\gamma_1(y_{(1,1)}, \ldots, y_{(1,n)}) = (y_{(1,1)}^{a_{(1,1)}}, \ldots, y_{(1,l)}^{a_{(1,l)}}, y_{(1,l+1)}^{b_{(1,l+1)}}, \ldots, y_{(1,n)}^{b_{(1,n)}}),$$

where we allow some of the $b_{(1,i)}$ to be equal to 1 for convenience of notation. We choose analytic-local coordinates on $\hat{X}_2$ such that

$$f(y_{(2,1)}, \ldots, y_{(2,n)}) = (y_{(2,1)}^{c_1}, \ldots, y_{(2,n)}^{c_n}).$$

Then we have

$$\gamma_2(y_{(2,1)}, \ldots, y_{(2,n)}) = (y_{(2,1)}^{a_{(2,1)}}, \ldots, y_{(2,l)}^{a_{(2,l)}}, y_{(2,l+1)}^{b_{(2,l+1)}}, \ldots, y_{(2,n)}^{b_{(2,n)}})$$

with $a_{(2,i)} = c_i a_{(1,i)}$ and $b_{(2,j)} = c_j b_{(1,j)}$ for all $i \in \{1, \ldots, l\}$ and all $j \in \{l+1, \ldots, n\}$ respectively, and these are again coordinates as in Remark 1.63 with respect to γ_2, up to possibly permuting some of the indices in $\{l + 1, \ldots, n\}$.

Suppose that $1 \in I$ is an index corresponding to some irreducible component of Δ^{orb}. Then we have $a_{(1,1)} = k_{(1,1)} m_1$ for some $k_{(1,1)} \in \mathbb{N}_{>0}$, and the corresponding generator of $\Omega^1_{(X,\Delta,\gamma_1)}$ is $y_{(1,1)}^{k_{(1,1)}-1} dy_{(1,1)}$. The pull-back of this generator along f is given by

$$y_{(2,1)}^{c_1 k_{(1,1)} - c_1} y_{(2,1)}^{c_1 - 1} dy_{(2,1)} = y_{(2,1)}^{c_1 k_{(1,1)} - 1} dy_{(2,1)},$$

which is the corresponding generator of $\Omega^1_{(X,\Delta,\gamma_2)}$ because

$$a_{(2,1)} = c_1 a_{(1,1)} = c_1 k_{(1,1)} m_1.$$

Suppose that $l \in I$ is an index corresponding to some irreducible component of $\Delta^{\log}$. Then the corresponding generator of $\Omega^1_{(X,\Delta,\gamma_1)}$ is $\frac{dy_{(1,l)}}{y_{(1,l)}}$. The pull-back of this generator along f is given by

$$\frac{d\left(y_{(2,l)}^{c_l}\right)}{y_{(2,l)}^{c_l}} = \frac{c_1 dy_{(2,l)}}{y_{(2,l)}},$$

again equal up to a non-zero scalar multiple to the corresponding generator of $\Omega^1_{(X,\Delta,\gamma_2)}$.

Suppose that the n-th coordinate does not correspond to an irreducible divisor in Δ. The last generator of $\Omega^1_{(X,\Delta,\gamma_1)}$ would then be $y_{(1,n)}^{b_{(1,n)}-1} dy_{(1,n)}$. Its pull-back along f is given by

$$y_{(2,n)}^{c_n b_{(1,n)} - c_n} d\left(y_{(2,n)}^{c_n}\right) = y_{(2,n)}^{c_n b_{(1,n)} - 1} dy_{(2,n)},$$

again equal to the corresponding generator of $\Omega^1_{(X,\Delta,\gamma_2)}$ because of the relation $b_{(2,n)} = c_n b_{(1,n)}$. ◻

Example 1.75. Let $X = \mathrm{Spec}(\mathbb{C}[x_1, x_2]) = \mathbb{A}^2$ and consider $\Delta = \frac{1}{2}\{x_1 = 0\} + \frac{1}{2}\{x_2 = 0\}$ as in Examples 1.54 and 1.59. Let $\gamma_1 \colon \hat{X}_1 \longrightarrow X$ be the cyclic cover considered in Example 1.59 and let $\gamma \colon \hat{X}_2 \longrightarrow X$ be the abelian cover considered in Example 1.54. Denote $A := \mathbb{C}[x_1, x_2]$, $K := A_{(0)}$, K_1 the function field of $\hat{X}_1$ and K_2 the function field of $\hat{X}_2$. We have seen in Examples 1.58 and 1.59 that we can write

$$K_1 = K[t]/(t^2 - x_1x_2) \quad \text{and} \quad K_2 = K[t_1, t_2]/(t_1^2 - x_1, t_2^2 - x_2).$$

The K-algebra morphism $K_1 \to K_2$ induced by

$$t \mapsto t_1t_2$$

and the universal property of the normalization show that there exist a morphism $f\colon \hat{X}_2 \to \hat{X}_1$ over X. Explicitly, writing

$$\hat{X}_2 = \{(x_1, x_2, t_1, t_2) \in \mathbb{A}^4 \mid t_1^2 = x_1, t_2^2 = x_2\} \subseteq \mathbb{A}^4$$

and

$$\hat{X}_1 = \{(x_1, x_2, t) \in \mathbb{A}^3 \mid t^2 = x_1x_2\} \subseteq \mathbb{A}^3,$$

the morphism f is induced by the morphism

$$\begin{aligned} \mathbb{A}^4 &\to \mathbb{A}^3 \\ (x_1, x_2, t_1, t_2) &\mapsto (x_1, x_2, t_1t_2) \end{aligned}$$

To clarify things a bit further, let us also consider the isomorphism $\psi\colon \operatorname{Spec}(\mathbb{C}[y_1, y_2]) \to \hat{X}_2$ induced by

$$(y_1, y_2) \mapsto (y_1^2, y_2^2, y_1, y_2).$$

Under this isomorphism, adapted cover γ_2 is identified with the morphism induced by $(y_1, y_2) \mapsto (y_1^2, y_2^2)$, and the morphism f is identified with the morphism induced by

$$(y_1, y_2) \mapsto (y_1^2, y_2^2, y_1y_2).$$

Therefore, $\hat{X}_1$ is the ordinary quadratic cone in $\mathbb{A}^3$, with its A_1 singularity, and $f\colon \hat{X}_2 \to \hat{X}_1$ is the quotient morphism for the $\mathbb{Z}/2\mathbb{Z}$-action on $\hat{X}_1 = \mathbb{A}^2$ given by $(y_1, y_2) \mapsto (-y_1, -y_2)$. Since both γ_1 and γ_2 are perfectly adapted, we have

$$\Omega^{[1]}_{(X,\Delta,\gamma_i)} = \Omega^{[1]}_{\hat{X}_i}$$

for all $i \in \{1, 2\}$, cf. Lemma 1.73. So an adapted (reflexive) differential on $\hat{X}_1$ is the same as a regular differential defined away from the singularity of the cone, and an adapted differential on $\hat{X}_2$ is the same as a usual differential on the affine plane. The pull-back of a differential defined away from the singularity gives a differential on the plane defined away from the origin, which is the restriction of a regular differential defined on the whole affine plane by reflexivity and normality.

Remark 1.76 ([KR]). In the setting of Lemma 1.74, assume that f is Galois with Galois group G. Then the G-action on $\hat{X}_2$ induces a G-sheaf structure on logarithmic differentials [GKKP11, Fact 10.5], hence on adapted differentials by Lemma 1.74.

Lemma 1.77. *Let $\gamma_1 : \hat{X}_1 \to X$ and $\gamma_2 : \hat{X}_2 \to X$ be two adapted morphisms and let $f : \hat{X}_2 \to \hat{X}_1$ be a morphism over X which is a Galois cover with Galois group G. Then the pull-back of rational differentials induces an isomorphism*

$$\Omega^{[p]}_{(X,\Delta,\gamma_1)} \cong \left(f_* \Omega^{[p]}_{(X,\Delta,\gamma_2)}\right)^G,$$

using the terminology of Item 13 of Notation 0.7.

Proof. The sheaf $(f_*\Omega^{[p]}_{(X,\Delta,\gamma_2)})^G$ is reflexive [GKKP11, Lemma A.4], so we may restrict our attention to big open subsets and assume that (X, Δ), $(\hat{X}_1, \gamma_1^*\Delta)$ and $(\hat{X}_2, \gamma_2^*\Delta)$ are all smooth. By Lemma 1.74, the pull-back of rational differentials induces an isomorphism

$$f^*\Omega^p_{(X,\Delta,\gamma_1)} \cong \Omega^p_{(X,\Delta,\gamma_2)}.$$

If ϕ_g denotes the automorphism of $\hat{X}_2$ corresponding to $g \in G$, then $f \circ \phi_g = f$, so the previous isomorphism is also an isomorphism of G-sheaves with respect to the G-sheaf structure induced on the pull-back. Hence

$$\Omega^p_{(X,\Delta,\gamma_1)} \cong \left(f_* f^* \Omega^p_{(X,\Delta,\gamma_1)}\right)^G \cong \left(f_*\Omega^p_{(X,\Delta,\gamma_2)}\right)^G.$$

□

Example 1.78. In the setting of Example 1.75, we rename things a bit and consider the commutative diagram

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, "q"] \arrow[dr, "\gamma"'] & Y \arrow[d, "\bar{\gamma}"] \\ & X \end{tikzcd}$$

in which γ is given by $(y_1, y_2) \mapsto (y_1^2, y_2^2)$ and q is the geometric quotient by the $G := \mathbb{Z}/2\mathbb{Z}$-action given by $(y_1, y_2) \mapsto (-y_1, -y_2)$. The quotient Y is the ordinary quadratic cone in $\mathbb{A}^3$. Adapted differentials on $\mathbb{A}^2$ (resp. on Y) are the same as usual differentials on $\mathbb{A}^2$ (resp. reflexive differentials on Y). A reflexive differential on Y pulls back to a G-invariant differential on $\mathbb{A}^2$. Conversely, a G-invariant differential on $\mathbb{A}^2$ restricts to a G-invariant differential on $\mathbb{A}^2 \setminus \{(0,0)\}$, which descends to a differential form defined away from the singularity in Y, i.e., a reflexive differential form.

To conclude this section, let us study the behavior of adapted differentials on products and on quotients. The main purpose of these results is to produce examples in the following chapters.

Lemma 1.79. *In the setting of Lemma 1.42, we have*

$$\Omega^{[1]}_{(X\times Y, \Delta_{X\times Y}, \gamma_X \times \gamma_Y)} \cong \left(p^*_{\hat{X}} \Omega^{[1]}_{(X,\Delta_X,\gamma_X)}\right) \oplus \left(p^*_{\hat{Y}} \Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)}\right).$$

Proof. Note that the projections from the product are flat, and the direct sum of two reflexive coherent sheaves is again a reflexive coherent sheaf. So both sides of the isomorphism are reflexive coherent sheaves.

Therefore, we may assume that we are in the situation of Definition-Lemma 1.62 as usual. It suffices to check the isomorphism analytic-locally, cf. proof of Lemma 1.74. We use analytic-local coordinates such that the following hold:

- We have $\gamma_X(x_1, \ldots, x_n) = (x_1^{a_1}, \ldots, x_n^{a_n})$ with $a_1, \ldots, a_n \in \mathbb{N}_{>0}$.
- We have $\gamma_Y(y_1, \ldots, y_m) = (y_1^{b_1}, \ldots, y_m^{b_m})$ with $b_1, \ldots, b_m \in \mathbb{N}_{>0}$.
- The irreducible components of the divisors Δ_X and Δ_Y are given by coordinate hyperplanes with respect to these coordinates.

With respect to the induced coordinates on the products $\hat{X} \times \hat{Y}$ and $X \times Y$, the morphism $\gamma_X \times \gamma_Y$ is given by

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1^{a_1}, \ldots, x_n^{a_n}, y_1^{b_1}, \ldots, y_m^{b_m}).$$

By Definition 1.18, the irreducible components of $\Delta_{X \times Y}$ are still given by the corresponding coordinate hyperplanes with respect to these coordinates. The claim follows then from the description of the local generators in Remark 1.63. □

Remark 1.80. In order to describe adapted p-forms on the product of adapted covers, for $p \geq 2$, one would need to use formulas of the type

$$\bigwedge^p (M \oplus N) = \bigoplus_{i+j=p} \left(\wedge^i M\right) \otimes \left(\wedge^j N\right).$$

Since we are mostly interested in Lemma 1.79 to produce examples of the adapted Albanese in Chapter 3, the case $p = 1$ suffices for our purposes.

Lemma 1.81 (cf. [KR]). *In the setting of Lemma 1.43, if $\gamma' \colon \hat{X} \to X'$ is adapted and $p \in \mathbb{N}$, then we have*

$$\Omega^{[p]}_{(X,\Delta,\gamma)} = \Omega^{[p]}_{(X',\Delta',\gamma')}.$$

Proof. It suffices to show the equality analytic-locally, cf. proof of Lemma 1.74. As in the proof of Lemma 1.73, we may assume that we are in the situation of Definition-Lemma 1.62 with $p = 1$. Pick analytic-local coordinates $y_1, \ldots, y_n$ on $\hat{X}$, $x_1, \ldots, x_n$ on X' and $z_1, \ldots, z_n$ on X such that the following hold:

- We have $\gamma'(y_1, \ldots, y_n) = (y_1^{a_1}, \ldots, y_n^{a_n})$ with $a_1, \ldots, a_n \in \mathbb{N}_{>0}$.
- We have $q(x_1, \ldots, x_n) = (x_1^{b_1}, \ldots, x_n^{b_n})$ with $b_1, \ldots, b_n \in \mathbb{N}_{>0}$.
- We have $I^{\mathrm{orb}}_{X'} = \{1, \ldots, l'\}$ and $I^{\log}_{X'} = \{l' + 1, \ldots, l\}$, say with $1 \leq l' < l < n$, and the irreducible components of Δ' are given by $D'_i = \{x_i = 0\}$ for all $i \in \{1, \ldots, l\}$.

It follows that the irreducible components D_i of Δ are given by $\{z_i = 0\}$ for $i \in \{1, \ldots, l\} \cup \{i \in \{1, \ldots, n\} \mid b_i > 1\}$, and we have

$$\mathrm{mult}_{C,D_i}(\Delta) = b_i \, \mathrm{mult}_{C,D'_i}(\Delta')$$

for all $i \in \{1, \ldots, n\}$. The logarithmic components give rise to the same generators automatically, cf. Remark 1.63. And if $i \in \{1, \ldots, n\} \setminus I_{X'}^{\log}$, then we have

$$a_i b_i = k_i \operatorname{mult}_{C,D_i'}(\Delta') b_i = k_i \operatorname{mult}_{C,D_i}(\Delta)$$

for some $k_i \in \mathbb{N}$ as in Remark 1.63. So the analytic-local computation in Remark 1.63 shows that each such component gives rise to the same generator, and the two sheaves agree. $\square$

CHAPTER 2

Adapted differentials as a qfh-sheaf

The definition of adapted differentials as in Definition 1.67 has a number of drawbacks. For example, one wants to talk about invariants of the C-pair (X, Δ) and not of any given adapted cover $\gamma\colon Y \to X$, so whenever we define an invariant using adapted differentials we need to worry about cover-independence issues.

Adapted covers are examples of *qfh-covers*, i.e., "open covers" with respect to the qfh-topology on the category of schemes. So we can think of adapted differentials as being defined *qfh-locally* on X. In this chapter we make this precise by showing that adapted differentials form a qfh-sheaf on the category $\mathbf{Sch}/X$ of schemes (of finite type[1]) over X. More precisely:

Theorem 2.1. *Let (X, Δ) be an algebraic C-pair and let $p \in \mathbb{N}$. Then there exists a presheaf $\Omega^p_{(X,\Delta)}$ on $\mathbf{Sch}/X$, unique up to isomorphism, with the following universal property:*

(1) *For every adapted morphism $\gamma\colon Y \to X$ there exists a morphism*
$$\Omega^p_{(X,\Delta)}(Y) \to \Omega^{[p]}_{(X,\Delta,\gamma)}(Y),$$
and these morphisms are compatible with the pull-back of differential forms.

(2) *For every presheaf $\mathcal{H}$ on $\mathbf{Sch}/X$ satisfying Item 1, there exists a unique morphism $\mathcal{H} \to \Omega^p_{(X,\Delta)}$ compatible with the morphisms from Item 1.*

Moreover, the morphisms in Item 1 are isomorphisms in the case of $\Omega^p_{(X,\Delta)}$, and $\Omega^p_{(X,\Delta)}$ is a sheaf with respect to the qfh-topology on $\mathbf{Sch}/X$.

Remark 2.2. In particular, we can recover $\Omega^{[p]}_{(X,\Delta,\gamma)}$ as the restriction of $\Omega^p_{(X,\Delta)}$ to the Zariski topology on Y, because every dense open subset of the domain of an adapted morphism induces another adapted morphism.

Remark 2.3. The universal property in Theorem 2.1 corresponds to the natural way of extending a presheaf on the full subcategory $\mathbf{Adpt}(X, \Delta)$ to a presheaf on the whole category $\mathbf{Sch}/X$. In the context of presheaves on topological spaces, this is somewhat analogous to the extension of a presheaf defined on a basis for the topology to a presheaf on the whole topological space, cf. [EGA I, Chap. 0, (3.2.1)].

[1] Recall Item 4 in Notation 0.7.

This chapter is extracted from [Núñ22], but with the appropriate modification to fit into this thesis and with some more details. In Section 1 we briefly review topologies on categories and sheaves in this setting. Sections 2, 3, 4 and 5 contain the proof of Theorem 2.1. In Section 6 we look at some properties of the sheaf constructed in Theorem 2.1, and in Section 7 we compute its cohomology and its sections over some quasi-finite schemes over X.

1. Sites and sheaves

Zariski-open subsets are too large for many purposes, cf. [SP, 03N3]. For this reason, it is sometimes useful to consider the analytic topology instead. In fact, we have already done this in Chapter 1, see for example Remark 1.63. But the analytic topology is not always an option, e.g., when working with schemes over more general fields. For this and other reasons it is desirable to abstract the notion of a topological space and replace it with a categorical analogue. In order to do geometry on schemes, we mostly care about sheaves on them. So the notion that we would really like to abstract is that of an “open cover”, which is all that is needed to define what being a sheaf means for a given presheaf. This leads to the following definition:

Definition 2.4 (Site, cf. [SP, 00VH])**.** A *family of morphisms with fixed target* in a category $\mathbf{C}$ consists of an object U of $\mathbf{C}$ and a set of morphisms $\{U_i \longrightarrow U\}_{i\in I}$ in $\mathbf{C}$. A *site*[2] consists of a category $\mathbf{C}$ together with a set $\mathrm{Cov}(\mathbf{C})$ of families of morphism with fixed target in $\mathbf{C}$, called *coverings of* $\mathbf{C}$, satisfying the following axioms:

(1) If $V \longrightarrow U$ is an isomorphism, then $\{V \longrightarrow U\} \in \mathrm{Cov}(\mathbf{C})$.
(2) If $\{U_i \longrightarrow U\}_{i\in I} \in \mathrm{Cov}(\mathbf{C})$ and for each $i \in I$ we have $\{V_{ij} \longrightarrow U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathbf{C})$, then $\{V_{ij} \longrightarrow U\}_{i\in I, j\in J} \in \mathrm{Cov}(\mathbf{C})$.
(3) If $\{U_i \longrightarrow U\}_{i\in I} \in \mathrm{Cov}(\mathbf{C})$ and $V \longrightarrow U$ is a morphism in $\mathbf{C}$, then $U_i \times_U V$ exists for all $i \in I$ and $\{U_i \times_U V \longrightarrow V\}_{i\in I} \in \mathrm{Cov}(\mathbf{C})$.

Remark 2.5. There are some set-theoretic issues that arise from this definition of a site and from considering sheaves on sites. We will not discuss these issues here. Instead, we refer to [SP, 00VI] for a detailed discussion.

Example 2.6 ([SP, 00VJ])**.** Let X be a topological space and let X_{Zar} be the category whose objects are the open subsets $U \subseteq X$ and whose morphisms are the inclusion maps, i.e.,

$$\mathrm{Mor}_{X_{\mathrm{Zar}}}(U, V) = \begin{cases} \{*\} & \text{if } U \subseteq V, \\ \varnothing & \text{if } U \not\subseteq V. \end{cases}$$

[2]We follow [SP] when it comes to sites and sheaves on them. In particular, our terminology differs from [SGA4], cf. [SP, 00V0].

We set $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X_{\mathrm{Zar}})$ if and only if $\cup_{i\in I} U_i = U$. Note that in X_{Zar} we have $U \times_X V = U \cap V$, so with these coverings we obtain a site on X_{Zar}.

Definition 2.7 (Presheaf, cf. [SP, 00V2]). A *presheaf* on a category $\mathbf{C}$ is a contravariant functor from $\mathbf{C}$ to the category $\mathbf{Set}$ of sets, i.e., a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. A *morphism of presheaves* is a natural transformation of functors. We denote by $\mathbf{PSh}(\mathbf{C})$ the category of presheaves on $\mathbf{C}$.

Definition 2.8 (Sheaf, cf. [SP, 00VM]). Let $\mathbf{C}$ be a site and let $\mathcal{F}$ be a presheaf on the category $\mathbf{C}$. We say that $\mathcal{F}$ is a *sheaf* if for every $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathbf{C})$ the diagram

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \overset{\mathrm{pr}_0^*}{\underset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{2.1}$$

represents the first arrow as an equalizer of pr_0^* and pr_1^*. A *morphism of sheaves* is a morphism of presheaves. We denote by $\mathbf{Sh}(\mathbf{C})$ the category of sheaves on $\mathbf{C}$.

Example 2.9 ([SP, 00VO]). Let X be a topological space and let X_{Zar} be the site from Example 2.6. Then a sheaf on X_{Zar} coincides with the usual notion of a sheaf on the topological space X.

Remark 2.10. Two different sites with the same underlying category may give rise to the same notion of sheaf. There is a more general way to define sheaves on categories which does not assume the existence of any fiber products. This is the notion of a topology on a category [SP, 00Z4]. Every site induces a topology, and a presheaf is a sheaf for this site if and only if it is a sheaf for this induced topology [SP, 00ZC]. In algebraic geometry it is often convenient to work with a given class of coverings [SP, 00V0], this is why we work with sites instead of with topologies. But topologies are conceptually better, because they are determined by the notion of sheaf they define [SP, 00ZP].

Remark 2.11. Let $\mathbf{C}$ be a site and $\mathcal{F}$ be a sheaf. We will sometimes say that $\mathcal{F}$ is a sheaf with respect to the topology on $\mathbf{C}$ to mean that $\mathcal{F}$ is a sheaf with respect to the induced topology on $\mathbf{C}$. By Remark 2.10, there is no ambiguity in doing so.

Remark 2.12. We will frequently consider (pre)sheaves with values in abelian categories, such as the category $\mathbf{Ab}$ of abelian groups or the category $\mathbf{Mod}(\mathbb{C})$ of vector spaces over $\mathbb{C}$. This also means that the restriction morphisms should be morphisms in those abelian categories. We will usually not make this explicit in the notation and continue using $\mathbf{Sh}(\mathbf{C})$ to refer to sheaves of abelian groups (resp. vector spaces over $\mathbb{C}$) if no confusion is likely to happen. We refer to [SP, 00YR] for more details on this.

The reason to consider sheaves with values in abelian categories is that we want to do homological algebra with them, e.g., in order to produce interesting invariants. For example, the category of sheaves of abelian groups on a site $\mathbf{C}$ is an abelian category with enough injectives [SP, 03NU], so we can define *sheaf cohomology* as the right-derived functor of the sections functor. Concretely, for an object U in our site $\mathbf{C}$, a sheaf of abelian groups $\mathcal{F}$ on $\mathbf{C}$ and an integer $i \in \mathbb{N}$, we have

$$H^i(U, \mathcal{F}) := R^i\Gamma(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)),$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution and $\Gamma(U, -)\colon \mathbf{Sh}(\mathbf{C}) \to \mathbf{Ab}$ is the functor given on objects by $\mathcal{F} \mapsto \mathcal{F}(U)$.

Example 2.13 ([SP, 03N1])**.** Let X be a variety. The *étale topology* on $\mathbf{Sch}/X$ is induced by the site whose coverings are jointly surjective families of étale morphisms $\{U_i \to U\}_{i\in I}$. The site $\mathbf{Sch}/X$ is sometimes called the *big étale site* on X, as opposed to the *small étale site* on X, which is the analogous site whose underlying category is the full subcategory of $\mathbf{Sch}/X$ whose objects are étale morphisms $U \to X$. We denote the small étale site on X by $X_{\text{ét}}$.

In order to verify that a presheaf on $X_{\text{ét}}$ is an étale sheaf, it suffices to check that the presheaf satisfies the sheaf condition for Zariski-open coverings and for étale coverings consisting of a single morphism between affine schemes [Mil13, Proposition 6.6]. We will see later that similar simplifications hold for qfh-sheaves, cf. Section 5.

Given an étale sheaf of abelian groups $\mathcal{F}$ on $\mathbf{Sch}/X$, we can restrict it to $X_{\text{ét}}$ to obtain an étale sheaf of abelian groups $\mathcal{F}|_{X_{\text{ét}}}$ on $X_{\text{ét}}$. Moreover, the étale cohomology groups of $\mathcal{F}$ on $\mathbf{Sch}/X$ agree with the étale cohomology groups of $\mathcal{F}|_{X_{\text{ét}}}$ on $X_{\text{ét}}$ [SP, 03YX]. More precisely, for every étale morphism $U \to X$ and every $i \in \mathbb{N}$ we have an isomorphism

$$H^i(U, \mathcal{F}|_{X_{\text{ét}}}) \cong H^i(U, \mathcal{F}),$$

and these isomorphism are functorial in $\mathcal{F}$ [SP, 03YU].

Definition 2.14 (qfh-topology, cf. [Voe96, Definition 3.1.2])**.** An *h-covering* on the category of schemes is a collection of the form $\{p_i\colon U_i \to X\}_{i\in I}$, where $\{p_i\}_{i\in I}$ is a finite family of morphisms of finite type such that the morphism $\sqcup_{i\in I} p_i\colon \sqcup_{i\in I} U_i \to X$ is a *universal topological epimorphism*. Being a topological epimorphism means that the corresponding continuous map is surjective and the target carries the quotient topology with respect to this continuous map, and a topological epimorphism $p\colon X \to Y$ is called a universal topological epimorphism if for any morphism $f\colon Z \to Y$ the projection $Z \times_Y X \to Y$ is a topological epimorphism.

A *qfh-covering* on the category of schemes is an h-covering $\{p_i\colon U_i \to X\}_{i\in I}$ such that each p_i is a quasi-finite morphism. If X is a finite type scheme over $\mathbb{C}$, then we define the *qfh-topology* on $\mathbf{Sch}/X$ as the topology induced by the qfh-coverings, and similarly for the h-topology.

2. Construction of the presheaf

For the remaining of this chapter, we fix an algebraic C-pair (X, Δ) and a natural number $p \in \mathbb{N}$.

Adapted differentials form a Zariski sheaf on the domain of each adapted morphism. Taking their global sections gives us a presheaf $\mathcal{F}$ on the full subcategory $\mathbf{Adpt}(X, \Delta) \subseteq \mathbf{Sch}/X$ of adapted morphisms:

Lemma 2.15. *If $\gamma_1 \colon Y_1 \to X$ and $\gamma_2 \colon Y_2 \to X$ are two adapted morphisms and $f \colon Y_2 \to Y_1$ is a morphism over X, then the pull-back of rational differentials induces a $\mathbb{C}$-linear morphism $f^* \colon \Omega^{[p]}_{(X,\Delta,\gamma_1)}(Y_1) \to \Omega^{[p]}_{(X,\Delta,\gamma_2)}(Y_2)$. Moreover:*

- *If $\gamma \colon Y \to X$ is an adapted morphism, then $\mathrm{id}_Y^* = \mathrm{id}_{\Omega^{[p]}_{(X,\Delta,\gamma)}(Y)}$.*
- *If $\gamma_1 \colon Y_1 \to X$, $\gamma_2 \colon Y_2 \to X$ and $\gamma_3 \colon Y_3 \to X$ are three adapted morphisms, and $f \colon Y_2 \to Y_1$ and $g \colon Y_3 \to Y_2$ are two morphisms over X, then $(f \circ g)^* = g^* \circ f^*$.*

Proof. The existence of f^* follows from [Har80, Proposition 1.6] and Lemma 1.74. Since the sheaves of adapted differentials are torsion-free, it suffices to check the desired identities over a dense open subset. But over a dense open subset these morphisms agree with the pull-back of Kähler differentials, hence the claim. $\square$

This presheaf can be extended to a presheaf $\mathcal{G}$ on $\mathbf{Sch}/X$ via the right adjoint to the restriction functor $\mathbf{PSh}(\mathbf{Sch}/X) \to \mathbf{PSh}(\mathbf{Adpt}(X, \Delta))$:

Construction 2.16. Let ${}_pi \colon \mathbf{PSh}(\mathbf{Adpt}(X, \Delta)) \to \mathbf{PSh}(\mathbf{Sch}/X)$ be the right adjoint to the restriction functor, as described in [SP, 00XF]. We consider the presheaf $\mathcal{G} := {}_pi(\mathcal{F})$. Explicitly, we have

$$\mathcal{G}(T) = \lim_{\substack{Y \to T \\ Y \to X \text{ adapted}}} \mathcal{F}(Y). \tag{2.2}$$

So a section $\sigma = (\sigma_t)_t \in \mathcal{G}(T)$ consists of a family of adapted differentials indexed by morphisms $t \colon Y \to T$ such that the composition $Y \to X$ is an adapted morphism. This family is moreover compatible in the sense that

$$f^*(\sigma_t) = \sigma_{t \circ f} \tag{2.3}$$

for every morphism $f \colon Y' \to Y$ such that the composition $Y' \to X$ is an adapted morphism. If $\pi \colon T' \to T$ is a morphism and we are given a section $\sigma \in \mathcal{G}(T)$, then $\mathcal{G}(\pi)(\sigma)$ is the family uniquely determined by the property that

$$\mathcal{G}(\pi)(\sigma)_{t'} = \sigma_{\pi \circ t'} \tag{2.4}$$

for every morphism $t' \colon Y \to T'$ such that the composition $Y \to X$ is an adapted morphism.

Notation 2.17. We denote by $\mathcal{F}$ the presheaf on $\mathbf{Adpt}(X, \Delta)$ with $\mathcal{F}(Y) := \Omega^{[p]}_{(X,\Delta,\gamma)}(Y)$ for all adapted morphisms $\gamma\colon Y \to X$ and with restriction maps given as in Lemma 2.15.

Lemma 2.18. *Let $\gamma_1\colon Y_1 \to X$ and $\gamma_2\colon Y_2 \to X$ be two adapted morphisms and let $f\colon Y_2 \to Y_1$ be a morphism over X. Then the restriction map $f^*\colon \mathcal{F}(Y_1) \to \mathcal{F}(Y_2)$ is injective.*

Proof. Again by torsion freeness it suffices to check this over a dense open subset, over which it follows from the corresponding statement for pull-back of Kähler differentials along the dominant morphism f. □

Lemma 2.19. *Let $\gamma\colon Y \to X$ be an adapted morphism and let $V \subseteq Y$ be an open subset. Then the restriction map $\mathcal{F}(Y) \to \mathcal{F}(V)$ is the usual restriction map of the sheaf $\Omega^{[p]}_{(X,\Delta,\gamma)}$.*

Proof. Follows from torsion freeness and the corresponding statement for Kähler differentials as before. □

Lemma 2.20. *Let $\gamma_1\colon Y_1 \to X$ and $\gamma_2\colon Y_2 \to X$ be two adapted morphisms and let $f\colon Y_2 \to Y_1$ be a morphism over X which is a Galois cover with Galois group G. Then the restriction map f^* induces a bijection $\mathcal{F}(Y_1) \to \mathcal{F}(Y_2)^G$.*

Proof. Follows from Lemma 1.77 by taking global sections. □

Remark 2.21. If we denote by $i^P\colon \mathbf{PSh}(\mathbf{Sch}/X) \to \mathbf{PSh}(\mathbf{Adpt}(X, \Delta))$ the restriction functor, then the universal property in Theorem 2.1 can be rephrased as follows:

(1) There exists a morphism $\varepsilon\colon i^P(\Omega^p_{(X,\Delta)}) \to \mathcal{F}$.
(2) For every presheaf $\mathcal{H}$ on $\mathbf{Sch}/X$ such that there exists a morphism $\varphi\colon i^P(\mathcal{H}) \to \mathcal{F}$, there exists a unique morphism $\psi\colon \mathcal{H} \to \Omega^p_{(X,\Delta)}$ such that $\varphi = \varepsilon \circ i^P(\psi)$.

In other words, $\Omega^p_{(X,\Delta)}$ is the right Kan extension of $\mathcal{F}$ along the inclusion $i\colon \mathbf{Adpt}(X, \Delta)^{\mathrm{op}} \to (\mathbf{Sch}/X)^{\mathrm{op}}$.

Lemma 2.22. *The presheaf $\mathcal{G}$ satisfies the universal property in Theorem 2.1. The uniqueness in Theorem 2.1 follows from the universal property as usual.*

Proof. The component at $\mathcal{F}$ of the counit of the adjunction $i^P \dashv {}_P i$ gives us a morphism $\varepsilon\colon i^P(\mathcal{G}) \to \mathcal{F}$. Let now $\mathcal{H}$ be a presheaf on $\mathbf{Sch}/X$ together with a morphism $\varphi\colon i^P(\mathcal{H}) \to \mathcal{F}$. It follows from the adjunction that there exists a unique morphism $\psi\colon \mathcal{H} \to \mathcal{G}$ such that $\varphi = \varepsilon \circ i^P(\psi)$. This shows that the universal property in Theorem 2.1, rephrased as in Remark 2.21, holds. □

Lemma 2.23. *Let $\gamma\colon Y \to X$ be an adapted morphism. Then the canonical morphism $\mathcal{G}(Y) \to \mathcal{F}(Y)$ is an isomorphism.*

Proof. Since the inclusion $i\colon \mathbf{Adpt}(X,\Delta) \longrightarrow \mathbf{Sch}/X$ is fully faithful, the indexing category in Equation (2.2) has an initial object, namely the identity on Y. Hence $\mathcal{G}(Y) \cong \mathcal{F}(Y)$. Explicitly, the isomorphism is given by sending a family $(\sigma_t)_t$ to the component σ_{id_Y} at id_Y, and the inverse is given by pulling back along the corresponding morphisms. □

After Lemma 2.22 and Lemma 2.23, it remains to show that $\mathcal{G}$ is a qfh-sheaf on $\mathbf{Sch}/X$. We will use the ideas in [SV96, §6] and reduce the question to checking the following two conditions:

(1) *(Zariski condition)* The presheaf $\mathcal{G}$ restricts to a Zariski sheaf on any integral normal scheme over X.
(2) *(Galois condition)* For any integral normal scheme T over X and every finite Galois extension $L/\mathbb{C}(T)$, the restriction along the normalization $T' \longrightarrow T$ in L gives a bijection $\mathcal{G}(T) \cong \mathcal{G}(T')^{\mathrm{Gal}(L/\mathbb{C}(T))}$, where the action of the Galois group on $\mathcal{G}(T')$ is given by pulling sections back along the corresponding automorphisms of T' over X.

3. Zariski condition

In this section we fix an X-scheme $T \longrightarrow X$, and we use $\mathcal{G}_T$ to denote the restriction of $\mathcal{G}$ to the Zariski topology on T, i.e., the induced presheaf on T_{Zar}, cf. Example 2.6. Our goal is to show that $\mathcal{G}_T$ is a Zariski sheaf.

Lemma 2.24. *The presheaf $\mathcal{G}_T$ is separated with respect to the Zariski topology.*

Proof. Let U be an open subset of T and let $\{U_j\}_{j\in J}$ be an open cover of U. Let $\iota_j\colon U_j \longrightarrow U$ denote the open immersions, and let $\sigma, \sigma' \in \mathcal{G}_T(U)$ be two sections such that $\mathcal{G}_T(\iota_j)(\sigma) = \mathcal{G}_T(\iota_j)(\sigma')$ for all $j \in J$. We need to show that $\sigma = \sigma'$, so let $u\colon Y \longrightarrow U$ be a morphism such that the composition $\gamma\colon Y \longrightarrow X$ is an adapted morphism. Let J' be the subset of all $j \in J$ such that $u^{-1}(U_j) \neq \varnothing$, so that the composition $u^{-1}(U_j) \longrightarrow X$ is an adapted morphism for all $j \in J'$. So for each $j \in J'$ we have a commutative diagram as follows:

$$\begin{tikzcd}
u^{-1}(U_j) \arrow[r, hook, "\iota'_j"] \arrow[d, "u_j"'] & Y \arrow[d, "u"] \arrow[dr, bend left, "\gamma"] & \\
U_j \arrow[r, hook, "\iota_j"] & U \arrow[r] & X.
\end{tikzcd}$$

The collection $\{u^{-1}(U_j)\}_{j\in J'}$ forms an open cover of Y. Using Lemma 2.19, Equation (2.3), Equation (2.4) and the assumption on σ and σ' we have

$$\begin{aligned}
\sigma_u|_{u^{-1}(U_j)} &= (\iota'_j)^*(\sigma_u)\\
&= \sigma_{u\circ\iota'_j}\\
&= \sigma_{\iota_j\circ u_j}\\
&= \mathcal{G}_T(\iota_j)(\sigma)_{u_j}\\
&= \mathcal{G}_T(\iota_j)(\sigma')_{u_j}\\
&= \sigma'_u|_{u^{-1}(U_j)}
\end{aligned}$$

for all $j\in J'$, so we can use separatedness of $\Omega^{[p]}_{(X,\Delta,\gamma)}$ as a sheaf on Y to conclude that $\sigma_u = \sigma'_u$. Hence $\sigma = \sigma'$. □

Lemma 2.25. *The presheaf $\mathcal{G}_T$ is a sheaf with respect to the Zariski topology.*

Proof. After Lemma 2.24, it remains only to show the gluing condition. So let U be an open subset of T and let $\{U_j\}_{j\in J}$ be an open cover of U. Let $\iota_j\colon U_j \to U$ denote the open immersions. For each $(j,k)\in J^2$ we set $U_{jk} := U_j\cap U_k$ and denote by $p_1^{(j,k)}\colon U_{jk}\to U_j$ and $p_2^{(j,k)}\colon U_{jk}\to U_k$ the inclusions in the corresponding cartesian square:

$$\begin{tikzcd}
U_{jk} \arrow[r, hook, "p_2^{(j,k)}"] \arrow[d, hook, "p_1^{(j,k)}"'] & U_k \arrow[d, hook, "\iota_k"] \\
U_j \arrow[r, hook, "\iota_j"] & U.
\end{tikzcd}$$

Let $(\sigma^j)_{j\in J}$ be a family of sections with $\sigma^j\in\mathcal{G}_T(U_j)$ for all $j\in J$ and with $\mathcal{G}_T(p_1^{(j,k)})(\sigma^j) = \mathcal{G}_T(p_2^{(j,k)})(\sigma^k)$ for all $(j,k)\in J^2$. We want to construct an appropriate section $\sigma\in\mathcal{G}_T(U)$, so let $u\colon Y\to U$ be a morphism such that the composition $\gamma\colon Y\to X$ is an adapted morphism. Consider again the subset $J'\subseteq J$ of all $j\in J$ such that $u^{-1}(U_j)\neq\varnothing$, so that the composition $u^{-1}(U_j)\to X$ is an adapted morphism for all $j\in J'$. Let us also denote by $u_j\colon u^{-1}(U_j)\to U_j$ the induced morphism for each $j\in J'$, so that for each $j\in J'$ we have again the following commutative diagram:

$$\begin{tikzcd}
u^{-1}(U_j) \arrow[r, hook, "\iota'_j"] \arrow[d, "u_j"'] & Y \arrow[d, "u"] \arrow[dr, bend left, "\gamma"] & \\
U_j \arrow[r, hook, "\iota_j"] & U \arrow[r] & X.
\end{tikzcd}$$

Then we consider the family of adapted differentials $(\sigma^j_{u_j})_{j\in J'}$, i.e., for each $j\in J'$ we take the component of the section $\sigma^j\in\mathcal{G}_T(U_j)$ corresponding to $u_j\colon u^{-1}(U_j)\to U_j$. For each $(j,k)\in J'^2$, with the notation given by the commutative diagram

$$
\begin{tikzcd}
u^{-1}(U_{jk}) \arrow[rr, "q_2^{(j,k)}"] \arrow[dr, "q_1^{(j,k)}"'] \arrow[dd, "u_{jk}"'] & & u^{-1}(U_k) \arrow[dr, "\iota'_k"] \arrow[dd, "u_k"] & \\
& u^{-1}(U_j) \arrow[rr, "\iota'_j"] \arrow[dd, "u_j"] & & Y \arrow[dd, "u"] \\
U_{jk} \arrow[rr, "p_2^{(j,k)}"] \arrow[dr, "p_1^{(j,k)}"'] & & U_k \arrow[dr, "\iota_k"] & \\
& U_j \arrow[rr, "\iota_j"] & & U
\end{tikzcd}
$$

and with the same arguments as in Lemma 2.24 we deduce that

$$
\begin{aligned}
\sigma^j_{u_j}|_{u^{-1}(U_{jk})} &= (q_1^{(j,k)})^*(\sigma^j_{u_j}) \\
&= \sigma^j_{u_j \circ q_1^{(j,k)}} \\
&= \sigma^j_{p_1^{(j,k)} \circ u_{jk}} \\
&= \mathcal{G}_T(p_1^{(j,k)})(\sigma^j)_{u_{jk}} \\
&= \mathcal{G}_T(p_2^{(j,k)})(\sigma^k)_{u_{jk}} \\
&= \sigma^k_{p_2^{(j,k)} \circ u_{jk}} \\
&= \sigma^k_{u_k \circ q_2^{(j,k)}} \\
&= \sigma^k_{u_k}|_{u^{-1}(U_{jk})},
\end{aligned}
$$

so we can glue this family of adapted differentials to obtain a global section σ_u of the sheaf $\Omega^{[p]}_{(X,\Delta,\gamma)}$ on Y.

We repeat this process with every morphism $u' \colon Y' \longrightarrow U$ such that the composition $Y' \to X$ is an adapted morphism to obtain a family $(\sigma_u)_u$. We check next that Equation (2.3) is satisfied, i.e., that $(\sigma_u)_u \in \mathcal{G}_T(U)$. Let $f \colon Y' \longrightarrow Y$ be a morphism over X and let $u' = u \circ f$. In order to show that $f^*(\sigma_u) = \sigma_{u'}$, it suffices to show that $f^*(\sigma_u)|_{u'^{-1}(U_j)} = \sigma_{u'}|_{u'^{-1}(U_j)}$ for all $j \in J'$. With the notation given by the commutative diagram

$$\begin{tikzcd}
u'^{-1}(U_j) \arrow[rr, "\iota''_j"] \arrow[dr, "f_j"] \arrow[dd, "u'_j"'] & & Y' \arrow[dr, "f"] \arrow[dd, "u'"] & \\
& u^{-1}(U_j) \arrow[rr, "\iota'_j"] \arrow[dd, "u_j"] & & Y \arrow[dd, "u"] \\
U_j \arrow[rr, "\iota_j"] \arrow[dr, equal] & & U \arrow[dr, equal] & \\
& U_j \arrow[rr, "\iota_j"] & & U
\end{tikzcd}$$

and using Lemma 2.19, Equation (2.3) and the defining property of σ_u and $\sigma_{u'}$, we obtain

$$\begin{aligned}
f^*(\sigma_u)|_{u'^{-1}(U_j)} &= (\iota''_j)^*(f^*(\sigma_u)) \\
&= f_j^*((\iota'_j)^*(\sigma_u)) \\
&= f_j^*(\sigma^j_{u_j}) \\
&= \sigma^j_{u_j \circ f_j} \\
&= \sigma^j_{u'_j} \\
&= \sigma_{u'}|_{u'^{-1}(U_j)}
\end{aligned}$$

for all $j \in J'$. Therefore, $\sigma := (\sigma_u)_u \in \mathcal{G}_T(U)$. It remains to show that $\mathcal{G}_T(\iota_j)(\sigma) = \sigma^j$ for all $j \in J$. But if we apply the recipe above to the morphism $\iota_j \circ u_j$ for a morphism $u_j \colon Y \to U_j$ such that the composition $Y \to X$ is an adapted morphism we obtain $\sigma_{\iota_j \circ u_j} = \sigma^j_{u_j}$ by construction, so

$$\mathcal{G}_T(\iota_j)(\sigma)_{u_j} = \sigma_{\iota_j \circ u_j} = \sigma^j_{u_j}$$

for all such u_j and thus $\mathcal{G}_T(\iota_j)(\sigma) = \sigma^j$ for all $j \in J$. □

4. Galois condition

In this section we fix a normal integral X-scheme $T \to X$ and a finite Galois extension $L/\mathbb{C}(T)$ of its function field $\mathbb{C}(T)$. We denote by G the Galois group of this extension and by $\pi \colon T' \to T$ the normalization of T in L [GW10, Definition 12.42].

Remark 2.26. Let Y and Y' be normal integral schemes and let $f \colon Y' \to Y$ be a morphism (of finite type). Then f is a geometric quotient of Y' by the action of a finite group H if and only if f induces a finite Galois extension of function fields with Galois group H and Y' is the normalization of Y in this extension of its function field. See Lemma B.22.

So in our current setting $\pi \colon T' \to T$ is the geometric quotient of T' by the induced G-action on T'. For each $g \in G$ we denote by ϕ_g the corresponding automorphism of T'. Then we have a G-action on $\mathcal{G}(T')$ given by pulling back along the corresponding automorphism. The restriction map $\mathcal{G}(\pi)\colon \mathcal{G}(T) \to \mathcal{G}(T')$ has image contained in $\mathcal{G}(T')^G$, because $\pi \circ \phi_g = \pi$ for all $g \in G$. Our goal in this section is to show that this restriction map is a bijection onto $\mathcal{G}(T')^G$.

Lemma 2.27. *The morphism $\mathcal{G}(\pi)\colon \mathcal{G}(T) \to \mathcal{G}(T')^G$ is injective.*

Proof. Let $\sigma, \sigma' \in \mathcal{G}(T)$ be two sections such that $\mathcal{G}(\pi)(\sigma) = \mathcal{G}(\pi)(\sigma')$. We want to show that $\sigma = \sigma'$, so let $t\colon Y \to T$ be a morphism such that the composition $Y \to X$ is an adapted morphism and let us show that $\sigma_t = \sigma'_t$. We consider the cartesian square

$$\begin{tikzcd} T' \times_T Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ T' \arrow[r, "\pi"] & T. \end{tikzcd}$$

Since π is finite and surjective, so is p_2, because both properties are stable under base change [GW10, Appendix C]. Therefore, some irreducible component of $T' \times_T Y$ surjects onto Y. We regard this component as a subscheme of the fiber product with its induced reduced structure and take its normalization, which we denote by Y'. We denote by $f\colon Y' \to Y$ the induced surjective morphism onto Y. Since closed immersions and normalizations (of finite type schemes over a field) are finite, the morphism f is again finite. Let $t'\colon Y' \to T'$ be the induced morphism, so that the composition $Y' \to X$ is an adapted morphism as well and the following diagram commutes:

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d, "t'"'] & Y \arrow[d, "t"] \\ T' \arrow[r, "\pi"] & T. \end{tikzcd}$$

It follows now from Equation (2.3), commutativity of the diagram and Equation (2.4) that

$$\begin{aligned} f^*(\sigma_t) &= \sigma_{t\circ f} \\ &= \sigma_{\pi\circ t'} \\ &= \mathcal{G}(\pi)(\sigma)_{t'} \\ &= \mathcal{G}(\pi)(\sigma')_{t'} \\ &= f^*(\sigma'_t), \end{aligned}$$

hence $\sigma_t = \sigma'_t$ by Lemma 2.18. □

Lemma 2.28. *The morphism $\mathcal{G}(\pi)\colon \mathcal{G}(T) \to \mathcal{G}(T')^G$ is surjective.*

Proof. Let $\sigma' \in \mathcal{G}(T')$ be a G-invariant section, i.e., $\mathcal{G}(\phi_g)(\sigma') = \sigma'$ for all $g \in G$, where ϕ_g denotes the automorphism of T' corresponding to g. We want to find a section $\sigma \in \mathcal{G}(T)$ such that $\mathcal{G}(\pi)(\sigma) = \sigma'$, so let $t\colon Y \to T$ be a morphism such that the composition $Y \to X$ is an adapted morphism. Let Y_0' be any irreducible component of the fiber product $T' \times_T Y$, considered as a subscheme with its induced reduced structure. Let G_0 be the stabilizer of Y_0' in G, so that the induced morphism $Y_0' \to Y$ is a geometric quotient of Y_0' by G_0, cf. [Hub00, Lemma 2.1.11] and [SV96, Corollary 5.10]. Let $\bar{Y}_0'$ be the normalization of Y_0'. Then G_0 acts on $\bar{Y}_0'$ as well and the induced morphism $f_0^Y : \bar{Y}_0' \to Y$ is still a geometric quotient for this action, cf. Remark 2.26. Let $t_0 \colon \bar{Y}_0' \to T'$ be the induced morphism, so that $t \circ f_0^Y = \pi \circ t_0$ and the composition $\bar{Y}_0' \to X$ is an adapted morphism:

$$
\begin{tikzcd}
\bar{Y}_0' \arrow[r, "f_0^Y"] \arrow[d, "\text{norm.}"] \arrow[ddd, bend right=60, "t_0"'] & Y \arrow[dd, equal] \\
Y_0' \arrow[d, hook] & \\
T' \times_T Y \arrow[r] \arrow[d] & Y \arrow[d, "t"] \\
T' \arrow[r, "\pi"] & T.
\end{tikzcd}
$$

The desired $\sigma \in \mathcal{G}(T)$ must then have the property that $(f_0^Y)^*(\sigma_t) = \sigma_{\pi \circ t_0} = \sigma'_{t_0}$. Let now $g \in G_0$ and let ψ_g denote the corresponding automorphism of $\bar{Y}_0'$. From Equation (2.3), the equality $t_0 \circ \psi_g = \phi_g \circ t_0$, Equation (2.4) and G-invariance of σ' we deduce that

$$
\begin{aligned}
\psi_g^*(\sigma'_{t_0}) &= \sigma'_{t_0 \circ \psi_g} \\
&= \sigma'_{\phi_g \circ t_0} \\
&= \mathcal{G}(\phi_g)(\sigma')_{t_0} \\
&= \sigma'_{t_0},
\end{aligned}
$$

hence σ'_{t_0} is a G_0-invariant adapted differential on $\bar{Y}_0'$. By Lemma 2.20 we can find an adapted differential σ_t on Y such that $(f_0^Y)^*(\sigma_t) = \sigma'_{t_0}$. Moreover, the resulting adapted differential is independent of the chosen irreducible component Y_0'. Indeed, let Y_1' be another irreducible component and let τ_t be an adapted differential on Y such that $(f_1^Y)^*(\tau_t) = \sigma'_{t_1}$, where f_1^Y and t_1 are defined as before but for the irreducible component Y_1' instead. Since G acts transitively on the irreducible components of the fiber product [SV96, Corollary 5.10], we can find some $g \in G$ such that the corresponding automorphism of the fiber product induces an isomorphism $Y_0' \to Y_1'$ over Y, hence also an isomorphism $\psi_g \colon \bar{Y}_0' \to \bar{Y}_1'$ over Y. From the defining property of σ_t, G-invariance of σ', the equality $\phi_g \circ t_0 = t_1 \circ \psi_g$, the equality $f_0^Y = f_1^Y \circ \psi_g$, the defining property of τ_t and the usual equations we have

$$\begin{aligned}(f_0^Y)^*(\sigma_t) &= \sigma'_{t_0} \\ &= \sigma'_{\phi_g \circ t_0} \\ &= \sigma'_{t_1 \circ \psi_g} \\ &= \psi_g^*(\sigma'_{t_1}) \\ &= \psi_g^*((f_1^Y)^*(\tau_t)) \\ &= (f_0^Y)^*(\tau_t),\end{aligned}$$

hence $\sigma_t = \tau_t$ by Lemma 2.18.

We show next that the family of adapted differentials σ constructed in this manner satisfies Equation (2.3), i.e., that $\sigma \in \mathcal{G}(T)$. Let $t\colon Y \to T$ be a morphism such that the composition $Y \to X$ is an adapted morphism and let $f\colon Z \to Y$ be a morphism such that the composition $Z \to X$ is an adapted morphism as well. By Remark 1.39, the morphism $f\colon Z \to Y$ is a quasi-finite morphism between normal varieties of the same dimension. Since $\pi\colon T' \to T$ is finite and surjective and these properties are stable under base change [GW10, Appendix C], the projections $T' \times_T Y \to Y$ and $T' \times_T Z \to Z$ are finite and surjective as well. By going up, we have

$$\dim(T' \times_T Y) = \dim(Y) = \dim(Z) = \dim(T' \times_T Z),$$

cf. [SP, 01WJ] and [SP, 0ECG]. Being quasi-finite is also stable under base change [GW10, Appendix C], so the induced map $T' \times_T Z \to T' \times_T Y$ is a quasi-finite morphism between finite type $\mathbb{C}$-schemes of equal dimension. Therefore, we may find an irreducible component of $T' \times_T Z$ which dominates an irreducible component of $T' \times_T Y$, say $Z'_0 \to Y'_0$. From the universal property of the normalization we obtain now a morphism $f_0\colon \bar{Z}'_0 \to \bar{Y}'_0$ such that $f \circ f_0^Z = f_0^Y \circ f_0$, where $f_0^Y\colon \bar{Y}'_0 \to Y$ and $f_0^Z\colon \bar{Z}'_0 \to Z$ are defined as in the last paragraph:

$$\begin{tikzcd} \bar{Z}'_0 \arrow[r, "f_0^Z"] \arrow[d, "f_0"'] & Z \arrow[d, "f"] \\ \bar{Y}'_0 \arrow[r, "f_0^Y"] \arrow[d, "t_0"'] & Y \arrow[d, "t"] \\ T' \arrow[r, "\pi"] & T. \end{tikzcd}$$

From Equation (2.3) for σ', the equality $f \circ f_0^Z = f_0^Y \circ f_0$ and the construction of σ we deduce that

$$\begin{aligned}
(f_0^Z)^*(f^*(\sigma_t)) &= f_0^*((f_0^Y)^*(\sigma_t)) \\
&= f_0^*(\sigma'_{t_0}) \\
&= \sigma'_{t_0 \circ f_0} \\
&= (f_0^Z)^*(\sigma_{t \circ f}),
\end{aligned}$$

hence $f^*(\sigma_t) = \sigma_{t \circ f}$ by Lemma 2.18.

To finish the proof we need to check that $\mathcal{G}(\pi)(\sigma) = \sigma'$, so let $s \colon Y \to T'$ be a morphism such that the composition $Y \to X$ is an adapted morphism. Let $f_0^Y \colon \bar{Y}'_0 \to Y$ and $t_0 \colon \bar{Y}'_0 \to T'$ be constructed as above with respect to the morphism $t := \pi \circ s$, fitting into the following commutative square:

$$\begin{tikzcd}
\bar{Y}'_0 \arrow[r, "f_0^Y"] \arrow[d, "t_0"'] & Y \arrow[d, "\pi \circ s"] \\
T' \arrow[r, "\pi"] & T.
\end{tikzcd}$$

For every $g \in G$, denote by E_g the equalizer of $\phi_g \circ t_0$ and $s \circ f_0^Y$, which is a closed subscheme of $\bar{Y}'_0$ because π is separated [GW10, Definition and Proposition 9.7.(ii)]. The diagram

$$\begin{tikzcd}
\bar{Y}'_0 \arrow[r, "f_0^Y"] \arrow[d, "t_0"'] & Y \arrow[dd, "\pi \circ s"] \arrow[ddl, "s"] \\
T' \arrow[d, "\phi_g"'] & \\
T' \arrow[r, "\pi"] & T
\end{tikzcd}$$

does not necessarily commute for a given $g \in G$, but we claim that $\bar{Y}'_0 = \cup_{g \in G} E_g$ as sets. Indeed, it suffices to show that every closed point of $\bar{Y}'_0$ belongs to E_g for some $g \in G$, because all E_g are closed subspaces and the set of closed points is dense in $\bar{Y}'_0$ [Har77, Exercise II.3.14]. But we are working over an algebraically closed field, so this follows from [GW10, Exercise 9.7] and from the equality $\pi \circ t_0 = \pi \circ s \circ f_0^Y$, in which π is a geometric quotient of T' by G. Since $\bar{Y}'_0$ is irreducible and G is finite, there exists some $g \in G$ such that $\bar{Y}'_0 = E_g$ as topological spaces. But $\bar{Y}'_0$ is reduced, so we must have $\bar{Y}'_0 = E_g$ and thus $\phi_g \circ t_0 = s \circ f_0^Y$. Hence, for this particular $g \in G$, the previous diagram does commute. Let now $\bar{Y}'_1$ be a normalized component of the fiber product such that g induces an isomorphism $\psi_g \colon \bar{Y}'_0 \to \bar{Y}'_1$ with $t_1 \circ \psi_g = \phi_g \circ t_0$, notation again as above. Then, the diagram

$$\begin{tikzcd}
\bar{Y}_0' \arrow[r, "\psi_g"] \arrow[d, "t_0"'] \arrow[rr, bend left, "f_0^Y"] & \bar{Y}_1' \arrow[r, "f_1^Y"] \arrow[d, "t_1"'] & Y \arrow[d, "t"] \arrow[dl, dashed, "s"'] \\
T' \arrow[r, "\phi_g"] & T' \arrow[r, "\pi"] & T
\end{tikzcd}$$

remains commutative after adding the dashed arrow, because ψ_g is an isomorphism, so it suffices to check the commutativity from $\bar{Y}_0'$. By construction of σ we have $(f_1^Y)^*(\sigma_t) = \sigma'_{t_1}$, and combining this with the usual equations and commutativity of the previous diagram we deduce that

$$\begin{aligned}
(f_1^Y)^*(\sigma_t) &= \sigma'_{t_1} \\
&= \sigma'_{s \circ f_1^Y} \\
&= (f_1^Y)^*(\sigma'_s).
\end{aligned}$$

It follows from Lemma 2.18 that $\sigma_t = \sigma'_s$. Since $t = \pi \circ s$, we conclude that $\mathcal{G}(\pi)(\sigma)_s = \sigma'_s$. Hence $\mathcal{G}(\pi)(\sigma) = \sigma'$. □

5. qfh-sheaf condition

If we denote by $i_1 \colon \mathbf{Adpt}(X, \Delta) \to \mathbf{Nor}/X$ and $i_2 \colon \mathbf{Nor}/X \to \mathbf{Sch}/X$ the inclusion functors, we have $\mathcal{G} = {}_p i_2({}_p i_1(\mathcal{F}))$. The arguments in [SV96, Theorem 6.2] do not use the assumption that we are working over a field, but only that we are working with finite type schemes over a field, cf. [SV96, Lemma 10.3]. Therefore, the Zariski and Galois conditions shown in the previous sections imply that $\mathcal{G}$ is a qfh-sheaf on $\mathbf{Sch}/X$. This, combined with Lemma 2.22, finishes the proof of Theorem 2.1.

6. Some properties

We may think of a morphism of varieties $f \colon X \to Y$ as its graph $\Gamma_f \subseteq X \times Y$, which is a closed integral subscheme of the product mapping isomorphically onto the first factor. We can generalize this and consider closed integral subschemes $Z \subseteq X \times Y$ mapping finitely onto the first factor. Taking $\mathbb{Q}$-linear combinations of such subschemes leads to the notion of a *finite*[3] *correspondence*. Finite correspondences can be composed in a meaningful way, so we can consider the category $\mathbf{Corr}/\mathbb{C}$ whose objects are $\mathbb{C}$-schemes of finite type and whose morphisms are given by finite correspondences. Sending a morphism to its graph gives a faithful functor

$$\mathbf{Sch}/\mathbb{C} \to \mathbf{Corr}/\mathbb{C}.$$

[3] There is a more general notion of *correspondence* [SP, 0FFZ], which is used to construct Chow motives [SP, 0FG9]. But in order to compose such correspondences one needs to restrict to categories of smooth varieties. On the other hand, finite correspondences can be composed in more general settings.

Roughly speaking, a presheaf of abelian groups $\mathcal{F}$ on $\mathbf{Sch}/\mathbb{C}$ is said to *admit transfer maps*, or is called a *presheaf with transfers*, if it factors through this functor. In other words, in addition to the usual functoriality that comes from being a presheaf, $\mathcal{F}$ should have a notion of pushforward along certain finite surjective morphisms, a.k.a. *transfer maps*, cf. [SV96, Definition 4.1]. The idea is that if $\mathcal{F}$ is a qfh-sheaf and $f\colon X \to S$ is a Galois cover of normal varieties with Galois group $G \subseteq \mathrm{Aut}_S(X)$, then we have $\mathcal{F}(S) = \mathcal{F}(X)^G$. Therefore, as explained in [SV96, §5], we can define a transfer map as follows:

$$\begin{aligned}\mathrm{Tr}_{X/S}\colon \mathcal{F}(X) &\to \mathcal{F}(S)\\ a &\mapsto \mathrm{Tr}_{X/S}(a) := \sum_{\phi\in G}\phi^*(a).\end{aligned}$$

Remark 2.29. Note that a familiar sheaf with this kind of functoriality is the canonical bundle on a smooth projective variety, cf. [Har77, Exercise III.7.2] or [For81, p. 164].

The story becomes more delicate when working over a more general base scheme X, but the general idea remains the same. One can use relative cycles [CD19, §8.1.a] to define finite X-correspondences [CD19, Definition 9.1.2], and one obtains a category $\mathbf{Corr}/X$ with a "graph" functor

$$\mathbf{Sch}/X \to \mathbf{Corr}/X$$

as before [CD19, Definition 9.1.8]. We can then define *presheaves with transfers* similarly as before [CD19, Definition 10.1.1], as additive presheaves of $\mathbb{Q}$-vector spaces on $\mathbf{Corr}/X$. We refer to [CD19] for a more precise discussion.

Any qfh-sheaf of $\mathbb{Q}$-vector spaces on $\mathbf{Sch}/X$ is naturally endowed with a unique structure of a Nisnevich[4] sheaf with transfers, and any morphism of qfh-sheaves is a morphism of sheaves with transfers [CD19, Proposition 10.5.9]. In particular, we have the following:

Corollary 2.30. *In the setting of Theorem 2.1, $\Omega^p_{(X,\Delta)}$ is a Nisnevich sheaf with transfers on* Sch/X.

Being a qfh-sheaf also has some nice implications in cohomology:

Corollary 2.31. *In the setting of Theorem 2.1, let T be a normal scheme over X and let $i \in \mathbb{N}$. Then the following canonical maps are isomorphisms*

$$H^i_{\mathrm{Nis}}(T, \Omega^p_{(X,\Delta)}) \xrightarrow{\cong} H^i_{\acute{e}t}(T, \Omega^p_{(X,\Delta)}) \xrightarrow{\cong} H^i_{\mathrm{qfh}}(T, \Omega^p_{(X,\Delta)}).$$

Proof. Follows from [CD19, Theorem 10.5.10] and [CD19, Proposition 10.5.12], because normal schemes are geometrically unibranch [SP, 0BQ3]. □

[4] We refer to [MVW06, §12] for the definition of this topology, which is not relevant for our purposes. Suffices to say that it is finer than the Zariski topology and coarser than the étale topology.

Presheaves with transfers were one of two important ingredients used by Voevodsky to construct his triangulated categories of motives in [Voe00]; the other ingredient was homotopy invariance. A presheaf $\mathcal{F}$ is said to be *homotopy invariant* if $\mathcal{F}(T) \cong \mathcal{F}(T \times \mathbb{A}^1)$ for any scheme T [MVW06, Definition 2.15], or in our relative setting, if $\mathcal{F}(T) \cong \mathcal{F}(T \times_X \mathbb{A}^1_X)$ for any X-scheme T. The sheaf $\Omega^p_{(X,\Delta)}$ is not homotopy invariant, see Remark 2.37. Sheaves of Kähler differentials are in fact a standard example of non homotopy invariant presheaves. However, sheaves of Kähler differentials are *reciprocity sheaves* [KSY16, Theorem A.6.2], which is a property weaker than homotopy invariance but still useful in this context, see [KSY16, § Introduction] and [KSY16, Definition 2.1.3]. Therefore, it is natural to ask:

Question 2.32. *In the setting of Theorem 2.1, is $\Omega^p_{(X,\Delta)}$ a reciprocity sheaf?*

7. Some computations

The value $\Omega^p_{(X,\Delta)}(T)$ can be described more explicitly for quasi-finite separated X-schemes $T \to X$. In Proposition 2.33 we treat the case of open immersions in some detail. As a consequence we can compute the qfh-cohomology groups of $\Omega^p_{(X,\Delta)}$ in terms of the usual cohomology groups of the sheaf of (reflexive) differentials on X, see Corollary 2.35. The cases $\dim(T) < \dim(X)$ and $\dim(T) = \dim(X)$ are discussed in Proposition 2.36 and Proposition 2.41 respectively.

Proposition 2.33. *In the setting of Theorem 2.1, let $\Omega^p_{(X,\Delta)}|_X$ be the restriction of $\Omega^p_{(X,\Delta)}$ to the Zariski topology on X. Then we have an isomorphism*

$$\Omega^{[p]}_X(\log \lfloor \Delta \rfloor) \cong \Omega^p_{(X,\Delta)}|_X.$$

Proof. We first define a presheaf morphism

$$\Omega^{[p]}_X(\log \lfloor \Delta \rfloor) \to \Omega^p_{(X,\Delta)}|_X.$$

We define the morphisms first on global sections. By definition we have

$$\Omega^p_{(X,\Delta)}(X) \cong \left\{ (\sigma_\gamma)_\gamma \in \prod_{\substack{\gamma\colon Y \to X\\ \text{adapted}}} \Omega^{[p]}_{(X,\Delta,\gamma)}(Y) \;\middle|\; \begin{array}{l} f^*(\sigma_\gamma) = \sigma_{\gamma\circ f} \text{ for all} \\ f\colon Y' \to Y \text{ over } X \end{array} \right\}.$$

On the other hand, $\Omega^{[p]}_X(\log \lfloor \Delta \rfloor)(X)$ are the reflexive logarithmic differentials on $(X, \lfloor \Delta \rfloor)$, which can be described as $i_*\Omega^p_U(\log \lfloor \Delta|_U \rfloor)$ for $U \subseteq X$ as in Definition 1.67. Indeed, given $\sigma \in \Omega^{[p]}_X(\log \lfloor \Delta \rfloor)(X) = \Omega^p_U(\log \lfloor \Delta|_U \rfloor)(U)$ and an adapted morphism $\gamma\colon Y \to X$, we can pull back σ along $\gamma|_{\gamma^{-1}(U)}\colon \gamma^{-1}(U) \to U$ and restrict the result to the open subset $\hat{U}$

in Definition 1.67 to obtain a section $\gamma^*\sigma \in \Omega^{[p]}_{(X,\Delta,\gamma)}(Y)$. By torsion freeness and functoriality of pull-back of logarithmic differentials, the resulting family $(\gamma^*\sigma)_\gamma$ is a section in $\Omega^p_{(X,\Delta)}(X)$. Since adapted morphisms are dominant, this defines an injective homomorphism $\Omega^{[p]}_X(\log\lfloor\Delta\rfloor)(X) \to \Omega^p_{(X,\Delta)}(X)$.

The same recipe works for any smaller open subset $W \subseteq X$. Suppose now that $\iota\colon W_1 \to W_2$ is an inclusion of open subsets of X. The restriction morphism of $\Omega^{[p]}_X(\log\lfloor\Delta\rfloor)$ is induced by the pull-back of rational differentials ι^* as in Lemma 2.15, so we want to show that the following diagram commutes:

$$\begin{tikzcd}
\Omega^{[p]}_X(\log\lfloor\Delta\rfloor)(W_2) \arrow[r] \arrow[d, "\iota^*"'] & \Omega^p_{(X,\Delta)}(W_2) \arrow[d, "\Omega^p_{(X,\Delta)}(\iota)"] \\
\Omega^{[p]}_X(\log\lfloor\Delta\rfloor)(W_1) \arrow[r] & \Omega^p_{(X,\Delta)}(W_1).
\end{tikzcd}$$

Let $\sigma \in \Omega^{[p]}_X(\log\lfloor\Delta\rfloor)(W_2)$ and let $w_1\colon Y \to W_1$ be a morphism such that the composition $\gamma_{w_1}\colon Y \to X$ is an adapted morphism. The upper horizontal isomorphism sends σ to the family $(w_2^*\sigma)_{w_2}$, indexed by morphisms $w_2\colon Z \to W_2$ such that the composition $Z \to X$ is an adapted morphism. Its image under $\Omega^p_{(X,\Delta)}(\iota)$ is the family whose component at w_1 is given by $(w_2^*\sigma)_{w_2=\iota\circ w_1}$, i.e., $w_1^*\iota^*\sigma$. Therefore, the diagram commutes.

So we have constructed an injective Zariski sheaf morphism

$$\Omega^{[p]}_X(\log\lfloor\Delta\rfloor) \to \Omega^p_{(X,\Delta)}|_X.$$

Let us show that this morphism is also surjective. The statement is local on X, so it suffices to show it for X and we may shrink X along the course of the argument. Suppose we are given a family $(\sigma_\gamma)_\gamma \in \Omega^p_{(X,\Delta)}(X)$. Let $U_0 \subseteq X$ be the largest open subset over which X and $\mathrm{Supp}(\Delta)$ are smooth, which is a big open subset contained in the open subset $U \subseteq X$ from Definition 1.67. By shrinking X we may assume that we are in the setting of [EV92, §3.5], so we can consider an adapted morphism $\gamma_0\colon Y_0 \to X$ factoring through $U_0 \subseteq X$ such that there exists a finite group G acting algebraically on Y_0 so that $Y_0 \to U_0$ is a geometric quotient of Y_0 by G. Equation (2.3) ensures that the logarithmic differential σ_{γ_0} is G-invariant, hence Hurwitz's formula [EV92, Lemma 3.16] implies that there exists a logarithmic differential $\sigma \in \Omega^p_{U_0}(\log\lfloor\Delta|_{U_0}\rfloor)(U_0)$ such that $\gamma_0^*\sigma = \sigma_{\gamma_0}$. This logarithmic differential defines a reflexive logarithmic differential in $\Omega^{[p]}_X(\log\lfloor\Delta\rfloor)(X)$ which is the desired preimage of the given section of $\Omega^p_{(X,\Delta)}$. □

Remark 2.34. In particular, the restriction of $\Omega^p_{(X,\Delta)}$ to the Zariski topology on X is isomorphic to $\Omega^{[p]}_X$ if $\Delta = 0$ and to $\Omega^{[p]}_X(\log \Delta)$ if $\Delta \neq 0$ and $\Delta - \lfloor \Delta \rfloor = 0$.

Corollary 2.35. *In the setting of Theorem 2.1, for any $i \in \mathbb{N}$, we have*

$$H^i_{\mathrm{qfh}}\left(X, \Omega^p_{(X,\Delta)}\right) \cong H^i\left(X, \Omega^{[p]}_X(\log \lfloor \Delta \rfloor)\right),$$

where the right-hand side denotes usual (Zariski) sheaf cohomology.

Proof. By Corollary 2.31, we have

$$H^i_{\text{ét}}\left(X, \Omega^p_{(X,\Delta)}\right) \cong H^i_{\mathrm{qfh}}\left(X, \Omega^p_{(X,\Delta)}\right).$$

As pointed out in Section 1, we can compute the étale cohomology group on the small étale site [SP, 03YX], hence

$$H^i_{\text{ét}}\left(X, \Omega^p_{(X,\Delta)}|_{X_{\text{ét}}}\right) \cong H^i_{\mathrm{qfh}}\left(X, \Omega^p_{(X,\Delta)}\right).$$

By Proposition 2.33, $\Omega^p_{(X,\Delta)}|_{X_{\mathrm{Zar}}}$ is a quasi-coherent sheaf on X which is also a sheaf for the étale topology, so the étale and Zariski cohomology groups agree [SP, 03DW] and thus

$$H^i\left(X, \Omega^{[p]}_X(\log \lfloor \Delta \rfloor)\right) \cong H^i\left(X, \Omega^p_{(X,\Delta)}|_X\right) \cong H^i_{\mathrm{qfh}}\left(X, \Omega^p_{(X,\Delta)}\right).$$

□

Proposition 2.36. *In the setting of Theorem 2.1, let $T \to X$ be an X-scheme such that* $\dim(T) < \dim(X)$*. Then we have*

$$\Omega^p_{(X,\Delta)}(T) = 0.$$

Proof. By construction we have

$$\Omega^p_{(X,\Delta)}(T) = \lim_{\substack{Y \to T \\ \gamma\colon Y \to X \text{ adapted}}} \Omega^{[p]}_{(X,\Delta,\gamma)}(Y),$$

see Equation (2.2). Since $\dim(T) < \dim(X)$, for any $t\colon Y \to T$ we have that the composition $Y \to X$ is not dominant. Therefore, the composition $Y \to X$ cannot be an adapted morphism, so $\Omega^p_{(X,\Delta)}(T) = 0$ is the limit over the empty diagram. □

Remark 2.37. Combining Proposition 2.33 and Proposition 2.36, we see that $\Omega^p_{(X,\Delta)}$ is not homotopy invariant. For example, let $X = \mathbb{A}^1$ and $\Delta = 0$ and consider $T = \mathrm{Spec}(\mathbb{C})$. Then we have $\Omega^1_{(X,\Delta)}(T) = 0$ by Proposition 2.36 and $\Omega^1_{(X,\Delta)}(\mathbb{A}^1) = \Omega^1_{\mathbb{A}^1}(\mathbb{A}^1) \neq 0$ by Proposition 2.33. Therefore

$$\Omega^1_{(X,\Delta)}(T) \neq \Omega^1_{(X,\Delta)}(T \times_{\mathbb{A}^1} \mathbb{A}^1_{\mathbb{A}^1}) = \Omega^1_{(X,\Delta)}(T \times \mathbb{A}^1) = \Omega^1_{(X,\Delta)}(\mathbb{A}^1).$$

Proposition 2.38. *Let $T \to X$ be an X-scheme with* $\dim(T) \leq \dim(X)$ *and let $\nu\colon \bar{T} \to T$ be its normalization. Then ν induces an isomorphism*

$$\Omega_{(X,\Delta)}(T) \cong \Omega_{(X,\Delta)}(\bar{T}).$$

Proof. With the notation from Section 2, we show that

$$\lim_{\substack{Y\to T\\ Y\to X \text{ adapted}}} \mathcal{F}(Y) = \lim_{\substack{Y\to \bar{T}\\ Y\to X \text{ adapted}}} \mathcal{F}(Y).$$

Indeed, it suffices to show that the two indexing categories are equal. Given an object $\bar{t}\colon Y \longrightarrow \bar{T}$ in the indexing category of the right-hand side, we obtain an object $t := \nu \circ \bar{t}$ in the indexing category of the left-hand side. And conversely, if $t\colon Y \longrightarrow T$ is an object in the indexing category of the left-hand side, then Y must dominate an irreducible component of T, because $\dim(T) \leq \dim(X)$ and $Y \longrightarrow X$ is dominant. Since Y is normal, the universal property of the normalization [SP, 035Q] implies that there exists a unique $\bar{t}\colon Y \longrightarrow \bar{T}$ such that $t = \nu \circ \bar{t}$. Hence the two indexing categories are isomorphic and so are the limits. □

Corollary 2.39. *Let $T \longrightarrow X$ be an X-scheme with $\dim(T) \leq \dim(X)$ and let $T = \cup_{i\in I} T_i$ be its decomposition into irreducible components. Then the normalization induces an isomorphism*

$$\Omega_{(X,\Delta)}(T) \cong \oplus_{i\in I}\Omega_{(X,\Delta)}(\bar{T}_i).$$

Proof. We have

$$\bar{T} = \sqcup_{i\in I}\bar{T}_i,$$

so the claim follows from Proposition 2.38 and from $\Omega_{(X,\Delta)}$ being a Zariski sheaf. □

We are now ready to compute the sections of $\Omega_{(X,\Delta)}$ over an arbitrary quasi-finite separated X-scheme. But before, let us briefly introduce the following definition from [KR]:

Definition 2.40 (Subadapted differentials)**.** In the setting of Definition-Lemma 1.62, drop the assumption of $\gamma^*(\Delta^{\mathrm{orb}})$ being a divisor with integer coefficients. Then we define the sheaf of *subadapted differentials* on $\hat{X}$ by replacing $\gamma^*\left(\frac{1}{m_i}D_i\right)$ with $\left\lceil \frac{1}{m_i}\gamma^*(D_i)\right\rceil$ in the definition. If $\gamma\colon \hat{X} \longrightarrow X$ is a quasi-finite morphism of normal varieties with $\dim(\hat{X}) = \dim(X)$, then we define the sheaf of *subadapted reflexive differentials* on $\hat{X}$ analogously to how we defined the sheaf of reflexive adapted differentials in Definition 1.67, and similarly for sheaves of subadapted differential p-forms. We also denote these sheaves by $\Omega^{[p]}_{(X,\Delta,\gamma)}$.

Proposition 2.41. *Let $\gamma\colon Y \longrightarrow X$ be a quasi-finite morphism of normal varieties with $\dim(Y) = \dim(X)$. Then, the restriction of $\Omega_{(X,\Delta)}$ to the Zariski topology on Y is isomorphic to the sheaf of subadapted reflexive differentials on Y.*

Proof. Same argument as in Proposition 2.33. □

Corollary 2.42. *Let $T \to X$ be a quasi-finite separated X-scheme and let $T = \cup_{i\in I} T_i$ be its decomposition into irreducible components. Let $J \subseteq I$ be the subset of indices such that $\dim(T_j) = \dim(X)$ for all $j \in J$. Then we have*

$$\Omega_{(X,\Delta)}(T) \cong \oplus_{j\in J}\Omega_{(X,\Delta)}(\bar{T}_j),$$

where the factors on the right-hand side consist of reflexive subadapted differentials as defined in Definition 2.40.

Proof. Follows from Corollary 2.39 and proposition 2.41. □

8. Morphisms of C-pairs and pull-back of adapted differentials

In [KR], Kebekus and Rousseau introduce the following notion:

Definition 2.43 (CKR-morphism)**.** Let (X, Δ_X) and (Y, Δ_Y) be $\mathbb{Q}$-factorial algebraic C-pairs and let $f\colon X \to Y$ be a morphism such that $f(X) \not\subseteq Y_{\mathrm{sing}} \cup \mathrm{Supp}(\Delta_Y)$. We say that f is a *CKR-morphism* if for every $p \in \mathbb{N}$, for every quasi-finite morphism $a\colon \hat{X} \to X$ of normal varieties of the same dimension, for every quasi-finite morphism $b\colon \hat{Y} \to Y$ of normal varieties of the same dimension and for every commutative square

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "a"'] & \hat{Y} \arrow[d, "b"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

the pull-back of rational differentials induces a pull-back morphism

$$\hat{f}^*\Omega^{[p]}_{(Y,\Delta_Y,b)} \to \Omega^{[p]}_{(X,\Delta_X,a)}.$$

When (Y, Δ_Y) is smooth, being a CKR-morphism is shown in [KR] to be equivalent[5] to inducing an orbifold morphism. We show that being a CKR-morphism is in turn a consequence of admitting a pull-back morphism at the level of qfh-sheaves:

Proposition 2.44. *Let (X, Δ_X) and (Y, Δ_Y) be $\mathbb{Q}$-factorial algebraic C-pairs and let $f\colon X \to Y$ be a morphism such that $f(X) \not\subseteq Y_{\mathrm{sing}} \cup \mathrm{Supp}(\Delta_Y)$. Suppose that for every $p \in \mathbb{N}$, the pull-back of rational differentials induces a pull-back morphism*

$$f^{-1}\Omega^p_{(Y,\Delta_Y)} \to \Omega^p_{(X,\Delta_X)}.$$

Then, f is a CKR-morphism.

Proof. Let $p \in \mathbb{N}$. Let $u\colon \mathbf{Sch}/Y \to \mathbf{Sch}/X$ be the functor given by $V \mapsto X \times_Y V =: f^{-1}(V)$. By definition, cf. [SP, 00WX] and [SP, 00X0], $f^{-1}\Omega_{(Y,\Delta_Y)}$ is the qfh-sheaf associated to the presheaf

$$u_P\Omega^p_{(Y,\Delta_Y)} \colon U \mapsto \operatorname*{colim}_{U \to f^{-1}(V)} \Omega^p_{(Y,\Delta_Y)}(V).$$

[5] Modulo some caveats having to do with locality issues in the analytic setting.

In particular, the pull-back of rational differentials induces a pull-back morphism

$$u_P\Omega^p_{(Y,\Delta_Y)} \to \Omega^p_{(X,\Delta_X)}.$$

Let now $a\colon U \to X$ be a quasi-finite morphism of normal varieties, let $b\colon V \to Y$ be a quasi-finite morphism of normal varieties and let $\hat{f}\colon U \to V$ be a morphism fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{f}} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y. \end{array}$$

By Proposition 2.41, we want to show that the pull-back of rational differentials induces a morphism

$$\hat{f}^*(\Omega^p_{(Y,\Delta_Y)}|_{V_{\mathrm{Zar}}}) \to \Omega^p_{(X,\Delta_X)}|_{U_{\mathrm{Zar}}}.$$

By adjunction, and since we may compose a and b with non-empty open immersions and still obtain quasi-finite morphisms of normal varieties of the same dimension, it suffices to show that the pull-back of rational differentials induces a morphism

$$\Omega^p_{(Y,\Delta_Y)}(V) \to \Omega^p_{(X,\Delta_X)}(U).$$

The universal property of the fiber product implies that the commutative square above induces a morphism $U \to f^{-1}(V)$ over X, so the existence of the desired morphism follows from the universal property of the colimit. $\square$

Question 2.45. *Is the converse of Proposition 2.44 true?*

Remark 2.46. Using the various universal properties involved, in order to show the converse of Proposition 2.44, it would suffice to show the existence of pull-back morphisms

$$\Omega^p_{(Y,\Delta_Y)}(T) \to \Omega^p_{(X,\Delta_X)}(U)$$

for all adapted morphisms $a\colon U \to X$ and all Y-schemes T with a morphism $U \to f^{-1}(T)$ over X. In particular, if $T \to Y$ is a quasi-finite morphism of normal varieties of the same dimension, then the existence of such a pull-back morphism follows directly from the assumption that f is a CKR-morphism.

CHAPTER 3

The adapted Albanese

Albanese varieties are higher-dimensional analogues of the Jacobian variety of a curve. Given a smooth projective variety X, its Albanese variety can be defined as the initial abelian variety to which X maps. This point of view was introduced by Serre in [Ser59]. More precisely, the Albanese variety of a pair (X, x_0), where X is a smooth projective variety and x_0 is a base point, is an abelian variety $\mathrm{Alb}(X)$ together with a morphism $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ sending x_0 to 0 such that for any abelian variety B and any morphism $b \colon X \to B$ sending x_0 to 0 there exists a unique morphism $c\colon \mathrm{Alb}(X) \to B$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}_X"] \arrow[dr, "b"'] & \mathrm{Alb}(X) \arrow[d, "c"] \\ & B. \end{tikzcd}$$

Since $c(0) = 0$, the induced morphism is even a group homomorphism on the underlying groups of closed points [Mil08, Corollary 1.2]. We could also forget about the base points and obtain a similar universal property, cf. Remark 3.5. The projectivity and smoothness assumptions can be ignored by taking compactifications and normalizations appropriately, as Serre does in [Ser59, Téorème 5]. However, by doing this, we are essentially ignoring the boundary of the compactification. In order to take the boundary of the compactification into account, one needs to replace abelian varieties by semi-abelian varieties, cf. [Ser59, Théorème 7].

Definition 3.1 (Semi-abelian variety, cf. [NW14, Definition 5.1.20])**.** A *semi-abelian variety* over $\mathbb{C}$ is a commutative[1] algebraic group G such that there exists an $r \in \mathbb{N}$ and an abelian variety A fitting into a short exact sequence

$$1 \to (\mathbb{G}_m)^{\times r} \to G \to A \to 1.$$

Such a short exact sequence is called a *presentation* of G.

Remark 3.2. It is shown in [Ser59, Théorème 8] that for larger "nice" classes of commutative algebraic groups the existence of a universal morphism as in the definition of the Albanese variety is not guaranteed.

[1] One can also define a semi-abelian variety in terms of its Barsotti–Chevalley decomposition [Mil17, Theorem 8.27], namely, as a connected algebraic group whose maximal connected affine algebraic subgroup is an algebraic torus [Fuj15, Definition 2.8]. Commutativity follows automatically from this definition [Fuj15, Lemma 2.11].

The meaning of the word nice is made precise in [Ser59, Propriétés I. et II.]. This is the reason to consider only semi-abelian varieties in the context of the Albanese variety.

Remark 3.3. The explicit construction that we will follow in the smooth quasi-projective setting, due to Iitaka [Iit76], uses logarithmic differential forms on a smooth compactification. For this reason, we will phrase the statements in terms of smooth projective log pairs, cf. Item 7 in Notation 0.7. See also Item 9 in Notation 0.7.

The goal in this chapter is to prove the following theorem, generalizing the main construction in [KR] to the singular setting:

Theorem 3.4. *Let (X, Δ) be a $\mathcal{C}$-pair with quotient singularities in which X is a projective variety. Let $\gamma\colon \hat{X} \to X$ be an adapted cover, let $\Delta_{\hat{X}} := \gamma^*(\lfloor\Delta\rfloor)_{\mathrm{red}}$, let $\hat{U} := \hat{X} \setminus \mathrm{Supp}(\Delta_{\hat{X}})$ and let $y_0 \in \hat{U}$ be a base point. Then there exists a semi-abelian variety* $\mathrm{Alb}(X, \Delta, \gamma)$ *and a morphism*

$$\mathrm{alb}_{(X,\Delta,\gamma)}\colon \hat{U} \to \mathrm{Alb}(X, \Delta, \gamma)$$

with the following universal property:

(1) *The morphism* $\mathrm{alb}_{(X,\Delta,\gamma)}$ *sends* y_0 *to* 0.
(2) *The pull-back of logarithmic differential* 1*-forms*

$$\mathrm{alb}^*_{(X,\Delta,\gamma)}\colon T_1(\mathrm{Alb}(X,\Delta,\gamma)) \to \Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$$

has image contained in $\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})$*, regarded as a subspace of* $\Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$ *as in Remark 1.70.*
(3) *For any semi-abelian variety G and any morphism $f\colon \hat{U} \to G$ sending y_0 to 0 and such that the pull-back of logarithmic differential* 1*-forms*

$$f^*\colon T_1(G) \to \Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$$

has image contained in $\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})$*, there exists a unique morphism $h\colon\ \mathrm{Alb}(X,\Delta,\gamma) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} \hat{U} \arrow[r, "\mathrm{alb}_{(X,\Delta,\gamma)}"] \arrow[dr, "f"'] & \mathrm{Alb}(X,\Delta,\gamma) \arrow[d, "h"] \\ & G. \end{tikzcd}$$

The semi-abelian variety $\mathrm{Alb}(X,\Delta,\gamma)$*, determined up to isomorphism by this universal property, is called the* adapted Albanese variety *of X with respect to the adapted cover $\gamma\colon \hat{X} \to X$. Moreover, we have*

$$\dim(\mathrm{Alb}(X,\Delta,\gamma)) \leq \dim_{\mathbb{C}}\left(\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})\right),$$

and the image of $\mathrm{alb}_{(X,\Delta,\gamma)}$ *generates* $\mathrm{Alb}(X,\Delta,\gamma)$ *in the sense of Item 15 in Notation 0.7.*

Remark 3.5. We could have phrased Theorem 3.4 without referencing to a base point. Indeed, the two formulations are equivalent, because we can translate by an appropriate element of the semi-abelian variety to force the image of any given point to be the origin. One can make this precise by using torsors instead of algebraic groups. But the construction of the adapted Albanese depends on the base point, so we make this explicit in the statement of Theorem 3.4. Another advantage of keeping track of the base point is that the induced morphism $h\colon \operatorname{Alb}(X,\Delta,\gamma) \to G$ must send the origin to the origin, so it is a morphism of algebraic groups by [NW14, Theorem 5.1.37].

In the proof of Theorem 3.4 we follow the strategy of the proof given by Kebekus and Rousseau in the analytic smooth setting in [KR]. To do so, there are some ingredients that need to be generalized to the singular setting, and this is done in Section 1. In Section 2 we review some of the necessary analytic and topological facts that will be used in the subsequent Albanese constructions. In Sections 3, 4 and 5 we review the already known Albanese constructions that Kebekus and Rousseau use to construct their adapted Albanese, and in Section 6 we review their construction of the adapted Albanese and prove Theorem 3.4.

1. Extension properties of adapted differentials

Our goal in this section is to prove an extension result for adapted differential 1-forms over $\mathcal{C}$-pairs with klt singularities. This extends a result of Kebekus and Rousseau in [KR] to the singular setting. In order to do this, we will need the following auxiliary result, inspired by Flenner's extension theorem [Fle88] and by some of the ideas and techniques in [KS21]:

Lemma 3.6. *Let X be a reduced complex space of pure dimension n. Let $\pi\colon \tilde{X} \to X$ be a resolution of singularities, let $p \in \{0,\dots,n\}$ and let $A \subseteq X$ be a closed complex subspace with* $\operatorname{codim}_X A \geq p+2$. *Then, sections of $\pi_*\Omega^p_{\tilde{X}}$ extend uniquely across A. That is, if we denote by $j\colon X\setminus A \to X$ the inclusion, then the canonical map*

$$\pi_*\Omega^p_{\tilde{X}} \to j_*j^*\pi_*\Omega^p_{\tilde{X}}$$

is an isomorphism.

Proof. The statement is local on X, so we may assume that X is a closed complex subspace of an open ball $B \subseteq \mathbb{C}^N$ for some $N \in \mathbb{N}$. Let $i\colon X \to B$ denote the inclusion and let $\nu\colon \tilde{X} \to B$ denote the composition $i \circ \pi$. The statement is independent of the resolution, cf. [KS21, §1.3]. So we may also assume that π is a *strong resolution*, i.e., a projective morphism which is an isomorphism over X_{reg}. In particular, ν is a projective morphism as well. The actual proof of Lemma 3.6 will require to use Saito's version of the Decomposition Theorem for the projective morphism ν [KS21, Corollary 4.11], as discussed in [KS21, §2]. But let us first explain the naive idea, which does not require to use this machinery.

We wish to apply [KS21, Proposition 6.4] to the complex subspace $A \subseteq B$ and to the cochain complex

$$\mathbf{R}\nu_*\Omega^p_{\tilde{X}} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_B),$$

where $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_B)$ denotes the bounded derived category of coherent $\mathcal{O}_B$-modules. This cochain complex has

$$H^m(\mathbf{R}\nu_*\Omega^p_{\tilde{X}}) = R^m\nu_*\Omega^p_{\tilde{X}} = 0$$

for all $m < 0$, so [KS21, Proposition 6.4] tells us that if

$$\dim\left(A \cap \mathrm{Supp}(R^k\mathcal{H}om_{\mathcal{O}_B}(\mathbf{R}\nu_*\Omega^p_{\tilde{X}}, \omega^\bullet_B))\right) \leq -(k+2) \text{ for all } k \in \mathbb{Z}, \tag{3.1}$$

with the understanding that $\dim(\varnothing) \leq k$ for all $k \in \mathbb{Z}$, then sections of $H^0(\mathbf{R}\nu_*\Omega^p_{\tilde{X}}) = \nu_*\Omega_{\tilde{X}}$ would extend uniquely across A, i.e., the canonical map

$$\nu_*\Omega^p_{\tilde{X}} \longrightarrow \iota_*\iota^*\nu_*\Omega^p_{\tilde{X}} = \iota_*\iota^* i_*\pi_*\Omega^p_{\tilde{X}}$$

would be an isomorphism, where $\iota\colon B\setminus A \to B$ is the open immersion. The immersion ι fits into a cartesian square

$$\begin{tikzcd} X \setminus A \arrow[r, "j"] \arrow[d, "i|_{X\setminus A}"'] & X \arrow[d, "i"] \\ B \setminus A \arrow[r, "\iota"] & B. \end{tikzcd}$$

By flat base change, which in this case amounts to compatibility of push-forward and restriction to open subsets, the canonical map

$$\nu_*\Omega^p_{\tilde{X}} \longrightarrow \iota_*(i|_{X\setminus A})_* j^*\pi_*\Omega^p_{\tilde{X}} = i_*j_*j^*\pi_*\Omega^p_{\tilde{X}}$$

would be an isomorphism as well. Applying i^* and using functoriality, we would deduce that the canonical map

$$i^*i_*\pi_*\Omega^p_{\tilde{X}} \longrightarrow i^*i_*j_*j^*\pi_*\Omega^p_{\tilde{X}}$$

is an isomorphism as well. Since i is the inclusion of a closed subspace, the counit of the adjunction $i^*i_* \to \mathrm{id}$ is an isomorphism, so the canonical map

$$\pi_*\Omega^p_{\tilde{X}} \longrightarrow j_*j^*\Omega^p_{\tilde{X}}$$

would be an isomorphism as well.

The problem with the previous argument is that Equation (3.1) is not necessarily true for the cochain complex $\mathbf{R}\nu_*\Omega^p_{\tilde{X}}$. So instead, we use Saito's version of the Decomposition Theorem for the projective morphism ν, as explained in [KS21, §2.4] and [KS21, §8], to obtain a (non-canonical) decomposition

$$\mathbf{R}\nu_*\Omega^p_{\tilde{X}} \cong K_p \oplus R_p$$

into two cochain complexes $K_p, R_p \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_B)$ with the following properties:

(1) The support of R_p is contained in the singular locus $X_{\mathrm{sing}} \subseteq B$. In particular, this implies that $H^0(R_p) = 0$, because it is a direct summand of the torsion-free sheaf $\pi_*\Omega_{\tilde{X}}$ on X and it is supported on the proper closed subspace $X_{\mathrm{sing}} \subseteq X$.
(2) One has $H^k(K_p) = 0$ for all $k \geq n - p + 1$.
(3) One has $\mathbf{R}\mathcal{H}om_{\mathcal{O}_B}(K_p, \omega_B^\bullet) \cong K_{n-p}[n]$.

Item 1 above implies that we may apply [KS21, Proposition 6.4] to the cochain complex K_p instead to obtain the same conclusion, because

$$H^0(\mathbf{R}\nu_*\Omega^p_{\tilde{X}}) \cong H^0(K_p).$$

Moreover, since $R^m\nu_*\Omega^p_{\tilde{X}} = 0$ for all $m < 0$, we must also have $H^m(K_p) = 0$ for all $m < 0$, because $R^m\nu_*\Omega^p_{\tilde{X}} \cong H^m(K_p) \oplus H^m(R_p)$ for all $m \in \mathbb{Z}$. So we can indeed apply [KS21, Proposition 6.4] to the cochain complex K_p. Therefore, it suffices to show that Equation (3.1) holds for K_p.

Since $\dim(A) \leq n - p - 2$, we have

$$\dim\Big(A \cap \mathrm{Supp}(R^k\mathcal{H}om_{\mathcal{O}_B}(K_p, \omega_B^\bullet))\Big) \leq -(p - n + 2) \leq -(k + 2)$$

for all $k \leq p - n$. In order to establish the inequality for $k \geq p - n + 1$, it suffices to show that $R^k\mathcal{H}om_{\mathcal{O}_B}(K_p, \omega_B^\bullet) = 0$. By Item 3 above, we may rewrite this sheaf as $H^k(K_{n-p}[n]) = H^{k+n}(K_{n-p})$. And Item 2 tells us that $H^k(K_{n-p}) = 0$ for all $k \geq p + 1$. Therefore, $H^{k+n}(K_{n-p}) = 0$ for all $k \in \mathbb{Z}$ such that $k + n \geq p + 1$, i.e., for all $k \geq p - n + 1$. □

Remark 3.7. Flenner-type extension results are known to fail in positive characteristic, cf. [Gra21a, Theorem 3].

We now prove the desired extension result in a nice setting, following the argument given in [KR]:

Lemma 3.8 (cf. [KR])**.** *Let (X, Δ) be a $\mathcal{C}$-pair, algebraic or analytic, and let $\gamma\colon \hat{X} \to X$ be an adapted morphism. Suppose that we can find a commutative diagram*

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\pi_Y"] \arrow[d, "\tilde{f}"] & \hat{Y} \arrow[r, "\gamma_Y"] \arrow[d, "\hat{f}"] & Y \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\pi"] & \hat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

in which (Y, Δ_Y) is smooth for $\Delta_Y := (f^*\lfloor\Delta\rfloor)_{\mathrm{red}}$, f is a perfectly adapted cover, π is a log resolution of $(\hat{X}, \Delta_{\hat{X}})$ for $\Delta_{\hat{X}} := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}$, $\hat{f}$ is a cover, γ_Y is a quasi-finite morphism between normal varieties of the same dimension, π_Y is a log resolution of $(\hat{Y}, \Delta_{\hat{Y}})$ for $\Delta_{\hat{Y}} := (\hat{f}^*\Delta_{\hat{X}})_{\mathrm{red}}$ and $\tilde{f}$ is a generically finite surjective morphism. Then, for any $p \in \mathbb{N}$ and for $\Delta_{\tilde{X}} := (\pi^*\Delta_{\hat{X}})_{\mathrm{red}}$, *the pull-back of rational differentials induces a morphism of $\mathcal{O}_{\tilde{X}}$-modules*

$$\pi^*\Omega^{[p]}_{(X,\Delta,\gamma)} \to \Omega^p_{\tilde{X}}(\log \Delta_{\tilde{X}}). \tag{3.2}$$

Proof. The analytic case is treated in [KR]. The same argument works in the algebraic case, and we include it here for the sake of completeness.

We show instead the existence of the adjoint morphism

$$\Omega^{[p]}_{(X,\Delta,\gamma)} \to \pi_*\Omega^p_{\tilde{X}}(\log \Delta_{\tilde{X}}).$$

Recall from Item 13 in Notation 0.7 that this morphism should be induced by the pull-back of rational differential forms, so let $V \subseteq \hat{X}$ be a non-empty open subset and let $\sigma_{\hat{X}} \in \Omega^{[p]}_{(X,\Delta,\gamma)}(V)$ be an adapted reflexive differential, seen as a rational differential form in $\mathcal{K}_{\hat{X}}(\Omega^p_{\hat{X}})(V)$ under the inclusions of Remark 1.70. By Lemma 1.74, the pull-back of rational differentials induces an isomorphism

$$\hat{f}^{[*]}\Omega^{[p]}_{(X,\Delta,\gamma)} \to \Omega^{[p]}_{(X,\Delta,\gamma\circ\hat{f})},$$

therefore

$$\hat{f}^*(\sigma_{\hat{X}}) \in \Omega^{[p]}_{(X,\Delta,\gamma\circ\hat{f})}(\hat{f}^{-1}(V)) = \Omega^{[p]}_{(X,\Delta,f\circ\gamma_Y)}(\hat{f}^{-1}(V)).$$

Again by Lemma 1.74, the pull-back of rational differentials induces an isomorphism

$$\gamma_Y^{[*]}\Omega^{[p]}_{(X,\Delta,f)}(\hat{f}^{-1}(V)) \cong \Omega^{[p]}_{(X,\Delta,f\circ\gamma_Y)}(\hat{f}^{-1}(V)),$$

and by Lemma 1.73 and Remark 1.71 we have $\Omega^{[p]}_{(X,\Delta,f)} \cong \Omega^p_Y(\log \Delta_Y)$, so we can find a unique section

$$\xi \in \gamma_Y^*\Omega^p_Y(\log \Delta_Y)(\hat{f}^{-1}(V))$$

whose image is $\hat{f}^*(\sigma_{\hat{X}})$. Since the statement that we are trying to show is local on X, we may assume that

$$\xi = \sum_i g_i\sigma_i \in \gamma_Y^*\Omega^p_Y(\log \Delta_Y)(\hat{Y})$$

for $g_i \in \mathcal{O}_{\hat{Y}}(\hat{Y})$ and $\sigma_i \in \Omega^p_Y(\log \Delta_Y)(Y)$, so that

$$\hat{f}^*(\sigma_{\hat{X}}) = \sum_i g_i\gamma_Y^*(\sigma_i) \in \Omega^{[p]}_{(X,\Delta,\gamma\circ\hat{f})}(\hat{Y}).$$

Now pulling back $\hat{f}^*(\sigma_{\hat{X}})$ to $\tilde{Y}$ along π_Y yields

$$\sigma_{\tilde{Y}} := \pi_Y^*\hat{f}^*(\sigma_{\hat{X}}) = \sum_i \pi_Y^*(g_i)(\gamma_Y \circ \pi_Y)^*(\sigma_i) \in \Omega^p_{\tilde{Y}}(\log \Delta_{\tilde{Y}})(\tilde{Y}).$$

Let now $\sigma_{\tilde{X}} := \pi^*(\sigma_{\hat{X}}) \in \mathcal{K}_{\tilde{X}}(\Omega^p_{\tilde{X}})(\tilde{X})$. In particular, note that $\tilde{f}^*(\sigma_{\tilde{X}}) = \sigma_{\tilde{Y}}$. The claim is that $\sigma_{\tilde{X}}$ is also a logarithmic differential form. Let $\tilde{X}^\circ \subseteq \tilde{X}$ be the maximal open subset with preimage $\tilde{Y}^\circ$ such that $\tilde{f}^\circ := \tilde{f}|_{\tilde{Y}^\circ} : \tilde{Y}^\circ \to \tilde{X}^\circ$ is a finite surjective morphism and the pairs

$(\tilde{Y}^\circ, \operatorname{Ram} \tilde{f}^\circ)$ and $(\tilde{X}^\circ, \operatorname{Branch}(\tilde{f}^\circ))$ are both smooth. Since $\sigma_{\tilde{Y}}$ is a logarithmic differential and $\sigma_{\tilde{Y}}|_{\tilde{Y}^\circ}$ is the pull-back of $\sigma_{\tilde{X}}|_{\tilde{X}^\circ}$ under $\tilde{f}^\circ$, the criterion in [GKK10, Corollary 2.12] implies that $\sigma_{\tilde{X}}$ is a logarithmic differential over the big open subset $\tilde{X}^\circ$. Hence $\sigma_{\tilde{X}}$ is a logarithmic differential. □

From this lemma we can deduce the following extension result for C-pairs with quotient singularities:

Corollary 3.9. *Let (X, Δ) be a C-pair with quotient singularities. Let $\gamma\colon \hat{X} \to X$ be an adapted morphism and let $\pi\colon \tilde{X} \to \hat{X}$ be a log resolution of $(\hat{X}, \Delta_{\hat{X}})$ for $\Delta_{\hat{X}} := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}$. Then, for any $p \in \mathbb{N}$ and for $\Delta_{\tilde{X}} := (\pi^*\Delta_{\hat{X}})_{\mathrm{red}}$, the pull-back of rational differentials induces a morphism of $\mathcal{O}_{\tilde{X}}$-modules*

$$\pi^*\Omega^{[p]}_{(X,\Delta,\gamma)} \to \Omega^p_{\tilde{X}}(\log \Delta_{\tilde{X}}).$$

Proof. The question is analytic-local on $\hat{X}$, hence also on X. Therefore, by Corollary 1.55, we may assume that (X, Δ) admits a perfectly adapted cover $f\colon Y \to X$ such that (Y, Δ_Y) is smooth for $\Delta_Y := (f^*\lfloor\Delta\rfloor)_{\mathrm{red}}$. Being finite and surjective is stable under base change, so we can find an irreducible component of $\hat{X} \times_X Y$ that surjects onto $\hat{X}$ along the projection from the fiber product. The inclusion of this irreducible component in the fiber product is a finite morphism, and the normalization of variety is a finite surjective morphism, so after picking a suitable irreducible component of the fiber product and normalizing it we obtain a commutative square

$$\begin{tikzcd} \hat{Y} \arrow[r, "\gamma_Y"] \arrow[d, "\hat{f}"] & Y \arrow[d, "f"] \\ \hat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

in which $\hat{f}$ is a cover and γ_Y is a quasi-finite morphism between normal varieties of the same dimension. Let now $\pi_Y\colon \tilde{Y} \to \hat{Y}$ be a log resolution of $(\hat{Y}, \Delta_{\hat{Y}})$ for $\Delta_{\hat{Y}} := (\gamma_Y^*\Delta_Y)_{\mathrm{red}}$. Then we have a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\pi_Y"] \arrow[d, dashed, "\tilde{f}"] & \hat{Y} \arrow[r, "\gamma_Y"] \arrow[d, "\hat{f}"] & Y \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\pi"] & \hat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

in which $\tilde{f}$ is a priori only a rational map, obtained as the composition of rational maps $\pi^{-1} \circ \hat{f} \circ \pi_Y$. After resolving the indeterminacy of $\tilde{f}$, we can apply Lemma 3.8 to obtain the result. □

Theorem 3.10. *Let (X, Δ) be a klt C-pair. Let $\gamma\colon \hat{X} \to X$ be an adapted morphism and let $\pi\colon \tilde{X} \to \hat{X}$ be a log resolution of singularities of $\hat{X}$. Then* *the pull-back of rational differentials induces a morphism of $\mathcal{O}_{\tilde{X}}$-modules*

$$\pi^*\Omega^{[1]}_{(X,\Delta,\gamma)} \to \Omega^1_{\tilde{X}}.$$

Proof. The idea is to use the fact that klt implies quotient singularities in codimension 2 [GKKP11, Proposition 9.3] to reduce to the case of quotient singularities. By Lemma 3.6, we may remove closed subsets of codimension at least 3 from $\hat{X}$, hence also from X because γ is a quasi-finite morphism between normal varieties of the same dimension. Moreover, the statement is analytic-local on X, so by Corollary 1.60 we may assume that there exists a perfectly adapted cover $f\colon Y \to X$ from a smooth variety Y. Taking fiber products, choosing suitable irreducible components, normalizing and resolving singularities and indeterminacies as in the proof of Corollary 3.9, we may assume that we are in the situation of Lemma 3.8, and the result follows. □

Remark 3.11. The definition of klt singularities and other kinds of singularities in the MMP has to do with extension of reflexive differential top-forms to resolutions of singularities. Roughly speaking, these singularities measure what kind of poles can such an extension have along the irreducible components of the exceptional divisors. The statement in Theorem 3.10 has to do with 1-forms, rather than with top-forms, so it is not a direct consequence of the definition. However, it has been established in [KS21] that the extension of p-forms is indeed related to the extension of forms of smaller degrees.

Remark 3.12. In the setting of Theorem 3.10, the singularities of $\hat{X}$ may be worse than those of X. For example, if $\hat{X}$ is the affine cone over a smooth curve of degree 4 in $\mathbb{P}^2$, then $\hat{X}$ has worse than log canonical singularities, and one can find a reflexive 1-form on $\hat{X}$ which does not extend to its standard resolution of singularities [GKKP11, Example 3.1]. So in general we cannot expect to be able to extend all reflexive 1-forms on $\hat{X}$ to its resolution of singularities. But Theorem 3.10 ensures that, at least for those reflexive 1-forms which are adapted with respect to a klt C-pair, the extension to the resolution is possible.

Remark 3.13. The assumption on the singularities in Theorem 3.10 is sharp, because if $\Delta = 0$, then the identity on X is an adapted cover and all reflexive differentials on X are adapted differentials. So sharpness of Theorem 3.10 in this sense follows from sharpness of [GKKP11, Theorem 1.4]. The sharpness of [GKKP11, Theorem 1.4] is discussed in [GKKP11, §3].

Remark 3.14. As pointed out in Remark 3.13, the usual extension of reflexive differentials on klt spaces is a particular case of Theorem 3.10. But this extension result is known to fail in positive characteristic [Gra21b, Example 10.2], so Theorem 3.10 also fails in positive characteristic.

In the case of C-pairs with quotient singularities we can also prove the following vanishing result:

Proposition 3.15 ([KR]). *In the situation of Corollary 3.9, let Z be a smooth variety and let $g\colon Z \to \tilde{X} \setminus \mathrm{Supp}(\Delta_{\tilde{X}})$ be a morphism whose image is contained in a fiber of π. For any $\sigma_{\hat{X}} \in \Omega^{[p]}_{(X,\Delta,\gamma)}(\hat{X})$, we have*

$$g^*\pi^*(\sigma_{\hat{X}}) = 0 \in \Omega^p_{\tilde{X}}(\log \Delta_{\tilde{X}})(\tilde{X}).$$

Proof. The analytic case is treated in [KR]. The same argument works in the algebraic case, and we include it here for the sake of completeness.

Let $\sigma_{\tilde{X}}$ be the rational differential $\pi^*(\sigma_{\hat{X}})$ on $\tilde{X}$. We check that $g^*(\sigma_{\tilde{X}}) = 0$. The statement is analytic-local on $\hat{X}$, so we may assume that we are in the situation of Lemma 3.8. Thus we have a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\pi_Y"] \arrow[d, "\tilde{f}"'] & \hat{Y} \arrow[r, "\gamma_Y"] \arrow[d, "\hat{f}"'] & Y \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\pi"] & \hat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

in which $\hat{f}$ is a finite surjective morphism between normal varieties, π and π_Y are log resolutions and $\tilde{f}$ is generically finite. By resolving the singularities of the fiber product $Z \times_{\tilde{X}} \tilde{Y}$ appropriately we can produce a commutative diagram

$$\begin{tikzcd} Z_Y \arrow[r, "p"] \arrow[d, "f_Z"'] & \tilde{Y} \arrow[r, "\pi_Y"] \arrow[d, "\tilde{f}"'] & \hat{Y} \arrow[r, "\gamma_Y"] \arrow[d, "\hat{f}"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & \tilde{X} \arrow[r, "\pi"] & \hat{X} \arrow[r, "\gamma"] & X \end{tikzcd}$$

in which f_Z and $\tilde{f}$ are generically finite and surjective. To show that $g^*(\sigma_{\tilde{X}}) = 0$, it suffices to show that $f_Z^* g^*(\sigma_{\tilde{X}}) = 0$, because we are in characteristic zero and f_Z is generically finite and surjective. Since $\sigma_{\tilde{X}} = \pi^*(\sigma_{\hat{X}})$ and the diagram commutes, it suffices in turn to show that $p^*\pi_Y^*(\hat{f}^*(\sigma_{\hat{X}})) = 0$. But it follows from the proof of Lemma 3.8 that, up to possibly shrinking $\hat{X}$ a bit more, the form $\hat{f}^*(\sigma_{\hat{X}})$ is of the form

$$\hat{f}^*(\sigma_{\hat{X}}) = \sum_i g_i \gamma_Y^*(\sigma_i)$$

with $g_i \in \mathcal{O}_{\hat{Y}}(\hat{Y})$ and $\sigma_i \in \Omega^p_Y(\log \Delta_Y)(Y)$. Therefore,

$$p^*\pi_Y^*(\hat{f}^*(\sigma_{\hat{X}})) = \sum_i (\pi_Y \circ p)^*(g_i)(\gamma_Y \circ \pi_Y \circ p)^*(\sigma_i).$$

But the image of Z in $\hat{X}$ is a point and $\hat{f}$ is a finite morphism, so the image of Z_Y in $\hat{Y}$ is a point as well, i.e., the composition $\gamma_Y \circ \pi_Y \circ p$ is constant and so $(\gamma_Y \circ \pi_Y \circ p)^*(\sigma_i) = 0$ for all i. $\square$

The extension result in Corollary 3.9 and the vanishing result in Proposition 3.15 for $p = 1$ are the necessary ingredients to generalize the construction of the adapted Albanese in [KR] to the singular setting. The extension works in the klt setting, cf. Theorem 3.10. If the vanishing also worked in the klt setting, then we could generalize the construction of the adapted Albanese to the klt setting. This motivates the following:

Question 3.16. *Can Proposition 3.15 be generalized to the klt setting?*

2. Analytical and topological preliminaries

The construction of the adapted Albanese in Section 6 will be based on one of the classic constructions of the Albanese of a smooth projective variety, which will in fact be a particular case, cf. Example 3.42. The Albanese of a smooth projective variety is an abelian variety, but we will use analytic methods to construct it, so we start by recalling some general facts about complex Lie groups and then move on to the construction of the Albanese in the next section. Everything in this section is well-known in the literature.

Definition 3.17 (Lattice, cf. [Neu99, Definition I.4.1])**.** Let $n \in \mathbb{N}$ and let V be an n-dimensional $\mathbb{R}$-vector space. A *lattice* in V is a subgroup of the form

$$\Lambda = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m,$$

where $v_1, \ldots, v_m \in V$ are $\mathbb{R}$-linearly independent vectors. The lattice is called *complete* if $m = n$. A (complete) lattice in a finite dimensional complex vector space is a (complete) lattice in the underlying real vector space.

Fact 3.18 ([Neu99, Proposition I.4.2])**.** *Let V be a finite dimensional real vector space. A subgroup $\Lambda \subseteq V$ is a lattice if and only if it is discrete.*

Remark 3.19. The property of being a (complete) lattice is invariant under $\mathbb{R}$-linear isomorphisms, i.e., if $\varphi \colon V \longrightarrow V'$ is an isomorphism of finite dimensional real vector spaces and $\Lambda \subseteq V$ is a (complete) lattice, then so is $\varphi(\Lambda)$.

Example 3.20. The subgroup $\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{R}$ is not a lattice, because it is not discrete.

Lemma 3.21 (cf. [Neu99, Exercise I.4.1])**.** *Let V be a finite dimensional real vector space and let $\Lambda \subseteq V$ be a discrete subgroup. Then Λ is a complete lattice if and only if V/Λ is compact.*

Idea of proof. In one direction we can use the homeomorphisms $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, and in the other direction the fact that a real vector space of strictly positive dimension is not compact. □

Definition 3.22 (Torus, cf. [NW14, Definition 5.1.1])**.** A *torus* is a complex Lie group which is the quotient of a finite dimensional complex vector space by a complete lattice.

Fact 3.23 ([NW14, Proposition 5.1.2])**.** *Every connected compact complex Lie group is a torus. In particular, the analytification of an abelian variety is a torus in the sense of Definition 3.22.*

Idea of proof. Compactness ensures that it is commutative, and this in turn ensures that the exponential map is a group homomorphism. Since the exponential is a local isomorphism, its kernel is a discrete subgroup of the tangent space at the origin. By connectedness and compactness, the exponential is also surjective, so the quotient of the tangent space at the origin by this lattice must be compact, hence the kernel is a complete lattice. See also proof of [Mil08, Proposition 2.1]. □

Remark 3.24. Conversely, a polarizable torus [Mil08, p. 14] is the analytification of an abelian variety [Mil08, Theorem 2.8]. In fact, the analytification functor induces an equivalence of categories between the category of abelian varieties with algebraic group morphisms and the category of polarizable tori with complex Lie group homomorphisms [Mil08, Theorem 2.9]. Therefore, in light of [Mil08, Corollary 1.2] and [NW14, Theorem 5.1.36], the analytification functor also induces an equivalence of categories between the category of abelian varieties with morphisms of varieties and the category of polarizable tori and holomorphic maps.

Definition 3.25 (Semi-torus, cf. [NW14, Definition 5.1.5])**.** A *semi-torus* is a connected commutative complex Lie group G that admits a short exact sequence

$$1 \to (\mathbb{C}^\times)^k \to G \to A \to 1,$$

where A is a torus in the sense of Definition 3.22 and $k \in \mathbb{N}$. Such a short exact sequence is called a *presentation* of G.

Fact 3.26 ([NW14, Proposition 5.1.8])**.** *A connected commutative complex Lie group is a semi-torus if and only if it is isomorphic to a quotient of the form $\mathbb{C}^n/\Lambda$ for some $n \in \mathbb{N}$ and some lattice $\Lambda \subseteq \mathbb{C}^n$ that generates $\mathbb{C}^n$ as a complex vector space.*

Fact 3.27 ([NW14, Proposition 5.1.21])**.** *The analytification of a semi-abelian variety is a semi-torus.*

Remark 3.28. The analogous of Remark 3.24 is not true for semi-abelian varieties. There are commutative algebraic groups which are not semi-abelian varieties, but whose analytifications are semi-tori, cf. [NW14, Example 5.1.44]. The presentation of a semi-abelian variety is unique [NW14, Proposition 5.1.30], whereas the presentation of a semi-torus is not [NW14, §5.1.5]. This can be used to construct semi-abelian varieties which are not isomorphic as algebraic groups but are isomorphic as complex Lie groups [NW14, §5.1.7]. Therefore, the analytification functor cannot be fully faithful on the category of semi-abelian varieties, because if it was, then it would reflect isomorphisms.

Lemma 3.29 (cf. [Bea96, p. 60])**.** *Let V be a finite dimensional complex vector space and let $\Lambda \subseteq V$ be a lattice. Let $G := V/\Lambda$ be the quotient complex Lie group and let $\pi\colon V \to G$ be the quotient map.*

(1) *The quotient map induces a canonical isomorphism $V \cong T_0G$, where T_0G is the real tangent space of G at the origin.*
(2) *The isomorphism in Item 1 identifies the quotient map $\pi\colon V \to G$ with the exponential map $\exp\colon T_0G \to G$.*
(3) *The quotient map induces a canonical isomorphism*
$$\Omega^1_G \cong V^\vee \otimes \mathcal{O}_G.$$
(4) *The differential $d\alpha$ of a linear form[2] $\alpha \in V^\vee$ induces a well-defined differential form $\delta\alpha \in \Omega^1_G(G)$, and this induces the isomorphism on global sections of the canonical isomorphism in Item 3. More precisely, the corresponding isomorphism on global sections is given by*
$$\begin{aligned} V^\vee \otimes_{\mathbb{C}} \mathcal{O}_G(G) &\to \Omega^1_G(G),\\ \sum_i \alpha_i \otimes f_i &\mapsto \sum_i f_i \delta\alpha_i. \end{aligned}$$

Proof. The underlying real vector space V is canonically isomorphic to its real tangent space at the origin [Lee13, Proposition 3.13] and $\Lambda \subseteq V$ is a discrete subgroup, so the assertion in Item 1 follows from [Lee13, Theorem 21.13]. This shows Item 1.

Since π is a Lie group homomorphism, we have a commutative diagram as follows [Lee13, Proposition 20.8 (g)]:
$$\begin{tikzcd} T_0V \arrow[r, "d\pi_0"] \arrow[d, "\exp"'] & T_0G \arrow[d, "\exp"] \\ V \arrow[r, "\pi"] & G. \end{tikzcd}$$
Under the canonical identification $T_0V = V$, the exponential becomes the identity [Lee13, Proposition 20.8 (e)]. This shows Item 2.

Since G is a complex Lie group, the holomorphic tangent bundle is canonically isomorphic to the trivial bundle $(T_0^{1,0}G) \times G$, where $T_0^{1,0}G$ denotes the holomorphic tangent space at the origin. Since π is holomorphic, we have $d\pi_0(T_0^{1,0}V) \subseteq T_0^{1,0}G$ [Huy05, Proposition 1.3.2], so the canonical isomorphism from Item 1 induces a canonical isomorphism $V \cong T_0^{1,0}G$. Therefore π induces a canonical isomorphism $V \otimes \mathcal{O}_G \cong \mathcal{T}_G$ and thus a canonical isomorphism $\Omega^1_G \cong V^\vee \otimes \mathcal{O}_G$. This shows Item 3.

We check next that $\delta\alpha$ in Item 4 is indeed well-defined. Fix an isomorphism $V \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$, so that we can write α as $\sum_{i=1}^n \lambda_i x_i$ for

[2] That is, the differential of the linear form regarded as a smooth function.

some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then we have

$$d\alpha = \sum_{i=1}^{n} \lambda_i dx_i,$$

and we want to show that this differential form on V induces a well-defined differential form on G. Let $\pi\colon V \longrightarrow G$ be the quotient map and consider an open subset V_j such that $\pi|_{V_j}\colon V_j \longrightarrow \pi(V_j)$ is an isomorphism. Since π is open, $U_j := \pi(V_j)$ is also open. We denote this isomorphism by $\pi_j\colon V_j \longrightarrow U_j$. Its inverse $\varphi_j\colon U_j \longrightarrow V_j$ is then by definition a complex chart on G. Consider another such complex chart $\varphi_{j'}\colon U_{j'} \longrightarrow V_{j'}$ such that $U_j \cap U_{j'} \neq \emptyset$. We check that

$$\varphi_j^*(d\alpha)|_{U_j \cap U_{j'}} = \varphi_{j'}^*(d\alpha)|_{U_j \cap U_{j'}}.$$

By $\mathbb{C}$-linearity, and since charts are isomorphisms, it suffices to show that

$$dx_1 = (\varphi_{j'} \circ \varphi_j^{-1})^*(dx_1).$$

Let $p \in U_j \cap U_{j'}$. Then we can write $\varphi_j(p) = v_j$ for some $v_j \in V_j$ and $\varphi_{j'}(p) = v_{j'}$ for some $v_{j'} \in V_{j'}$. Since both v_j and $v_{j'}$ come from the same point p, we have $v_{j'} - v_j \in \Lambda$. Writing $V_{jj'} := V_j \cap V_{j'}$ and $\varphi_{jj'} := \varphi_{j'} \circ \varphi_j^{-1}$, this implies that the continuous function

$$\varphi_{jj'} - \mathrm{id}_{V_{jj'}}\colon V_{jj'} \longrightarrow V_{jj'}$$

takes values in Λ, so it must be constant because Λ is discrete. Hence there exists some $\lambda \in \Lambda$ such that $\varphi_{jj'}(z) = z + \lambda$ for all $z \in V_{jj'}$. Therefore $\varphi_{jj'}^*(dx_1) = dx_1$. This shows Item 4. ☐

Remark 3.30. In particular, in Item 4 in Lemma 3.29, if G is compact, then we obtain a canonical isomorphism

$$\delta\colon V^\vee \xrightarrow{\cong} \Omega_G^1(G).$$

If G is a semi-abelian variety, then we can use a suitable compactification as in [NW14, Proposition 5.3.4] to obtain a canonical isomorphism

$$\delta\colon V^\vee \xrightarrow{\cong} T_1(G),$$

see [NW14, Proposition 5.4.3].

Lemma 3.31 (cf. [Bea96, p. 60])**.** *Let V be a finite dimensional complex vector space and let $\Lambda \subseteq V$ be a lattice. Let $G = V/\Lambda$ be the quotient complex Lie group.*

(1) *The quotient map $\pi\colon V \longrightarrow G$ is the universal cover of G. In particular, the fundamental group of G is isomorphic to Λ.*
(2) *An explicit isomorphism $h_G\colon \Lambda \longrightarrow \pi_1(G, 0)$ is given by sending $\lambda \in \Lambda$ to the equivalence class of the path*

$$\begin{aligned} h_G(\lambda)\colon [0,1] &\longrightarrow V/\Lambda \\ t &\longmapsto \lambda t + \Lambda. \end{aligned}$$

(3) *For any $\alpha \in V^\vee$ and $\lambda \in \Lambda$, using the notation of Lemma 3.29, we have*
$$\int_{h_G(\lambda)} \delta\alpha = \alpha(\lambda).$$

PROOF. The action of Λ on V is properly discontinuous [Bre93, Definition III.7.1] and V is contractible, so we can apply [Bre93, Proposition III.7.2] and [Bre93, Corollary III.7.3] to deduce Item 1. The assertion in Item 2 follows from the explicit description of the isomorphism between fundamental group and group of deck transformations of the cover in [Bre93, Theorem 6.8], because the path $\gamma_\lambda\colon [0,1] \to V$ lifting the path $t \mapsto \lambda t + \Lambda$ has end point $\gamma(1) = \lambda$. The assertion in Item 3 follows from the equalities
$$\begin{aligned}\int_{h_G(\lambda)} \delta\alpha &= \int_0^1 h_G(\lambda)^*(\delta\alpha)\\ &= \int_0^1 \gamma_\lambda^*\pi^*(\delta\alpha)\\ &= \int_0^1 \gamma_\lambda^* d\alpha\\ &= \int_0^1 d(\alpha\circ\gamma_\lambda)\\ &= (\alpha\circ\gamma_\lambda)(1) - (\alpha\circ\gamma_\lambda)(0) = \alpha(\lambda).\end{aligned}$$
□

Lemma 3.32. *Let V_1 and V_2 be finite dimensional complex vector spaces and let $\Lambda_1 \subseteq V_1$ and $\Lambda_2 \subseteq V_2$ be lattices. Let $\pi_1\colon V_1 \to G_1$ and $\pi_2\colon V_2 \to G_2$ be the corresponding quotients. Let $f\colon G_1 \to G_2$ be a smooth map such that $f(0) = 0$ and let $\tilde{f}\colon V_1 \to V_2$ be the (unique) lift of f to the universal covers with $\tilde{f}(0) = 0$ [Bre93, Theorem III.4.1]. The following are equivalent:*

(1) *The map f is a group homomorphism.*
(2) *After the identifications from Item 1 in Lemma 3.29 we have a commutative square as follows:*
$$\begin{tikzcd} V_1 \arrow[r, "df_0"] \arrow[d, "\pi_1"'] & V_2 \arrow[d, "\pi_2"] \\ G_1 \arrow[r, "f"] & G_2. \end{tikzcd}$$
(3) *The lift $\tilde{f}$ agrees with the differential $df_0\colon T_0G_1 \to T_0G_2$ under the identifications from Item 1 in Lemma 3.29.*
(4) *The lift $\tilde{f}$ is a group homomorphism.*

In particular, any Lie group homomorphism $f\colon G_1 \to G_2$ is determined by its differential at the origin.

Proof. Suppose f is a group homomorphism. Then, using the identifications from Item 2 in Lemma 3.29, the square in Item 2 commutes by [Lee13, Proposition 20.8 (g)]. If this square commutes, then the lift $\tilde{f}$ agrees with df_0 as a consequence of the uniqueness of the lift in [Bre93, Theorem III.4.1] and linear maps between finite dimensional real vector spaces being continuous. Suppose now that the lift $\tilde{f}$ is a group homomorphism and let $g, g' \in G_1$. Pick $v, v' \in V_1$ such that $\pi_1(v) = g$ and $\pi_1(v') = g'$. Then we have

$$\begin{aligned} f(g+g') &= f(\pi_1(v) + \pi_1(v')) \\ &= f(\pi_1(v+v')) \\ &= \pi_2(\tilde{f}(v+v')) \\ &= \pi_2(\tilde{f}(v) + \tilde{f}(v')) \\ &= \pi_2(\tilde{f}(v)) + \pi_2(\tilde{f}(v')) \\ &= f(\pi_1(v)) + f(\pi_1(v')) \\ &= f(g) + f(g'), \end{aligned}$$

so f is a group homomorphism.

If any of the equivalent conditions hold, then $f(g) = \pi_2(df_0(v))$ for any $g \in G_1$ and any $v \in \pi_1^{-1}(g)$, so f is determined by df_0. □

Fact 3.33 ([NW14, Theorem 5.1.36]). *Let G be a semi-torus without non-constant holomorphic functions, let H be a complex Lie group, and let $f\colon G \to H$ be a holomorphic map with $f(0) = 0$. Then f is a group homomorphism.*

Note that Fact 3.33 applies in particular when G is a torus. We also have the analogous statement for semi-abelian varieties:

Fact 3.34 ([NW14, p. 5.1.37]). *Let G and H be semi-abelian varieties and let $f\colon G \to H$ be a morphism with $f(0) = 0$. Then f is a morphism of algebraic groups.*

Remark 3.35. The proof of Fact 3.34 in [NW14, p. 5.1.37] uses that there are no rational curves on abelian varieties. This can be proven directly in the algebraic setting without using any Albanese machinery [Mil08, Proposition 3.9]. It can also be proven directly in the analytic setting as follows[3]. Let $f\colon X \to Y$ be a holomorphic map between connected complex manifolds. Assume that X is simply connected and that the universal cover $\pi\colon \tilde{Y} \to Y$ is a local isomorphism of complex manifolds. Then the lift $\tilde{f}\colon X \to \tilde{Y}$ is again holomorphic, because this can be checked locally on $\tilde{Y}$ and π is a local biholomorphism. If X is compact and $\tilde{Y}$ is Stein, the lift must be constant, so f is constant as well. This argument applies in particular to $X = \mathbb{P}^1$ and Y a torus in the sense of Definition 3.22.

[3]I would like to thank Andreas Demleitner for explaining this argument to me.

Corollary 3.36. *Let G and H be semi-abelian varieties and let $f\colon G \to H$ be a morphism. Then f is uniquely determined by $f(0)$ and by its differential df_0 at the origin, or equivalently, by the pull-back of differential forms*

$$f^*\colon \Omega^1_H(H) \to \Omega^1_G(G).$$

Proof. We may assume that $f(0) = 0$, so that f is a morphism of algebraic groups by Fact 3.34. Moreover, since G and H are varieties over the algebraically closed field $\mathbb{C}$, the morphism f is uniquely determined by what it does on closed points [Har77, Proposition II.2.6]. So f is uniquely determined by its analytification, and its analytification is uniquely determined by the differential at the origin by Lemma 3.32. Finally, by Lemma 3.29, the differential at the origin df_0 is uniquely determined by the pull-back of global differential 1-forms. □

3. The Albanese of a smooth projective variety

In this section we recall the construction of the Albanese of a smooth projective variety, following mostly the exposition in [Bea96, §V]. In the construction, we will also need the following:

Lemma 3.37. *Let V be a finite dimensional real vector space and let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Complex conjugation induces an $\mathbb{R}$-linear isomorphism of $V_{\mathbb{C}}$, and we denote the image of an $\mathbb{R}$-linear subspace $W \subseteq V_{\mathbb{C}}$ by $\overline{W}$. Suppose that we have a decomposition $V_{\mathbb{C}} = W \oplus \overline{W}$ for some $\mathbb{R}$-linear subspace $W \subseteq V_{\mathbb{C}}$ and let $\Lambda \subseteq V$ be a lattice. Then the image of Λ in $V_{\mathbb{C}}$ projects onto a lattice in W.*

Proof. Let $\iota\colon V \to V_{\mathbb{C}}$ be the natural inclusion, given by $v \mapsto v \otimes 1$, and let $p\colon V_{\mathbb{C}} \to W$ be the projection induced by the decomposition $V_{\mathbb{C}} = W \oplus \overline{W}$. In light of Remark 3.19 it suffices to show that the composition $p \circ \iota\colon V \to W$ is an isomorphism. Since both spaces have the same dimension, it suffices to show that it is injective. So let $v \in V$ such that $p(v \otimes 1) = 0$. Then $v \otimes 1 \in \overline{W}$. But $v \otimes 1$ is invariant under complex conjugation, so $v \otimes 1 \in W$ as well. Therefore $v \otimes 1 = 0$, and since ι is injective, $v = 0$. □

Theorem 3.38 (cf. [Bea96, Theorem V.13])**.** *Let X be a smooth projective variety and let $x_0 \in X$ be a base point. Then there exists an abelian variety* $\mathrm{Alb}(X)$ *and a morphism*

$$\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$$

with the following universal property:

(1) *The morphism alb_X sends x_0 to 0.*
(2) *For any semi-abelian variety G and any morphism $f\colon X \to G$ sending x_0 to 0, there exists a unique morphism $h\colon \mathrm{Alb}(X) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}_X"] \arrow[dr, "f"'] & \mathrm{Alb}(X) \arrow[d, "h"] \\ & G. \end{tikzcd}$$

The abelian variety $\mathrm{Alb}(X)$*, determined up to isomorphism by this universal property, is called the* Albanese variety *of* X*. The morphism* alb_X *induces an isomorphism*

$$\mathrm{alb}_X^*\colon \Omega^1_{\mathrm{Alb}(X)}(\mathrm{Alb}(X)) \to \Omega^1_X(X).$$

Proof. We will follow the argument given in [Bea96, Theorem V.13]. Let ι be the function $\pi_1(X, x_0) \to \Omega^1_X(X)^\vee$ defined by

$$\iota([\gamma])(\sigma) = \int_\gamma \sigma$$

for all $[\gamma] \in \pi_1(X, x_0)$ and all $\sigma \in \Omega^1_X(X)$, which is well-defined because holomorphic differential forms are closed. Let $\Lambda \subseteq \Omega^1_X(X)^\vee$ denote the image of ι. Since X is projective, the $\mathbb{C}$-vector space $\Omega^1_X(X)^\vee$ is finite dimensional. We claim that $\Lambda \subseteq \Omega^1_X(X)^\vee$ is a complete lattice. It follows from the universal property of the abelianization that Λ is also the image of the induced morphism $H_1(X, \mathbb{Z}) \to \Omega^1_X(X)^\vee$. Using Serre duality and Remark 3.19 we may replace $\Omega^1_X(X)^\vee$ by $H^n(X, \Omega^{n-1}_X)$. The universal coefficient theorem and the Hodge decomposition imply that

$$H^{2n-1}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^{2n-1}(X, \mathbb{C}) = H^n(X, \Omega^{n-1}_X) \oplus \overline{H^n(X, \Omega^{n-1}_X)}.$$

By the universal coefficient theorem and Poincaré duality, the image of $H_1(X, \mathbb{Z})$ in $H^{2n-1}(X, \mathbb{R})$ is a complete lattice, so the claim follows from Lemma 3.37.

Therefore, we can consider the complex Lie group

$$\mathrm{Alb}(X) := \Omega^1_X(X)^\vee/\Lambda,$$

which is a torus in the sense of Definition 3.22. Projectivity of X implies that $\mathrm{Alb}(X)$ is polarizable [Voi02, Corollary 12.12], so $\mathrm{Alb}(X)$ is an abelian variety by Remark 3.24.

Let us now construct the morphism $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$. For $x \in X$, let γ_x be a path joining x_0 to x. We set

$$\mathrm{alb}_X(x) := \left(\sigma \mapsto \int_{\gamma_x} \sigma\right) + \Lambda.$$

The value $\mathrm{alb}_X(x)$ does not depend on the choice of γ_x, because if δ_x is another path joining x_0 to x, then $\delta_x = \gamma_x \cdot \gamma$ for some loop γ based at x_0, so

$$\left(\sigma \mapsto \int_{\delta_x} \sigma\right) = \left(\sigma \mapsto \int_{\gamma_x} \sigma + \int_\gamma \sigma\right) = \left(\sigma \mapsto \int_{\gamma_x} \sigma\right) + \iota([\gamma]).$$

Therefore, we get a well-defined function $\mathrm{alb}_X\colon X \longrightarrow \mathrm{Alb}(X)$. We show that it is a holomorphic function. Let $\sigma_1, \dots, \sigma_q$ be a basis of $\Omega^1_X(X)$ over $\mathbb{C}$. With respect to the dual basis, we can write alb_X as

$$x \longmapsto \left(\int_{\gamma_x} \sigma_1, \dots, \int_{\gamma_x} \sigma_q \right) + \Lambda.$$

Let $U \subseteq X$ be an analytic-open neighborhood of x in X biholomorphic to a ball B of radius 1 around 0 in $\mathbb{C}^n$, say via a biholomorphic map $\varphi\colon U \longrightarrow B$, with $\varphi(x) = 0$. For any $y \in U$ we let s_y be the preimage in U of the path sending $t \in [0,1]$ to $t\varphi(y)$, i.e., the preimage in U of the segment joining the images of x and y in $\mathbb{C}^n$. After possibly shrinking U a bit, we may write $(\mathrm{alb}_X)|_U$ as the composition $\pi \circ a$, where $\pi\colon \Omega^1_X(X)^\vee \longrightarrow \mathrm{Alb}(X)$ is the (holomorphic) quotient morphism and $a\colon U \longrightarrow \Omega^1_X(X)^\vee$ is given with respect to the dual basis above by

$$y \longmapsto \left(\int_{\gamma_x} \sigma_1 + \int_{s_y} \sigma_1, \dots, \int_{\gamma_x} \sigma_q + \int_{s_y} \sigma_q \right).$$

Therefore it suffices to show that the map $U \longrightarrow \Omega^1_X(X)^\vee$ given by

$$y \longmapsto \left(\int_{s_y} \sigma_1, \dots, \int_{s_y} \sigma_q \right)$$

is holomorphic, for which in turn it suffices to show that for every holomorphic 1-form σ on B, the map $B \longrightarrow \mathbb{C}$ given by

$$z \longmapsto \int_{\overline{0z}} \sigma$$

is holomorphic, where $\overline{0z}$ denotes the path $t \longmapsto tz$ for $t \in [0,1]$. Let $z_1, \dots, z_n$ be holomorphic coordinates on B around 0 and let σ be a holomorphic 1-form on B. By complex analyticity of holomorphic functions [Voi02, Theorem 1.17] and linearity of integrals we may assume that $\sigma = z_1^{i_1} \cdots z_n^{i_n} dz_1$ for some $i_1, \dots, i_n \in \mathbb{N}$. But then we have

$$(z_1, \dots, z_n) \longmapsto \int_{\overline{0z}} \sigma = \int_0^1 (tz_1)^{i_1} \cdots (tz_n)^{i_n} z_1 dt = \frac{z_1^{i_1+1} \cdots z_n^{i_n}}{i_1 + \cdots + i_n + 1}.$$

Therefore $\mathrm{alb}_X\colon X \longrightarrow \mathrm{Alb}(X)$ is holomorphic, and since X is projective, it follows that alb_X is a morphism of algebraic varieties [Ser56, Proposition 15].

We show next that $\mathrm{alb}_X^*\colon \Omega^1_{\mathrm{Alb}(X)}(\mathrm{Alb}(X)) \longrightarrow \Omega^1_X(X)$ is an isomorphism. Let $\delta\colon \Omega^1_X(X)^{\vee\vee} \longrightarrow \Omega^1_{\mathrm{Alb}(X)}(\mathrm{Alb}(X))$ be the isomorphism from Remark 3.30 and let $j\colon \Omega^1_X(X) \longrightarrow \Omega^1_X(X)^{\vee\vee}$ be the canonical isomorphism. For $\sigma \in \Omega^1_X(X)$, we will also denote $(-)(\sigma) := j(\sigma)$. Since $\delta \circ j$ is an isomorphism, it suffices to show that the following diagram commutes:

$$\begin{tikzcd} \Omega^1_X(X) \arrow[r, "\delta\circ j"] \arrow[dr, equal] & \Omega^1_{\mathrm{Alb}(X)}(\mathrm{Alb}(X)) \arrow[d, "\mathrm{alb}_X^*"] \\ & \Omega^1_X(X). \end{tikzcd}$$

It suffices to show this analytic-locally on $\mathrm{Alb}(X)$, so we may assume again that we are over a small enough analytic-open subset U with a factorization $\mathrm{alb}_X = \pi \circ a$ as above. Over U, π restricts to the inverse of a standard complex chart. So for $\sigma \in \Omega^1_X(X)$ we have

$$\mathrm{alb}_X^*(\delta j(\sigma)) = a^*\pi^*(\delta((-)(\sigma))) = a^*d((-)(\sigma)),$$

because $\delta\alpha$ was defined by pulling back $d\alpha$ along the local chart inverse to the corresponding restriction of π for all linear forms α, cf. Lemma 3.29. The dual basis of $\sigma_1, \ldots, \sigma_q$ yields an identification $\Omega^1_X(X)^\vee \cong \mathbb{C}^q$ as in the proof of Lemma 3.29. Let $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$ such that $\sigma = \sum_{i=1}^q \lambda_i\sigma_i$. Under this identification we have

$$d((-)(\sigma)) = \sum_{i=1}^q \lambda_i dx_i.$$

It follows from the description of $a\colon U \longrightarrow \Omega^1_X(X)^\vee$ above that for all $i \in \{1, \ldots, q\}$ we have

$$a^*dx_i = d\left(\int_{\overline{0z}} \sigma_i\right) = \sigma_i$$

under the identifications of U with B and $\Omega^1_X(X)^\vee$ with $\mathbb{C}^q$. Therefore, for all $y \in U$ we have

$$a^*d((-)(\sigma)) = \sigma,$$

hence $\mathrm{alb}_X^*(\delta(j(\sigma))) = \sigma$.

It remains to show that $\mathrm{Alb}(X)$ satisfies the desired universal property. Let $f\colon X \longrightarrow G$ be a morphism to a semi-abelian variety sending x_0 to 0, with $G = V/\Lambda'$ with V a finite dimensional complex vector space and $\Lambda' \subseteq V$ a lattice. Any factorization $h\colon \mathrm{Alb}(X) \to G$ would be a morphism between semi-abelian varieties sending $0 \in \mathrm{Alb}(X)$ to $f(x_0) = 0$, hence a Lie group homomorphism by Fact 3.34. By Lemma 3.32, if h existed, then the following diagram should commute:

$$\begin{tikzcd} \Omega^1_X(X)^\vee \arrow[r, "dh_0"] \arrow[d, "\pi"'] & V \arrow[d, "\pi_G"] \\ \mathrm{Alb}(X) \arrow[r, "h"] & G. \end{tikzcd}$$

As pointed out in Corollary 3.36, h is uniquely determined by h^*. But alb_X^* is an isomorphism, so $h^* = (\mathrm{alb}^*)^{-1} \circ f^*$ is in turn uniquely determined by f^*. Hence such a factorization would be unique.

To show the existence of the factorization, we need to check that $dh_0(\Lambda) \subseteq \Lambda'$. Let $(dh_0)^\vee$ denote the transpose of the differential at the origin. That is, writing $\langle \alpha, v\rangle$ for the evaluation of a linear form α at a vector v, we have

$$\langle -, dh_0(-)\rangle = \langle (dh_0)^\vee(-), -\rangle.$$

Under the identifications of Lemma 3.29 we would obtain the following commutative square:

$$\begin{tikzcd} V^\vee \arrow[r, "(dh_0)^\vee"] \arrow[d, "\delta_G"'] & \Omega^1_X(X)^{\vee\vee} \arrow[d, "\delta"] \\ \Omega^1_G(G) \arrow[r, "h^*"] & \Omega^1_{\mathrm{Alb}(X)}(\mathrm{Alb}(X)). \end{tikzcd}$$

Let $[\gamma] \in \pi_1(X, x_0)$. We want to show that $dh_0(\iota([\gamma])) \in \Lambda'$, i.e., that there exists some $\lambda' \in \Lambda'$ with $dh_0(\iota([\gamma])) = \lambda'$. It suffices to show that there exists some $\lambda' \in \Lambda'$ such that for all $\alpha \in V^\vee$ we have

$$\langle \alpha, dh_0(\iota([\gamma]))\rangle = \langle \alpha, \lambda'\rangle.$$

Since $f^* = \mathrm{alb}_X^* \circ h^*$ and $(\mathrm{alb}_X^*)^{-1} = \delta \circ j$, we have $h^* = \delta \circ j \circ f^*$. Therefore we have

$$\begin{aligned} \langle \alpha, dh_0(\iota([\gamma]))\rangle &= \langle (dh_0)^\vee(\alpha), \iota([\gamma])\rangle \\ &= \langle \delta^{-1} \circ h^* \circ \delta_G(\alpha), \iota([\gamma])\rangle \\ &= \langle j \circ f^* \circ \delta_G(\alpha), \iota([\gamma])\rangle \\ &= \iota([\gamma])(f^* \circ \delta_G(\alpha)) \\ &= \int_\gamma f^*(\delta_G \alpha) \\ &= \int_{f\circ\gamma} \delta_G \alpha. \end{aligned}$$

Moreover, by Lemma 3.31 we have

$$\int_{f\circ\gamma} \delta_G \alpha = \langle \alpha, h_G^{-1}([f \circ \gamma])\rangle$$

and $h_G^{-1}([f\circ\gamma]) \in \Lambda'$. Therefore, $dh_0(\Lambda) \subseteq \Lambda'$ and dh_0 induces the desired morphism $h\colon \mathrm{Alb}(X) \to G$. □

Example 3.39. Let X be an abelian variety, and pick 0 as the base point. Then, the universal property shows that the identity on X is an Albanese morphism.

Example 3.40. Consider the projective space $\mathbb{P}^n$ for some $n \in \mathbb{N}$ and let $x_0 \in \mathbb{P}^n$ be a base point. Then $\Omega^1_{\mathbb{P}^n}(\mathbb{P}^n) = 0$, so the construction of the Albanese in Theorem 3.38 shows that $\mathrm{Alb}(\mathbb{P}^n) = \mathrm{Spec}(\mathbb{C})$ is a point. The same applies to any smooth projective variety X with $h^{1,0} = 0$.

Proposition 3.41 (cf. [Bea96, Remarks V.14]). *Let $f: X \to Y$ be a morphism of smooth projective varieties, let $x_0 \in X$ be a base point and consider the base point $y_0 := f(x_0) \in Y$. Then there exists a unique morphism $g: \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{alb}_X"'] & Y \arrow[d, "\mathrm{alb}_Y"] \\ \mathrm{Alb}(X) \arrow[r, "g"] & \mathrm{Alb}(Y). \end{tikzcd}$$

In particular, $(X, x_0) \mapsto \mathrm{Alb}(X)$ defines a functor from the category of pointed smooth projective varieties to the category of abelian varieties with morphisms of algebraic groups. This functor is left adjoint to the forgetful functor that sends an abelian variety to the underlying smooth projective variety with the origin as base point.

Proof. The functoriality statement follows from the universal property in Theorem 3.38, using implicitly that any morphism of abelian varieties that sends the origin to the origin is a morphism of algebraic groups [Mil08, Corollary 1.2]. The fact that the resulting functor is left adjoint to the forgetful functor is just a way to rephrase the universal property in Theorem 3.38. □

To conclude this section, we show that the Albanese of a pointed smooth projective variety already serves as the adapted Albanese in a very particular case:

Example 3.42. Let X be a smooth projective variety and let $x_0 \in X$ be a base point. Consider the C-pair $(X, 0)$ and the adapted cover $\gamma = \mathrm{id}_X$. Then the abelian variety $\mathrm{Alb}(X)$ and the morphism $\mathrm{alb}_X: X \to \mathrm{Alb}(X)$ satisfy the universal property of the adapted Albanese for id_X, so that $\mathrm{Alb}(X, 0, \mathrm{id}_X) = \mathrm{Alb}(X)$ and $\mathrm{alb}_{(X,0,\mathrm{id}_X)} = \mathrm{alb}_X$. Indeed, we have $\Omega^{[1]}_{(X,0,\mathrm{id}_X)}(X) = \Omega^1_X(X)$, so Item 2 in Theorem 3.4 becomes vacuous.

4. The Albanese of a smooth projective log pair

In this section we recall Iitaka's construction of the Albanese of a smooth projective log pair [Iit76], which will also be a necessary ingredient and a particular case of the construction of the adapted Albanese, cf. Example 3.51.

Lemma 3.43 (cf. [Fuj15, Lemma 3.1]). *Let (X, Δ) be a smooth projective log pair, cf. Item 7 in Notation 0.7. Let $U := X \setminus \mathrm{Supp}(\Delta)$ and let $\iota: U \to X$ denote the inclusion. Then the pushforward map*

$$\iota_*: H_1(U, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

is surjective.

Proof. Let $n := \dim(X)$ and let $D := \mathrm{Supp}(\Delta)$. We may assume that $n > 0$. The pair (X, U) gives rise to a long exact sequence [Bre93, Example IV.5.7]

$$\cdots \longrightarrow H_1(U, \mathbb{Z}) \xrightarrow{\iota_*} H_1(X, \mathbb{Z}) \longrightarrow H_1(X, U; \mathbb{Z}) \longrightarrow \cdots$$

By Poincaré–Lefschetz Duality [Bre93, Corollary VI.8.4] we have

$$H_1(X, U; \mathbb{Z}) \cong H^{2n-1}(D, \mathbb{Z}).$$

So it suffices to show that $H^{2n-1}(D, \mathbb{Z}) = 0$. We proceed by induction on the number r of irreducible components of D, which we denote by $\{D_1, \ldots, D_r\}$. If $r = 1$, then $D = D_1$ is a smooth projective variety itself, and it has dimension $n-1$. So it is a compact manifold of dimension $2n-2$ and therefore $H^{2n-1}(D, \mathbb{Z}) = 0$. If $r > 1$, we consider $D' := D_1 \cup \ldots \cup D_{r-1}$, so that $D = D' \cup D_r$. The long exact sequence of the pair (D, D') yields

$$\cdots \longrightarrow H^{2n-1}(D, D'; \mathbb{Z}) \longrightarrow H^{2n-1}(D, \mathbb{Z}) \longrightarrow H^{2n-1}(D', \mathbb{Z}) \longrightarrow \cdots$$

By induction hypothesis, $H^{2n-1}(D', \mathbb{Z}) = 0$, so it suffices to show that $H^{2n-1}(D, D'; \mathbb{Z}) = 0$ as well. The analytification of a projective variety admits triangulations compatible with any given finite collection of closed subvarieties [Hir75], so we can apply excision

$$H^{2n-1}(D, D'; \mathbb{Z}) \cong H^{2n-1}(D_r, D' \cap D_r; \mathbb{Z})$$

and it suffices to show that $H^{2n-1}(D_r, D' \cap D_r; \mathbb{Z}) = 0$. The long exact sequence of the pair $(D_r, D' \cap D_r)$ yields

$$\cdots \longrightarrow H^{2n-2}(D' \cap D_r, \mathbb{Z}) \longrightarrow H^{2n-1}(D_r, D' \cap D_r; \mathbb{Z}) \longrightarrow H^{2n-1}(D_r, \mathbb{Z}) \longrightarrow \cdots$$

We have $H^{2n-1}(D_r, \mathbb{Z}) = 0$ as before, because D_r is a $2n-2$-dimensional compact manifold. Moreover, since $(D_r, D' \cap D_r)$ is a smooth projective log pair with $\dim(D_r) = n-1$ and $D' \cap D_r$ has less irreducible components than D, the induction hypothesis implies that

$$H^{2n-2}(D' \cap D_r, \mathbb{Z}) = 0.$$

Therefore $H^{2n-1}(D_r, D' \cap D_r; \mathbb{Z}) = 0$, and this finishes the proof. □

Lemma 3.44 ([Iit76, Proposition 2]). *Let (X, Δ) be a smooth projective log pair, let $U := X \setminus \mathrm{Supp}(\Delta)$ and let $\iota \colon U \to X$ denote the inclusion. Let $\sigma \in \Omega^1_X(\log \Delta)(X)$ be a differential form such that for any $\eta \in \ker(\iota_*) \subseteq H_1(U, \mathbb{Z})$ we have*

$$\int_\eta \sigma = 0.$$

Then $\sigma \in \Omega^1_X(X)$.

Proof. By definition, σ can have at worst simple poles along the irreducible components of Δ. We will show that σ does not have them. Let D

be an irreducible component of Δ. Let $z_1, \dots, z_n$ be analytic-local coordinates around a general point of D, and assume that $D = \{z_1 = 0\}$ in these coordinates. Then we can write

$$\sigma = \alpha \frac{dz_1}{z_1} + \beta,$$

where α is a holomorphic function and β is a holomorphic 1-form. By Weierstrass division theorem [Huy05, Proposition 1.1.17], we may assume that α does not depend on z_1. Indeed, the theorem ensures that, up to shrinking our coordinate domain a bit further, we can write $\alpha = z_1 h + r$ with r a holomorphic function which does not depend on z_1, so that

$$\alpha \frac{dz_1}{z_1} + \beta = h dz_1 + r \frac{dz_1}{z_1} + \beta$$

and we can replace β by $hdz_1 + \beta$ and α by r. Therefore, we may assume that α does not depend on z_1. Logarithmic forms are closed [Del71, Corollaire (3.2.14)], so we deduce that

$$0 = d\sigma = d\alpha \wedge \frac{dz_1}{z_1} + 0,$$

hence $d\alpha = 0$ and α must be constant. We show that $\alpha = 0$. For this, we consider a small loop γ around the center of our coordinate domain going once around D. This loop defines a 1-cycle that becomes null-homologous when regarded inside X, i.e., an element in $\ker(\iota_*)$. Then we have

$$0 = \int_\gamma \sigma = \alpha \int_\gamma \frac{dz_1}{z_1} + 0 = \alpha 2\pi\sqrt{-1},$$

so $\alpha = 0$ and σ does not have a pole along D. $\square$

Theorem 3.45 (cf. [Fuj15, Theorem 3.16]). *Let (X, Δ) be a smooth projective log pair, let $U := X \setminus \operatorname{Supp}(\Delta)$ and let $x_0 \in U$ be a base point. Then there exists a semi-abelian variety $\operatorname{Alb}(X, \Delta)$ and a morphism*

$$\operatorname{alb}_{(X,\Delta)} \colon U \longrightarrow \operatorname{Alb}(X, \Delta)$$

with the following universal property:

(1) *The morphism $\operatorname{alb}_{(X,\Delta)}$ sends x_0 to 0.*
(2) *For any semi-abelian variety G and any morphism $f \colon U \to G$ sending x_0 to 0, there exists a unique morphism $h \colon \operatorname{Alb}(X, \Delta) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "\operatorname{alb}_{(X,\Delta)}"] \arrow[dr, "f"'] & \operatorname{Alb}(X, \Delta) \arrow[d, "h"] \\ & G. \end{tikzcd}$$

The semi-abelian variety $\operatorname{Alb}(X,\Delta)$*, determined up to isomorphism by this universal property, is called the* Albanese variety *of* (X,Δ)*. The pull-back of logarithmic differential* 1*-forms along* $\mathrm{alb}_{(X,\Delta)}$ *is an isomorphism*

$$\mathrm{alb}^*_{(X,\Delta)} : T_1(\operatorname{Alb}(X,\Delta)) \to \Omega^1_X(\log\Delta)(X) = T_1(U).$$

Proof. Recall that the construction of $\operatorname{Alb}(X)$, discussed in Section 3, was as follows. Integration of holomorphic 1-forms along singular 1-cycles in X yields a group homomorphism $H_1(X,\mathbb{Z}) \to \Omega^1_X(X)^\vee$ given by

$$\xi \mapsto \left(\sigma \mapsto \int_\xi \sigma\right).$$

Let $\sigma_1,\dots,\sigma_q$ be a basis of $\Omega^1_X(X)$ over $\mathbb{C}$. The dual basis yields an isomorphism $\Omega^1_X(X)^\vee \cong \mathbb{C}^q$, and with respect to this identification the group homomorphism above is given by

$$\xi \mapsto \left(\int_\xi \sigma_1,\dots,\int_\xi \sigma_q\right).$$

Let $\Lambda \subseteq \mathbb{C}^q$ be the image of this group homomorphism. Fixing a basis $\xi_1,\dots,\xi_{2q}$ of the free part of $H_1(X,\mathbb{Z})$ over $\mathbb{Z}$, we can write explicitly

$$\Lambda = \left\{ \sum_{i=1}^{2q} m_i \left(\int_{\xi_i}\sigma_1,\dots,\int_{\xi_i}\sigma_q\right) \,\middle|\, m_1,\dots,m_{2q}\in\mathbb{Z}\right\} \subseteq \mathbb{C}^q.$$

Then, the Albanese of X is $\operatorname{Alb}(X) := \mathbb{C}^q/\Lambda$.

We enlarge the basis $\sigma_1,\dots,\sigma_q$ of $\Omega^1_X(X)$ to a basis $\sigma_1,\dots,\sigma_q,\tau_1,\dots,\tau_t$ of $\Omega^1_X(\log\Delta)(X)$ over $\mathbb{C}$. Let $\iota\colon U \to X$ denote the inclusion of the complement of the divisor. By Lemma 3.43 can choose $\eta_1,\dots,\eta_{t'} \in \ker(\iota_*)$ such that $\tilde\xi_1,\dots,\tilde\xi_{2q},\eta_1,\dots,\eta_{t'}$ forms a basis of the free part of $H_1(U,\mathbb{Z})$ over $\mathbb{Z}$, where $\tilde\xi_i$ is a preimage of ξ_i for each $i \in \{1,\dots,2q\}$. Moreover, Deligne's theory of mixed Hodge structures ensures that the complex dimension of $\Omega^1_X(\log\Delta)(X)/\Omega^1_X(X)$ equals the difference of Betti numbers $b_1(U)-b_1(X)$, cf. [Iit76, Proposition 2] and [Fuj15, Lemma 3.2]. Therefore, we have $t' = t$. For each $i \in \{1,\dots,2q\}$ we consider the vector

$$A_i := \left(\int_{\xi_i}\sigma_1,\dots,\int_{\xi_i}\sigma_q,\int_{\tilde\xi_i}\tau_1,\dots,\int_{\tilde\xi_i}\tau_t\right) \in \mathbb{C}^{q+t};$$

and for each $j \in \{1,\dots,t\}$ we consider the vector

$$B_j := \left(\int_{\eta_j}\iota^*\sigma_1,\dots,\int_{\eta_j}\iota^*\sigma_q,\int_{\eta_j}\tau_1,\dots,\int_{\eta_j}\tau_t\right) \in \mathbb{C}^{q+t}.$$

Note that the first q-components of each B_j vanish, because $\eta_j \in \ker(\iota_*)$ and

$$\int_{\eta_j}\iota^*\sigma = \int_{\iota_*\eta_j}\sigma = 0$$

for all $\sigma \in \Omega^1_X(X)$. Moreover, we may choose the $\tau_1, \ldots, \tau_t$ above so that

$$\int_{\eta_j} \tau_i = 2\pi\sqrt{-1}\delta_{ij}$$

for all $i, j \in \{1, \ldots, t\}$. Indeed, let W be the $\mathbb{C}$-vector subspace of $\Omega^1_X(\log \Delta)(X)$ generated by the τ_i, and let M be the subgroup of $\ker(\iota_*)$ generated by the η_j. Let $M_{\mathbb{C}} := M \otimes_{\mathbb{Z}} \mathbb{C}$. Integration induces a pairing as follows:

$$\Phi\colon W \times M_{\mathbb{C}} \to \mathbb{C}.$$

We show that this is a perfect pairing. Since the vector spaces involved are of finite dimension, it suffices to show that it is non-degenerate. Let $\tau \in W$ such that $\Phi(\tau, \eta) = 0$ for all $\eta \in M_{\mathbb{C}}$. In particular, $\Phi(\tau, \eta_j) = 0$ for all $j \in \{1, \ldots, t\}$. Note that $\eta_1, \ldots, \eta_t$ generate the free part of $\ker(\iota_*)$, because $\iota_*\tilde{\xi}_i = \xi_i \neq 0$ for all $i \in \{1, \ldots, 2q\}$. Therefore $\Phi(\tau, \eta) = 0$ for all η in the free part of $\ker(\iota_*)$. The integral of a differential form along a torsion cycle must vanish anyway, so from Lemma 3.44 we deduce that $\tau \in \Omega^1_X(X)$. But since τ is in the subspace generated by the τ_i, this implies that $\tau = 0$. So the pairing is perfect, as claimed. Therefore, we may replace the original τ_i's by the dual basis of the η_j's scaled by a factor of $2\pi\sqrt{-1}$, and this proves the original claim.

We set now

$$\hat{A}_i := \left(\int_{\xi_i} \sigma_1, \ldots, \int_{\xi_i} \sigma_q\right)$$

for each $i \in \{1, \ldots, 2q\}$ and

$$\hat{B}_j := \left(\int_{\eta_j} \tau_1, \ldots, \int_{\eta_j} \tau_t\right)$$

for each $j \in \{1, \ldots, t\}$. The subgroup $\Lambda := \sum_{i=1}^{2q} \mathbb{Z}\hat{A}_i \subseteq \mathbb{C}^q$ is a complete lattice and has $\mathbb{C}^q/\Lambda = \mathrm{Alb}(X)$, as discussed in Section 3 and recalled above. On the other hand, setting $\Lambda_0 := \sum_{j=1}^{t} \mathbb{Z}\hat{B}_j \subseteq \mathbb{C}^t$, the complex exponential yields a group isomorphism $\mathbb{C}^t/\Lambda_0 \cong (\mathbb{G}_m)^{\times t}$. Finally, we consider the subgroup $\tilde{\Lambda} := \sum_{i=1}^{2q} \mathbb{Z}A_i + \sum_{j=1}^{t} \mathbb{Z}B_j \subseteq \mathbb{C}^{q+t}$. This subgroup is discrete, because the discussion above implies that it is generated by $\mathbb{R}$-linearly independent vectors in a finite dimensional real vector space. Moreover, from Λ being a complete lattice and the discussion about the perfect pairing above we also deduce that $\tilde{\Lambda}$ generates $\mathbb{C}^{q+t}$ as a complex vector space, so the quotient $\mathrm{Alb}(X, \Delta) := \mathbb{C}^{q+t}/\tilde{\Lambda}$ is a semi-torus by Fact 3.26. It fits into the following short exact sequence of complex Lie groups:

$$1 \to (\mathbb{G}_m)^{\times t} \to \mathrm{Alb}(X, \Delta) \to \mathrm{Alb}(X) \to 1,$$

in which $\mathrm{Alb}(X)$ is an abelian variety. It is shown in [Fuj15, Lemma 3.8] that $\mathrm{Alb}(X, \Delta)$ is an algebraic group and the short exact sequence is a short exact sequence of algebraic groups, so $\mathrm{Alb}(X, \Delta)$ is a semi-abelian variety.

We describe now the morphism $\mathrm{alb}_{(X,\Delta)}$. Let $\sigma \in \Omega^1_X(\log\Delta)(X)$ be a logarithmic 1-form. If we regard σ as a regular differential form on $\Omega^1_U(U)$ by restricting it to this open subset, then $d\sigma = 0$ by [Del71, Corollaire (3.2.14)]. So if $x \in U$ is another point, then Stokes' theorem implies that the value of the integral

$$\int_{x_0}^{x} \sigma$$

only depends on the path from x_0 to x up to homotopy in U. We can thus proceed similarly as in Section 3 and define $\mathrm{alb}_{(X,\Delta)} \colon U \longrightarrow \mathrm{Alb}(X,\Delta)$ by the formula

$$\mathrm{alb}_{(X,\Delta)}(x) := \left(\int_{x_0}^{x} \sigma_1, \ldots, \int_{x_0}^{x} \sigma_q, \int_{x_0}^{x} \tau_1, \ldots, \int_{x_0}^{x} \tau_t\right) + \tilde{\Lambda}.$$

This function satisfies $\mathrm{alb}_{(X,\Delta)}(x_0) = 0$ and it is an algebraic morphism [Fuj15, Lemma 3.10], so Item 1 in Theorem 3.45 holds.

The isomorphism on logarithmic differential forms can be shown again with Remark 3.30 as in Theorem 3.38 after passing to a suitable compactification of $\mathrm{Alb}(X,\Delta)$, say, as in [NW14, Proposition 5.3.4]. Lemma 3.29 does not apply directly anymore, but [NW14, Proposition 5.4.3] shows that the cotangent bundle of the compactification can be trivialized in way which is compatible with the trivialization in Lemma 3.29. See also [Fuj15, Lemma 3.12] for a proof using algebraic topology and mixed Hodge structures.

It remains to show that Item 2 in Theorem 3.45 holds. The desired morphism can be constructed analogously to the one in Theorem 3.38. We refer to [Fuj15, Lemma 3.14] for a proof that the resulting morphism is algebraic. □

Generalizing Example 3.39, we have the following:

Example 3.46. Let G be a semi-abelian variety, and pick 0 as the base point. Then, the universal property shows that the identity on X is an Albanese morphism.

Example 3.47. Consider the affine line $\mathbb{A}^1$ with its origin as base point. The only possible smooth compactification is $(\mathbb{P}^1, \Delta)$, where Δ is the point at infinity. Since $\Omega^1_{\mathbb{P}^1}(\log\Delta)(\mathbb{P}^1) = 0$, we have $\mathrm{Alb}(\mathbb{P}^1, \Delta) = \mathrm{Spec}(\mathbb{C})$.

Remark 3.48. The Albanese of a smooth projective log pair does not depend on the boundary divisor, but only on its complement, cf. [Fuj15] or [Iit76]. For this reason, we will sometimes write $\mathrm{Alb}(U)$ instead of $\mathrm{Alb}(X,\Delta)$ and alb_U instead of $\mathrm{alb}_{(X,\Delta)}$ in the setting of Theorem 3.45.

As in the case of the Albanese of a smooth projective variety, we have the following functoriality:

Proposition 3.49 (cf. [Fuj15, Lemma 3.15]). *The assignment* $(U, x_0) \mapsto \mathrm{Alb}(U)$ *defines a functor from the category of pointed smooth varieties to* *the category of semi-abelian varieties with morphisms of algebraic groups. This functor is left adjoint to the forgetful functor that sends a semi-abelian variety to the underlying smooth variety with the origin as base point.*

Remark 3.50 ([Fuj15, Corollary 3.17])**.** Combining Proposition 3.49 with Example 3.47 and Example 3.46 we deduce that every morphism from $\mathbb{A}^1$ to a semi-abelian variety G is constant. Note that this is just a reality check and no new information, because one can show (without the Albanese) that an algebraic group is semi-abelian if and only if it is connected and commutative and every unipotent algebraic subgroup is trivial [NW14, Proposition 5.1.23].

Finally, generalizing Example 3.42, we have the following:

Example 3.51. Let (X, Δ) be a smooth projective log pair and let $x_0 \in X \setminus \operatorname{Supp}(\Delta)$ be a base point. The identity id_X is an adapted cover with respect to the $\mathcal{C}$-pair (X, Δ), and we can take $\operatorname{Alb}(X, \Delta, \mathrm{id}_X) = \operatorname{Alb}(X, \Delta)$ and $\mathrm{alb}_{(X,\Delta,\mathrm{id}_X)} = \mathrm{alb}_{(X,\Delta)}$, constructed with respect to the base point x_0. The reason is again that Item 2 in Theorem 3.4 is automatically satisfied for any morphism into a semi-abelian variety.

5. The Albanese for a subspace of differentials

The last main ingredient for the construction of the adapted Albanese is the construction of the Albanese with respect to a subspace of (logarithmic) differential forms, which is also not a new construction. We recall the details here following the exposition in [Zuo99, §4.2].

We will need the following observation from linear algebra:

Lemma 3.52. *Let $f\colon V \to W$ be a linear map between vector spaces over a field, and let $f^\vee\colon W^\vee \to V^\vee$ denote the dual map. Then we have*

$$\ker(f^\vee) = \operatorname{im}(f)^\perp := \{\beta \in W^\vee \mid \beta(u) = 0 \text{ for all } u \in \operatorname{im}(f)\}.$$

Proof. Let $\beta \in \ker(f^\vee)$ and let $u = f(v) \in \operatorname{im}(f)$. Then we have

$$\beta(u) = \beta(f(v)) = f^\vee(\beta)(v) = 0,$$

so $\beta \in \operatorname{im}(f)^\perp$. And conversely, if $\beta \in \operatorname{im}(f)^\perp$ and $v \in V$, then

$$f^\vee(\beta)(v) = \beta(f(v)) = 0,$$

so $\beta \in \ker(f^\vee)$. □

We will also need the following observation:

Lemma 3.53. *Let $f\colon X \to Y$ be a morphism between varieties. If the pullback of Kähler differentials $f^*\Omega_Y \to \Omega_X$ is zero (as a morphism of sheaves), then f is a constant morphism.*

Proof. We have an exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0,$$

so the assumption implies that

$$\Omega_X \cong \Omega_{X/Y}.$$

Therefore, by [Har77, Theorem II.8.6.A], the fiber over the image over any closed point in X has dimension equal to the dimension of X, so it must be all of X by irreducibility.

An alternative argument, using heavier machinery but perhaps more geometric, is the following. By continuity, it suffices to show that the restriction of f to some dense open subset is constant, so we may assume that X is smooth. The image of f is irreducible, so its closure is irreducible as well. Replacing Y by the closure of the image with its reduced subscheme structure, we may assume that f is dominant. By generic smoothness [Har77, Corollary III.10.7], we may then assume that f is smooth. So it suffices to show the assertion for smooth dominant morphisms between smooth varieties. The isomorphism above shows that it is a smooth morphism of relative dimension equal to the dimension of X [Har77, Proposition III.10.4], so $\dim(Y) = 0$. □

In particular, we will use the following consequence of Lemma 3.53:

Lemma 3.54. *Let $f\colon X \to Y$ be a morphism between varieties. Assume that Ω_Y is globally generated and that the morphism*

$$f^*\colon \Omega_Y(Y) \to \Omega_X(X)$$

induced by the composition $\Omega_Y \to f_*f^*\Omega_Y \to f_*\Omega_X$ [Iit82, §5.4.b] is zero. Then f is a constant morphism.

Proof. Let $Z \subseteq Y$ be the closure of the image of f, with its induced reduced subscheme structure and let $j\colon Z \to Y$ denote the closed immersion. Then Ω_Z is globally generated. Indeed, Ω_Z being globally generated follows from the exact sequence [Har77, Proposition II.8.12]

$$j^*\Omega_Y \to \Omega_Z \to 0,$$

Ω_Y being globally generated and pull-backs being right exact. Therefore Ω_Z is again globally generated.

Let $f_Z\colon X \to Z$ be the induced morphism, so that $f = j \circ f_Z$. We claim that $f_Z^* = 0$ as well. Indeed, since Ω_Y is globally generated, the assumption implies that the composition

$$\Omega_Y \to f_*f^*\Omega_Y \to f_*\Omega_X$$

is zero as a sheaf morphism. Therefore, the composition

$$j^*\Omega_Y \to j^*f_*f^*\Omega_Y \to j^*f_*\Omega_X$$

is also zero. Since j is a closed immersion, we have a natural isomorphism $j^*j_* \cong \mathrm{id}$. Combining this with functoriality, we deduce that the following composition is the zero sheaf morphism:

$$j^*\Omega_Y \to (f_Z)_*f_Z^*j^*\Omega_Y \to (f_Z)_*\Omega_X.$$

We have the following commutative diagram:

$$\begin{tikzcd}
j^*\Omega_Y \arrow[r] \arrow[d, two heads] & (f_Z)_* f_Z^* j^* \Omega_Y \arrow[r] \arrow[d] & (f_Z)_*\Omega_X \arrow[d, equal] \\
\Omega_Z \arrow[r] & (f_Z)_* f_Z^* \Omega_Z \arrow[r] & (f_Z)_*\Omega_X.
\end{tikzcd}$$

Since the top row is zero and the left vertical arrow is surjective, the bottom row is zero as well. Therefore, $f_Z^* \colon \Omega_Z(Z) \to \Omega_X(X)$ is zero as well.

As in the proof of Lemma 3.53, we may assume that Z is smooth. But then the morphism $\Omega_Z \to (f_Z)_* f_Z^* \Omega_Z$ is injective on global sections [Iit82, Lemma 2.35], so $f_Z^* = 0$ implies that $f_Z^*\Omega_Z \to \Omega_X$ is zero on global sections. Since pull-back is right exact, $f_Z^*\Omega_Z$ is also globally generated. Therefore, the morphism $f_Z^*\Omega_Z \to \Omega_X$ is zero and f_Z is constant by Lemma 3.53. ☐

Following the ideas in [Zuo99, §4.2], we construct the Albanese with respect to a subspace of differentials:

Proposition 3.55. *Let (X, Δ) be a smooth projective log pair, let $U := X \setminus \operatorname{Supp}(\Delta)$ and let $x_0 \in U$ be a base point. Let $V \subseteq \Omega_X^1(\log \Delta)(X) = T_1(U)$ be a $\mathbb{C}$-vector subspace. Then there exists a semi-abelian variety $\operatorname{Alb}(X, \Delta, V)$ and a morphism*

$$\operatorname{alb}_{(X,\Delta,V)} \colon U \to \operatorname{Alb}(X, \Delta, V)$$

with the following universal property:

1. *The morphism $\operatorname{alb}_{(X,\Delta,V)}$ sends x_0 to 0.*
2. *The pull-back of logarithmic differential 1-forms*
$$\operatorname{alb}^*_{(X,\Delta,V)} \colon T_1(\operatorname{Alb}(X, \Delta, V)) \to T_1(U)$$
has image contained in V.
3. *For any semi-abelian variety G and any morphism $f \colon U \to G$ sending x_0 to 0 and such that the pull-back of logarithmic differential 1-forms*
$$f^* \colon T_1(G) \to T_1(U)$$
has image contained in V, there exists a unique morphism $h\colon \operatorname{Alb}(X, \Delta, V) \to G$ such that the following diagram commutes:
$$\begin{tikzcd}
U \arrow[r, "\operatorname{alb}_{(X,\Delta,V)}"] \arrow[dr, "f"'] & \operatorname{Alb}(X, \Delta, V) \arrow[d, "h"] \\
& G.
\end{tikzcd}$$

The semi-abelian variety $\operatorname{Alb}(X, \Delta, V)$, determined up to isomorphism by this universal property, is called the Albanese variety *of the tuple (X, Δ, V).*

Proof. Let $\operatorname{alb}_{(X,\Delta)} \colon U \to \operatorname{Alb}(X, \Delta)$ be the Albanese of the smooth projective log pair (X, Δ), as constructed in Section 4, so that $\operatorname{Alb}(X, \Delta)$ is the quotient of the complex vector space $\Omega_X^1(\log \Delta)(X)^\vee$ by a discrete subgroup. We consider the annihilator subspace

$$V^\perp = \{\alpha \in \Omega_X^1(\log \Delta)(X)^\vee \mid \alpha(\sigma) = 0 \text{ for all } \sigma \in V\}$$

and we let $H \subseteq \mathrm{Alb}(X, \Delta)$ be the smallest algebraic subgroup that contains the image of $V^\perp$ in the quotient. Note that H has to be connected, because the image of $V^\perp$ is connected, so it is contained in the connected component of the identity of any algebraic subgroup containing it. Therefore, H is again a semi-abelian variety and so is the quotient

$$\mathrm{Alb}(X, \Delta, V) := \mathrm{Alb}(X, \Delta)/H,$$

see [NW14, Proposition 5.1.26]. Take $\mathrm{alb}_{(X,\Delta,V)}$ to be the composition

$$U \xrightarrow{\mathrm{alb}_{(X,\Delta)}} \mathrm{Alb}(X, \Delta) \xrightarrow{q} \mathrm{Alb}(X, \Delta, V),$$

where q is the quotient morphism. By construction, the morphism $\mathrm{alb}_{(X,\Delta,V)}$ sends x_0 to 0, so it remains to show that

$$\mathrm{alb}_{(X,\Delta,V)} \colon U \longrightarrow \mathrm{Alb}(X, \Delta, V)$$

satisfies Items 2 and 3 in Proposition 3.55.

We check first that the property in Item 2 holds. We want to check that the pull-back of logarithmic differentials with respect to a suitable compactification has image contained in V. Let $\Lambda \subseteq \Omega^1_X(\log \Delta)(X)^\vee$ denote the lattice such that $\mathrm{Alb}(X, \Delta) = \Omega^1_X(\log \Delta)(X)^\vee/\Lambda$, which was called $\tilde{\Lambda}$ in Section 4. Then, the image of $V^\perp$ in $\mathrm{Alb}(X, \Delta)$ is the complex Lie group $A := V^\perp/\Lambda'$, where $\Lambda' := \Lambda \cap V^\perp$ is also a lattice in the finite dimensional complex vector space $V^\perp$. Let $\iota\colon A \longrightarrow \mathrm{Alb}(X, \Delta)$ denote the inclusion. By Lemma 3.29 and Lemma 3.32 we have the following commutative square:

$$\begin{tikzcd}
V^\perp \arrow[r, hook, "d\iota_0"] \arrow[d, "\exp"'] & \Omega^1_X(\log \Delta)(X)^\vee \arrow[d, "\exp"] \\
A \arrow[r, hook, "\iota"] & \mathrm{Alb}(X, \Delta).
\end{tikzcd}$$

Under the identifications of Lemma 3.29, as in the proof of Theorem 3.38, we have the following commutative square:

$$\begin{tikzcd}
\Omega^1_X(\log \Delta)(X)^{\vee\vee} \arrow[r, "(d\iota_0)^\vee"] \arrow[d, "\delta"'] & (V^\perp)^\vee \arrow[d, "\delta_A"] \\
T_1(\mathrm{Alb}(X, \Delta)) \arrow[r, "\iota^*"] & T_1(A).
\end{tikzcd}$$

Here ι^* denotes the pull-back of logarithmic differential forms and the vertical arrows are the isomorphisms from Remark 3.30. By Lemma 3.52 we have $\ker((d\iota_0)^\vee) = V^{\perp\perp}$, therefore

$$\ker(\iota^*) = (\delta^{-1})^{-1}(V^{\perp\perp}) = \delta(V^{\perp\perp}).$$

By construction, the composition

$$A \xrightarrow{\iota} \mathrm{Alb}(X, \Delta) \xrightarrow{q} \mathrm{Alb}(X, \Delta, V)$$

is constant. Therefore we have $(q \circ \iota)^* = \iota^* \circ q^* = 0$, so $\mathrm{im}(q^*) \subseteq \ker(\iota^*) = \delta(V^{\perp\perp})$. Denoting by $j\colon \Omega^1_X(\log \Delta)(X) \to \Omega^1_X(\log \Delta)(X)^{\vee\vee}$ the canonical isomorphism, we have $\mathrm{alb}^*_{(X,\Delta)} = j^{-1} \circ \delta^{-1}$ as in the proof of Theorem 3.38. Therefore, we have

$$\mathrm{im}(\mathrm{alb}^*_{(X,\Delta,V)}) = \mathrm{im}(\mathrm{alb}^*_{(X,\Delta)}\, q^*) = \mathrm{im}(j^{-1}\delta^{-1}q^*) \subseteq j^{-1}\delta^{-1}\delta(V^{\perp\perp}) = V.$$

We check next that the property in Item 3 holds. Let G be a semi-abelian variety and let $f\colon U \to G$ be a morphism sending x_0 to 0 such that the image of

$$f^*\colon T_1(G) \to \Omega^1_X(\log \Delta)(X)$$

is contained in V. The universal property of $\mathrm{Alb}(X,\Delta)$ ensures the existence of a unique morphism $\tilde{h}\colon\ \mathrm{Alb}(X,\Delta) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\mathrm{alb}_{(X,\Delta)}"] \arrow[dr, "f"'] & \mathrm{Alb}(X,\Delta) \arrow[d, "\tilde{h}"] \\ & G. \end{tikzcd}$$

We check that $\tilde{h}$ factors uniquely as $\tilde{h} = h \circ q$ for some $h\colon\ \mathrm{Alb}(X,\Delta,V) \to G$. Since $\tilde{h}$ preserves the neutral element, it is a group morphism by [NW14, Theorem 5.1.37]. Thus it suffices to check that $H \subseteq \ker(\tilde{h})$ inside $\mathrm{Alb}(X,\Delta)$. By definition of H, since $\ker(\tilde{h})$ is also an algebraic subgroup of $\mathrm{Alb}(X,\Delta,V)$, this is equivalent to showing that $A \subseteq \ker(\tilde{h})$, where A denotes the image of $V^\perp$ in $\mathrm{Alb}(X,\Delta)$ as above. Let $\iota\colon A \to \mathrm{Alb}(X,\Delta)$ denote the inclusion as above. Then, by the universal property of the kernel, it suffices to show that $\tilde{h} \circ \iota = 0$, i.e., that $\tilde{h} \circ \iota$ is the constant algebraic group morphism:

$$\begin{tikzcd} & A \arrow[dl, dashed, hook, "\exists ?"'] \arrow[d, hook, "\iota"] & \\ \ker(\tilde{h}) \arrow[r, hook] & \mathrm{Alb}(X,\Delta) \arrow[r, "\tilde{h}"] & G. \end{tikzcd}$$

The chosen compactifications of A and G are log-parallelizable [NW14, Corollary 5.4.5]. That is, the logarithmic cotangent bundles are trivial, in particular globally generated. Suppose that we can show that

$$(\tilde{h} \circ \iota)^*\colon T_1(G) \to T_1(A)$$

is zero. Then, since the compactification of G is log-parallelizable, this implies that the pull-back morphism on global sections

$$(\tilde{h} \circ \iota)^*\colon \Omega^1_G(G) \to \Omega^1_A(A)$$

is zero as well. Since Ω^1_G is still globally generated, Lemma 3.54 would imply that $\tilde{h} \circ \iota$ is a constant morphism. Therefore, it suffices to show that $(\tilde{h} \circ \iota)^* = 0$.

We show that $\mathrm{im}(\tilde{h}^*) \subseteq \ker(\iota^*)$. We have seen above that $\ker(\iota^*) = \delta(V^{\perp\perp}) = \delta(j(V))$, and that $\mathrm{alb}^*_{(X,\Delta)} = j^{-1} \circ \delta^{-1}$. Since

$$V \supseteq \mathrm{im}(f^*) = \mathrm{im}(\mathrm{alb}^* \circ \tilde{h}^*) = j^{-1}\delta^{-1}(\mathrm{im}(\tilde{h}^*)),$$

the claim follows. □

Remark 3.56. In the setting of Proposition 3.55, suppose that $\Delta = 0$. Then we have the following direct consequences of the construction:

(1) The semi-abelian variety $\mathrm{Alb}(X, 0, V)$ is an abelian variety, i.e., projective. Indeed, it is the surjective image of the projective variety $\mathrm{Alb}(X) = \mathrm{Alb}(X, 0)$, so it is itself projective.
(2) The morphism $\mathrm{alb}_{(X,0,V)}$ is proper. Indeed, both X and $\mathrm{Alb}(X, 0, V)$ are proper over $\mathbb{C}$, so the claim follows from [GW10, Appendix C].
(3) If alb_X is surjective and has connected fibers, then the same is true for $\mathrm{alb}_{(X,0,V)}$. Indeed, since the subgroup H in the proof of Proposition 3.55 is connected, the fibers of the quotient morphism q are connected. Since $\mathrm{alb}_{(X,0,V)}$ is proper, its underlying continuous function is closed. So it is a closed, surjective and continuous function with connected fibers. Therefore, the preimage of any non-empty connected subset is connected, and this implies that the fibers of the composition $\mathrm{alb}_{(X,0,V)} = q \circ \mathrm{alb}_X$ are connected.

The Albanese constructed in Proposition 3.55 has the following additional properties:

Lemma 3.57 (cf. [KR])**.** *In the setting of Proposition 3.55, we have the following:*

(1) *The pull-back of logarithmic differential* 1*-forms*

$$\mathrm{alb}^*_{(X,\Delta,V)} : T_1(\mathrm{Alb}(X, \Delta, V)) \longrightarrow T_1(U)$$

is injective.
(2) *We have* $\dim(\mathrm{Alb}(X, \Delta, V)) \leq \dim_{\mathbb{C}}(V)$.
(3) *The image of* $\mathrm{alb}_{(X,\Delta,V)}$ *generates* $\mathrm{Alb}(X, \Delta, V)$ *in the sense of Item 15 in Notation 0.7.*
(4) *Let Z be a smooth variety and let $g: Z \longrightarrow U$ be a morphism such that $g^*(\sigma) = 0$ for all $\sigma \in V$, where g^* denotes the morphism*

$$g^* : T_1(U) \longrightarrow T_1(Z).$$

Then $\mathrm{alb}_{(X,\Delta,V)} \circ g : Z \longrightarrow \mathrm{Alb}(X, \Delta, V)$ *is constant.*

Proof. With the notation from the proof of Proposition 3.55, we have

$$\mathrm{alb}^*_{(X,\Delta,V)} = \mathrm{alb}^*_{(X,\Delta)} \circ q^*.$$

By Theorem 3.45, the pull-back $\mathrm{alb}^*_{(X,\Delta)}$ is an isomorphism. Since q is surjective, the pull-back q^* is injective [Iit82, Theorem 11.2], so Item 1 holds.

Since $\mathrm{Alb}(X, \Delta, V)$ is semi-abelian, we have

$$\dim_{\mathbb{C}}(T_1(\mathrm{Alb}(X, \Delta, V))) = \dim(\mathrm{Alb}(X, \Delta, V)),$$

because any suitable compactification is log-parallelizable [NW14, Corollary 5.4.5]. Therefore, Item 2 follows from Item 1.

By [Ser59, Théorème 2], the image of $\mathrm{alb}_{(X,\Delta)}$ generates $\mathrm{Alb}(X, \Delta)$. Since q is surjective, the image of $\mathrm{alb}_{(X,\Delta,V)}$ generates $\mathrm{Alb}(X, \Delta, V)$ as well, so Item 3 holds.

Let us check Item 4. Let $f := \mathrm{alb}_{(X,\Delta,V)} \circ g$. By [Iit82, Theorem 11.2], we can write f^* as the composition

$$T_1(\mathrm{Alb}(X, \Delta, V)) \xrightarrow{\mathrm{alb}^*_{(X,\Delta,V)}} T_1(U) \xrightarrow{g^*} T_1(Z).$$

By assumption we have $\mathrm{im}(\mathrm{alb}^*_{(X,\Delta,V)}) \subseteq \ker(g^*)$, so $f^* = 0$. The claim follows now using Lemma 3.54 as in the proof of Item 3 in Proposition 3.55. $\square$

Example 3.58. In the setting of Proposition 3.55, if we take $V = 0$, then $\mathrm{Alb}(X, \Delta, V) = 0$; and if we take $V = \Omega^1_X(\log \Delta)(X)$, then $\mathrm{Alb}(X, \Delta, V) = \mathrm{Alb}(X, \Delta)$.

6. Construction of the adapted Albanese

In this section we prove Theorem 3.4 following the construction in [KR] in the analytic smooth setting.

Proof of Theorem 3.4. Let $\pi\colon \tilde{X} \to \hat{X}$ be a log resolution of $(\hat{X}, \Delta_{\hat{X}})$ and let $\Delta_{\tilde{X}} := (\pi^*\Delta_{\hat{X}})_{\mathrm{red}}$. By Corollary 3.9, the pull-back of rational differentials induces a $\mathcal{O}_{\tilde{X}}$-module morphism

$$\pi^*\Omega^{[1]}_{(X,\Delta,\gamma)} \to \Omega^1_{\tilde{X}}(\log \Delta_{\tilde{X}}).$$

Let $V \subseteq \Omega^1_{\tilde{X}}(\log \Delta_{\tilde{X}})(\tilde{X})$ be the image of the composition

$$\Omega^{[1]}_{(X,\Delta,\gamma)} \to \pi_*\pi^*\Omega^{[1]}_{(X,\Delta,\gamma)} \to \pi_*\Omega^1_{\tilde{X}}(\log \Delta_{\tilde{X}})$$

on global sections. We consider

$$\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}\colon \tilde{U} \to \mathrm{Alb}(\tilde{X}, \Delta_{\tilde{X}}, V),$$

where $\tilde{U} := \tilde{X} \setminus \mathrm{Supp}(\Delta_{\tilde{X}})$, and where the base point is any closed point in $\pi^{-1}(y_0)$. Our first goal is to show that $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}$ factors through $\pi|_{\tilde{U}}$, i.e., that there exists a morphism $\varphi\colon \hat{U} \to \mathrm{Alb}(\tilde{X}, \Delta_{\tilde{X}}, V)$ such that the following diagram commutes:

$$\begin{tikzcd} \tilde{U} \arrow[d, "\pi|_{\tilde{U}}"'] \arrow[dr, "\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}"] & \\ \hat{U} \arrow[r, "\varphi"'] & \mathrm{Alb}(\tilde{X}, \Delta_{\tilde{X}}, V). \end{tikzcd}$$

To find such a morphism, it is enough to show that $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}$ is constant of the fibers of $\pi|_{\tilde{U}}$. So let $y \in \hat{U}$ be a closed point. By Zariski's main theorem [Har77, Corollary III.11.4], the fiber $\pi^{-1}(y)$ is a connected closed subscheme of $\tilde{U}$. Therefore its image in $\mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V)$ is connected, so it suffices to show that $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}$ is constant on each irreducible component of $\pi^{-1}(y)$. Hence we may assume that $\pi^{-1}(y)$ is irreducible. Since we want to show an assertion about the underlying topological map, we may also assume that $\pi^{-1}(y)$ is reduced. So we may assume that $\pi^{-1}(y)$ is a variety. Let $g\colon Z \to \pi^{-1}(y)$ be a resolution of singularities. Then it suffices in turn to show that the composition $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)} \circ g$ is constant. This follows from Proposition 3.15 and Item 4 in Lemma 3.57. Therefore, $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}$ is constant on the fibers of $\pi|_{\tilde{U}}$ and we obtain the desired morphism $\varphi\colon \hat{U} \to \mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V)$.

We set $\mathrm{Alb}(X,\Delta,\gamma) := \mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V)$ and $\mathrm{alb}_{(X,\Delta,\gamma)} := \varphi$ and check that the desired universal property holds. We have chosen the base point in $\tilde{U}$ such that Item 1 in Theorem 3.4 follows from Item 1 in Proposition 3.55, so it remains to show that Items 2 and 3 in Theorem 3.4 hold.

It follows from Item 2 in Proposition 3.55 that the pull-back of logarithmic differentials

$$\mathrm{alb}^*_{(\tilde{X},\Delta_{\tilde{X}},V)}\colon T_1(\mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V)) \to T_1(\tilde{U}) = \Omega^1_{\tilde{X}}(\log \Delta_{\tilde{X}})(\tilde{X})$$

has image contained in V. Given $\sigma \in T_1(\mathrm{Alb}(X,\Delta,\gamma)) = T_1(\mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V))$, we want to show that its pull-back $\mathrm{alb}^*_{(X,\Delta,\gamma)}(\sigma)$ as a rational differential is in the image of $\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})$ in $\mathcal{K}_{\hat{X}}(\Omega^1_{\hat{X}})$, cf. Item 13 in Notation 0.7. It suffices to show this over a big open subset, so we may look at the restriction of $\mathrm{alb}^*_{(X,\Delta,\gamma)}(\sigma)$ to the complement of the center of $\pi|_{\tilde{U}}$ in $\hat{U}$ instead. It is then enough to check that this restriction pulls back to the restriction of a differential form in V to the complement of the exceptional locus of $\pi|_{\tilde{U}}$ in $\tilde{U}$, and this is the case by construction. Therefore, Item 2 in Theorem 3.4 holds.

Let now G be a semi-abelian variety and let $f\colon \hat{U} \to G$ be a morphism sending y_0 to 0 such that

$$f^*\colon T_1(G) \to \Omega^{[1]}_{\hat{X}}(\log \Delta_{\hat{X}})(\hat{X})$$

has image contained in $\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})$. Then the pull-back of logarithmic differentials along $f \circ \pi|_{\tilde{U}}\colon \tilde{U} \to G$ has image contained in V, so that Item 3 in Proposition 3.55 implies that there exists a unique morphism $h\colon \mathrm{Alb}(X,\Delta,\gamma) \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U} \arrow[r, "\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}"] \arrow[d, "\pi|_{\tilde{U}}"'] & \mathrm{Alb}(\tilde{X},\Delta_{\tilde{X}},V) \arrow[d, equal] \\
\hat{U} \arrow[r, "\mathrm{alb}_{(X,\Delta,\gamma)}"] \arrow[dr, "f"'] & \mathrm{Alb}(X,\Delta,\gamma) \arrow[d, "h"] \\
 & G.
\end{tikzcd}$$

Suppose that h' was another morphism making the triangle on the bottom commute. Then the whole diagram would commute as well, so $h' = h$ by the aforementioned uniqueness. Therefore, Item 3 in Theorem 3.4 holds as well.

Since $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is reflexive and π is surjective, the composition

$$\Omega^{[1]}_{(X,\Delta,\gamma)} \to \pi_*\pi^*\Omega^{[1]}_{(X,\Delta,\gamma)} \to \pi_*\Omega^1_{\tilde{X}}(\log \Delta_{\tilde{X}})$$

is injective, because the kernel is torsion-free and the composition is generically injective. Therefore,

$$\dim_{\mathbb{C}}(V) = \dim_{\mathbb{C}}\left(\Omega^{[1]}_{(X,\Delta,\gamma)}(\hat{X})\right).$$

By Item 2 in Lemma 3.57, we have

$$\dim(\mathrm{Alb}(X,\Delta,V)) \leq \dim_{\mathbb{C}}(V),$$

so this gives the desired bound on the dimension.

Since $\pi|_{\tilde{U}}$ is surjective, the image of $\mathrm{alb}_{(X,\Delta,\gamma)}$ equals the image of $\mathrm{alb}_{(\tilde{X},\Delta_{\tilde{X}},V)}$. By Item 3 in Lemma 3.57, this image generates $\mathrm{Alb}(X,\Delta,\gamma)$. □

Remark 3.59. In the setting of Theorem 3.4, suppose that $\lfloor\Delta\rfloor = 0$. Then the adapted Albanese is an abelian variety, cf. Remark 3.56. Moreover, if we can find a resolution of the cover whose Albanese morphism is surjective with connected fibers, then the adapted Albanese morphism

$$\mathrm{alb}_{(X,\Delta,\gamma)} \colon \hat{X} \to \mathrm{Alb}(X,\Delta,\gamma)$$

is surjective and has connected fibers, cf. Remark 3.56.

Proposition 3.60 (cf. [KR])**.** *Let (X,Δ) be a projective C-pair with quotient singularities. Let $\gamma_1 \colon \hat{X}_1 \to X$ and $\gamma_2 \colon \hat{X}_2 \to X$ be two adapted covers, and denote $\hat{U}_i := \hat{X}_i \setminus \mathrm{Supp}(\gamma_i^*\lfloor\Delta\rfloor)$ for each $i \in \{1,2\}$. Let $f \colon \hat{X}_1 \to \hat{X}_2$ be a morphism over X. Let $y_1 \in \hat{U}_1$ be a base point on $\hat{U}_1$, and consider the base point $f(y_1) \in \hat{U}_2$ on $\hat{U}_2$. Then there exists a unique morphism $g \colon \mathrm{Alb}(X,\Delta,\gamma_1) \to \mathrm{Alb}(X,\Delta,\gamma_2)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\hat{U}_1 \arrow[r, "f|_{\hat{U}_1}"] \arrow[d, "\mathrm{alb}_{(X,\Delta,\gamma_1)}"'] & \hat{U}_2 \arrow[d, "\mathrm{alb}_{(X,\Delta,\gamma_2)}"] \\
\mathrm{Alb}(X,\Delta,\gamma_1) \arrow[r, "g"] & \mathrm{Alb}(X,\Delta,\gamma_2).
\end{tikzcd}$$

Moreover, g is a surjective algebraic group morphism.

Proof. We want to apply the universal property of the adapted Albanese to the morphism $\mathrm{alb}_{(X,\Delta,\gamma_2)} \circ f|_{\hat{U}_1}$, so we need to show that the pullback of logarithmic differential 1-forms

$$(\mathrm{alb}_{(X,\Delta,\gamma_2)} \circ f|_{\hat{U}_1})^* \colon T_1(\mathrm{Alb}(X,\Delta,\gamma_2)) \to \Omega^{[1]}_{\hat{X}_1}(\log \Delta_{\hat{X}_1})(\hat{X}_1)$$

has image contained in $\Omega^{[1]}_{(X,\Delta,\gamma_1)}(\hat{X}_1)$. Since all sheaves involved are torsion-free and the sheaf morphisms inducing $(\mathrm{alb}_{(X,\Delta,\gamma_2)} \circ f|_{\hat{U}_1})^*$ and $(f|_{\hat{U}_1})^* \,\mathrm{alb}^*_{(X,\Delta,\gamma_2)}$ agree generically, the two linear maps agree. The image of $\mathrm{alb}^*_{(X,\Delta,\gamma_2)}$ is contained in $\Omega^{[1]}_{(X,\Delta,\gamma_2)}(\hat{X}_2)$ by Item 2 in Theorem 3.4. By Lemma 1.74, the image of $\Omega^{[1]}_{(X,\Delta,\gamma_2)}(\hat{X}_2)$ under $f|^*_{\hat{U}_1}$ is contained in $\Omega^{[1]}_{(X,\Delta,\gamma_1)}(\hat{X}_1)$. So the universal property of the adapted Albanese implies the existence and uniqueness of the desired morphism.

The fact that g is a morphism of algebraic groups follows from [NW14, Theorem 54.1.37], as pointed out in Remark 3.5. So it remains to show that g is surjective. Since $f|_{\hat{U}_1} \colon \hat{U}_1 \to \hat{U}_2$ is surjective and the image of $\mathrm{alb}_{(X,\Delta,\gamma_2)}$ generates $\mathrm{Alb}(X,\Delta,\gamma_2)$, the image of g must also generate $\mathrm{Alb}(X,\Delta,\gamma_2)$ by commutativity of the square. But g is a group homomorphism, so its image generating the target implies that it is surjective. □

7. Some properties and examples

The first thing that we note is that the adapted Albanese depends very much on the chosen adapted cover:

Example 3.61. Let X be a smooth projective curve of genus $g \geq 2$. Then X is not simply connected, so it admits a non-trivial finite étale morphism of degree $n \geq 2$, which we may assume to be irreducible. Hence we can find a finite étale morphism $f_1 \colon X_1 \to X =: X_0$ of smooth projective curves. By Riemann–Hurwitz [Har77, Corollary IV.2.4], we have

$$g_1 := g(X_1) > g(X_0) =: g_0.$$

Iterating this process, we find finite étale morphisms of smooth projective curves $f_i \colon X_i \to X_{i-1}$ such that $g_i := g(X_i) > g_{i-1}$ for all $i \in \mathbb{N}_{>0}$. For each $i \in \mathbb{N}_{>0}$ we define

$$\gamma_i := f_1 \circ f_2 \circ \cdots \circ f_i,$$

so that $\gamma_i \colon X_i \to X$ is a perfectly adapted cover for $(X, 0)$. By Lemma 1.73, we have

$$\Omega^1_{(X,0,\gamma_i)} = \Omega^1_{X_i}.$$

Since X_i is already smooth, the adapted Albanese (with respect to some base point) is given by

$$\mathrm{Alb}(X,0,\gamma_i) = \mathrm{Alb}(X_i) \quad \text{and} \quad \mathrm{alb}(X,0,\gamma_i) = \mathrm{alb}_{X_i}.$$

But $\mathrm{Alb}(X_i)$ is the Jacobian of X_i, which has dimension g_i. Therefore, we have found a sequence of adapted covers $\{\gamma_n\}_{n\in\mathbb{N}}$ of $(X, 0)$ such that

$$\lim_{n\to\infty} \dim(\mathrm{Alb}(X, 0, \gamma_n)) = \infty.$$

Remark 3.62. In [KR], Kebekus and Rousseau show that if (X, Δ) is special and $\gamma\colon \hat{X} \to X$ is an adapted cover, then $\mathrm{alb}_{(X,\Delta,\gamma)}$ is dominant. In particular, it is not possible to find a sequence of adapted covers as in Example 3.61. One may express this by saying that *special pairs have bounded irregularity.*

Remark 3.63. The dependence on the cover should not come as a surprise, because the adapted Albanese was defined in terms of the sheaves of adapted differential 1-forms, and those depend on the cover, as shown in Example 3.61. Moreover, this example also shows that restricting our attention to perfectly adapted covers will not be enough to get rid of this ambiguity. See also Example 1.58.

In order to produce examples more easily, let us study the behavior of the adapted Albanese under products and quotients.

Lemma 3.64. *Let (X, Δ_X) and (Y, Δ_Y) be two projective C-pairs with quotient singularities, and assume that $\lfloor\Delta_X\rfloor = 0$ and $\lfloor\Delta_Y\rfloor = 0$. Let $\gamma_X\colon \hat{X} \to X$ and $\gamma_Y\colon \hat{Y} \to Y$ be adapted covers. Then the adapted Albanese*[4] *of the product C-pair $(X \times Y, \Delta_{X\times Y})$ with respect to the adapted cover $\gamma_X \times \gamma_Y$ and a base point $(x_0, y_0) \in \hat{X} \times \hat{Y}$ exists and is given by*

$$\hat{X} \times \hat{Y} \xrightarrow{\mathrm{alb}_{(X,\Delta_X,\gamma_X)} \times \mathrm{alb}_{(Y,\Delta_Y,\gamma_Y)}} \mathrm{Alb}(X, \Delta_X, \gamma_X) \times \mathrm{Alb}(Y, \Delta_Y, \gamma_Y),$$

where the adapted Albanese of γ_X (resp. γ_Y) is constructed with respect to the base point x_0 (resp. y_0).

Proof. We show that the universal property in Theorem 3.4 is satisfied. To make the notation lighter, denote $A_X := \mathrm{Alb}(X, \Delta_X, \gamma_X)$, $a_X := \mathrm{alb}_{(X,\Delta_X,\gamma_X)}$, $A_Y := \mathrm{Alb}(Y, \Delta_Y, \gamma_Y)$ and $a_Y := \mathrm{alb}_{(Y,\Delta_Y,\gamma_Y)}$. By construction, Item 1 in Theorem 3.4 holds.

In order to check Item 2 in Theorem 3.4, we first note that A_X and A_Y are already projective, because $\hat{X}$ and $\hat{Y}$ are, cf. proof of Proposition 3.55. Therefore, using additivity of global sections and the projection formula, we have

$$\begin{aligned} T_1(A_X \times A_Y) &= \Omega^1_{A_X\times A_Y}(A_X \times A_Y) \\ &\cong \left((\pi_X^*\Omega^1_{A_X}) \oplus (\pi_Y^*\Omega^1_{A_Y})\right)(A_X \times A_Y) \\ &\cong \left((\pi_X^*\Omega^1_{A_X})(A_X \times A_Y)\right) \oplus \left((\pi_Y^*\Omega^1_{A_Y})(A_X \times A_Y)\right) \\ &\cong \Omega^1_{A_X}(A_X) \oplus \Omega^1_{A_Y}(A_Y). \end{aligned}$$

[4] Defined by the same universal property as in Theorem 3.4.

Under these identifications, the pull-back of differentials $(a_X\times a_Y)^*$ is given by

$$\begin{aligned}\Omega^1_{A_X}(A_X)\oplus\Omega^1_{A_Y}(A_Y)&\to\mathcal{K}_{\hat{X}\times\hat{Y}}(\Omega^1_{\hat{X}\times\hat{Y}})\\ (\sigma_X,\sigma_Y)&\mapsto p_X^*a_X^*(\sigma_X)+p_Y^*a_Y^*(\sigma_Y).\end{aligned}$$

Since the image of a_X^* (resp. a_Y^*) is contained in $\Omega^{[1]}_{(X,\Delta_X,\gamma_X)}(\hat{X})$ (resp. in $\Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)}(\hat{Y})$), the image of $(a_X\times a_Y)^*$ is contained in

$$\left(p_X^*\Omega^{[1]}_{(X,\Delta_X,\gamma_X)}\right)\oplus\left(p_Y^*\Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)}\right)(\hat{X}\times\hat{Y}).$$

By Lemma 1.79, we have

$$\Omega^{[1]}_{(X\times Y,\Delta_{X\times Y},\gamma_X\times\gamma_Y)}(\hat{X}\times\hat{Y})\cong\left(p_X^*\Omega^{[1]}_{(X,\Delta_X,\gamma_X)}\right)\oplus\left(p_Y^*\Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)}\right)(\hat{X}\times\hat{Y}),$$

so Item 2 in Theorem 3.4 holds.

To check Item 3 in Theorem 3.4, we consider the following commutative diagram:

$$\begin{tikzcd}
\hat{X}\times\hat{Y} \arrow[rr, "a_X\times a_Y"] \arrow[d, "p_Y"'] \arrow[ddr, "p_X"] \arrow[dddrrr, "f"] & & A_X\times A_Y \arrow[d, "\pi_Y"'] \arrow[ddr, "\pi_X"] & \\
\hat{Y} \arrow[rr, "a_Y"] & & A_Y & \\
& \hat{X} \arrow[rr, "a_X"] & & A_X \\
& & & G.
\end{tikzcd}$$

It follows from the rigidity theorem [Mil08, Theorem 1.1] that $f=f_X\circ p_X+f_Y\circ p_Y$ for some $f_X\colon\hat{X}\to G$ with $f_X(x_0)=0$ and some $f_Y\colon\hat{Y}\to G$ with $f_Y(y_0)=0$, cf. proof of [Mil08, Corollary 1.5]. Denoting by $i_X\colon\hat{X}\to\hat{X}\times\hat{Y}$ and $i_Y\colon\hat{Y}\to\hat{X}\times\hat{Y}$ the morphisms given on closed points by $i_X(x)=(x,y_0)$ and $i_Y(y)=(x_0,y)$ respectively, we must have $f_X=f\circ i_X$ and $f_Y=f\circ i_Y$, because $p_X\circ i_X=\mathrm{id}_{\hat{X}}$, $p_Y\circ i_Y=\mathrm{id}_{\hat{Y}}$, $p_X\circ i_Y\equiv x_0$ and $p_Y\circ i_X\equiv y_0$. By assumption, the pull-back of logarithmic 1-forms

$$f^*\colon T_1(G)\to\Omega^{[1]}_{\hat{X}\times\hat{Y}}(\hat{X}\times\hat{Y})$$

has image contained in $\Omega^{[1]}_{(X\times Y,\Delta_{X\times Y},\gamma_X\times\gamma_Y)}(\hat{X}\times\hat{Y})$. By Lemma 1.79, we have

$$\Omega^{[1]}_{(X\times Y,\Delta_{X\times Y},\gamma_X\times\gamma_Y)}(\hat{X}\times\hat{Y})\cong(p_X^*\Omega^{[1]}_{(X,\Delta_X,\gamma_X)})\oplus(p_Y^*\Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)})(\hat{X}\times\hat{Y}).$$

The pull-back of any generator coming from $\hat{Y}$ along i_X vanishes, because the composition $p_Y\circ i_X$ is constant. Therefore, the pull-back of logarithmic 1-forms

$$f_X^*=(f\circ i_X)^*\colon T_1(G)\to\Omega^{[1]}_{\hat{X}}(\hat{X})$$

has image contained in $\Omega^{[1]}_{(X,\Delta_X,\gamma_X)}(\hat{X})$. Similarly, the pull-back of logarithmic 1-forms

$$f_Y^*\colon T_1(G) \to \Omega^{[1]}_{\hat{Y}}(\hat{Y})$$

has image contained in $\Omega^{[1]}_{(Y,\Delta_Y,\gamma_Y)}(\hat{Y})$. Therefore, we can find morphisms $g_X\colon A_X \to G$ and $g_Y\colon A_Y \to G$ such that $g_X \circ a_X = f_X$ and $g_Y \circ a_Y = f_Y$. Define $g\colon A_X \times A_Y \to G$ by the formula

$$g(u,v) := g_X(u) + g_Y(v)$$

on closed points. We check that $g \circ (a_X \times a_Y) = f$ on closed points, so let $(x,y) \in \hat{X} \times \hat{Y}$ be a closed point. Then we have

$$\begin{aligned} g \circ (a_X \times a_Y)(x,y) &= g_X a_X(x) + g_Y a_Y(y) \\ &= f_X(x) + f_Y(y) \\ &= f_X \circ p_X(x,y) + f_Y \circ p_Y(x,y) \\ &= f(x,y). \end{aligned}$$

Hence the desired equality, and this shows the existence part of Item 3 in Theorem 3.4. For the uniqueness part, recall that any morphism $A_X \times A_Y \to G$ is a group homomorphism on closed points. So any such g is determined by the compositions $g \circ \pi_X$ and $g \circ \pi_Y$, which in turn are determined by the universal property of Theorem 3.4. □

Lemma 3.65. *Let (X', Δ') be a projective C-pair with quotient singularities and let G be a finite group acting on (X', Δ'). Assume that we can form the quotient of X' by G. Let $q\colon X' \to X$ be the quotient morphism and let (X, Δ) be the quotient C-pair. Let $\gamma'\colon \hat{X} \to X'$ be an adapted cover and let $\hat{U} = \hat{X} \setminus \mathrm{Supp}((\gamma')^* \lfloor \Delta' \rfloor)$. Then the adapted Albanese[5] of (X, Δ) with respect to the adapted cover $\gamma := q \circ \gamma'$ and a base point $x_0 \in \hat{U}$ exists, and agrees with the adapted Albanese of (X', Δ') with respect to the adapted cover γ' and the base point x_0.*

Proof. Follows immediately from Lemma 1.81. □

In the following examples, if no confusion is likely to happen, we omit the base points in the discussion.

Example 3.66. Let $X = \mathbb{P}^1$ and let Δ be the divisor from Example 0.3, i.e.,

$$\Delta = \sum_{i=0}^{5} \frac{1}{2} P_i$$

with $P_i = \zeta_6^i \in \mathbb{A}^1 \subseteq \mathbb{P}^1$ for all $i \in \{0, \ldots, 5\}$. By construction of Δ, the morphism $\gamma\colon C \to \mathbb{P}^1$ from the genus 2 curve in Example 0.3 is a perfectly adapted cover. Therefore, the adapted Albanese with respect to γ agrees with the Jacobian of C.

[5] Defined by the same universal property as in Theorem 3.4.

Example 3.67. Let E be an elliptic curve and let $X := E^{(2)}$ be its symmetric power. Let $q\colon E \times E \to X$ be the quotient by the action of the symmetric group. If we equip X with the quotient $\mathcal{C}$-pair structure coming from $(E \times E, 0)$, i.e., $\Delta = \frac{1}{2}\{(x, y) \in E \times E \mid x = y\}$, then q becomes a perfectly adapted cover and we have

$$\mathrm{Alb}(X, \Delta, q) = E \times E$$

as a consequence of Corollary 1.45, Lemma 1.73 and $E \cong \mathrm{Alb}(E)$. On the other hand, if we consider the $\mathcal{C}$-pair $(X, 0)$ instead, then id_X is already a perfectly adapted cover. By [MS11, Example 1.1] we have $h^{0,1}(X) = 1$, so in this case we have[6]

$$\dim(\mathrm{Alb}(X, 0, \mathrm{id}_X)) = \dim(\mathrm{Alb}(X)) = 1 < \dim(\mathrm{Alb}(X, \Delta, q)).$$

Remark 3.68. More generally, let $n \in \mathbb{N}_{>1}$ and let X be a smooth projective variety such that $\dim(\mathrm{Alb}(X)) > 0$. Then we can form the n-th cartesian square of the $\mathcal{C}$-pair $(X, 0)$, which is given by $(X^n, 0)$, where X^n is the n-th cartesian square of X. The symmetric group on n elements acts on $(X^n, 0)$, so we may consider the quotient $\mathcal{C}$-pair $(X^{(n)}, \Delta)$, where $X^{(n)}$ is the usual symmetric power of the smooth projective variety X. The Albanese of $X^{(n)}$ exists and agrees with the Albanese of X, i.e., we have

$$\mathrm{Alb}(X^{(n)}) = \mathrm{Alb}(X).$$

On the other hand, the quotient morphism $q\colon X^n \to X^{(n)}$ is perfectly adapted for $(X, 0)^{(n)}$, so we have

$$\mathrm{Alb}(X^{(n)}, \Delta, q) = \mathrm{Alb}(X^n) = \mathrm{Alb}(X)^n \neq \mathrm{Alb}(X).$$

Example 3.69. Let A be the Jacobian of a hyperelliptic curve of genus 2 and let X be its Kummer variety, i.e., the quotient of A by the involution $a \mapsto -a$. Then X is a projective surface with quotient singularities, and its minimal resolution $\pi\colon \tilde{X} \to X$ is given by a K3 surface. The quotient morphism $q\colon A \to X$ is quasi-étale, so it is a perfectly adapted cover for $(X, 0)$. Thinking of X as a projective variety with rational singularities, we would say that its Albanese is given by

$$\mathrm{Alb}(X) := \mathrm{Alb}(\tilde{X}) = 0,$$

because K3 surfaces have irregularity zero. On the other hand, thinking of X as the $\mathcal{C}$-pair $(X, 0)$, we could consider its adapted Albanese with respect to the perfectly adapted cover $q\colon A \to X$, which is then given by

$$\mathrm{Alb}(X, 0, q) = A \neq 0.$$

Example 3.70. There are also scenarios similar to the one in Example 3.69 in which things do fit nicely together. For instance, one can construct an Inoue surface S as a Galois cover of Cayley's nodal cubic X, via a quotient morphism $q\colon S \to X$, cf. [MP01, Example 4.1]. If we let (X, Δ) be the

[6] In fact, using invariance of Albanese under symmetric powers of smooth projective varieties, we can explicitly compute $\mathrm{Alb}(X, 0, \mathrm{id}_X) = E$.

quotient C-pair of $(S, 0)$, then q becomes a perfectly adapted cover. Inoue surfaces also have irregularity zero, and in this case the minimal resolution $\tilde{X}$ of X is a rational surface [MP01, Example 4.1], so we have

$$\mathrm{Alb}(\tilde{X}) = \mathrm{Alb}(X) = 0 = \mathrm{Alb}(X, \Delta, q).$$

8. The Albanese of a C-pair: a possible Ansatz

Let (X, Δ) be a C-pair. Defining the adapted Albanese on an adapted cover gives raise to the same issues as defining adapted differentials on an adapted cover. The result depends on the cover, and it would be desirable to have an object on X which is independent of the choice of a cover. The general consensus in the field seems to be that *the Albanese of* (X, Δ) should be, at least in nice situations, a quotient of the adapted Albanese of an adapted cover with the appropriate C-pair structure taken into account. This is the approach taken in [KR]. However, this construction is only possible in the case of *bounded irregularity*, i.e., if the dimension of the adapted Albanese varieties is bounded when one ranges over all possible adapted covers.

In [BMS23], the Albanese is constructed from a more motivic point of view. Namely, any semi-abelian variety, seen as an étale sheaf on the category $\mathbf{Sm}/\mathbb{C}$ of smooth $\mathbb{C}$-schemes of finite type, has a natural structure of étale sheaf with transfers. This allows to define the Albanese functor, roughly speaking, as a left adjoint to the inclusion of a full abelian subcategory of the category of étale sheaves with transfers $\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(\mathbb{C}, \mathbb{Q})$, cf. [BMS23, Theorem 1.1]. This motivates the following:

Question 3.71. *Is it possible to use the adapted Albanese construction to define an Albanese of* (X, Δ) *as a functor*

$$\mathrm{Alb}_{(X,\Delta)} \colon \mathbf{Shv}^{tr}_{ét}(X, \mathbb{Q}) \to \mathbf{Shv}^{tr}_{ét}(X, \mathbb{Q})$$

in the spirit of Chapter 2?

In particular, some desirable properties would be:

(1) For any adapted cover $\gamma \colon Y \to X$, the sheaf $\mathrm{Alb}_{(X,\Delta)}(Y)$ is represented by the corresponding adapted Albanese variety.
(2) If (X, Δ) has bounded irregularity, then $\mathrm{Alb}_{(X,\Delta)}(X)$ is represented by the variety underlying the Albanese C-pair constructed by Kebekus and Rousseau in [KR].

APPENDIX A

Reflexive sheaves

The main references for this appendix are [SP, 0AVT] and [Har80, §1]. Throughout this appendix, let X be an integral locally noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We denote by $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ its *dual*. There is a natural map $j_\mathcal{F}\colon \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ given affine locally by sending $m \in M$ to the A-linear evaluation map $\varphi \mapsto \varphi(m)$, where $M = \mathcal{F}(X)$ and $X = \operatorname{Spec}(A)$.

Definition A.1 (Reflexive sheaf)**.** A coherent sheaf $\mathcal{F}$ on an integral locally noetherian scheme X is called *reflexive* if the natural map $j_\mathcal{F}\colon \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is an isomorphism.

If $\mathcal{F}$ is any coherent sheaf on the integral locally noetherian scheme X, then the sheaf $\mathcal{F}^{\vee\vee}$ is a reflexive sheaf called the *reflexive hull* of $\mathcal{F}$ [SP, 0AVU]. Reflexive sheaves are torsion-free, because $\mathcal{F}$ is torsion-free if and only if the natural map $j_\mathcal{F}\colon \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$ is injective [SP, 0AY2]. And if X is moreover normal, then $\mathcal{F}$ is reflexive if and only if it is torsion-free and satisfies condition (S_2) [SP, 031O]. We are going to be dealing mostly with normal varieties, so let us summarize all the different characterizations of reflexivity in this setting:

Fact A.2. *Let X be an integral normal locally noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. The following are equivalent:*

(1) *The sheaf $\mathcal{F}$ is reflexive.*
(2) *The sheaf $\mathcal{F}$ can be locally included in a short exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

in which $\mathcal{E}$ is locally free and $\mathcal{G}$ is torsion-free.
(3) *The sheaf $\mathcal{F}$ is torsion-free and has property (S_2).*
(4) *There exists an open subscheme $i\colon U \longrightarrow X$ such that every irreducible component of $X \setminus U$ has codimension ≥ 2 in X, $i^*\mathcal{F}$ is locally free and $\mathcal{F} \cong i_*i^*\mathcal{F}$ via the canonical map.*
(5) *The sheaf $\mathcal{F}$ is torsion-free and for each open subscheme $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension ≥ 2, we have $\mathcal{F}|_U \cong i_*i^*(\mathcal{F}|_U)$, where $i\colon U \setminus Y \longrightarrow U$ is the open immersion.*
(6) *For each closed subset $Y \subseteq X$ of codimension ≥ 2, we have $\mathcal{H}^1_Y(\mathcal{F}) = 0$, where $\mathcal{H}^1_Y$ denotes the local cohomology sheaf, i.e., $\mathcal{H}^1_Y(\mathcal{F})$ is the sheaf associated to the presheaf $V \mapsto H^1_{Y\cap V}(V, \mathcal{F})$, cf. [Har67, Proposition 1.2].*

Proof. See [SP, 0AY6] and [Har80, Proposition 1.6]. □

If X is moreover regular, then any reflexive sheaf on X is locally free except along a closed subset of codimension ≥ 3 [Har80, Corollary 1.4]. Another useful fact is the following:

Fact A.3. *Let $f\colon X \to Y$ be a morphism of integral normal noetherian schemes.*

(1) *If f is flat and $\mathcal{F}$ is a reflexive coherent sheaf on Y, then $f^*\mathcal{F}$ is a reflexive coherent sheaf on X.*
(2) *If f is a proper dominant morphism with all fibers of the same dimension and $\mathcal{F}$ is a coherent reflexive sheaf on X, then $f_*\mathcal{F}$ is a coherent reflexive sheaf on Y.*

Proof. See [Har80, Corollary 1.7] and [Har80, Proposition 1.8]. □

Remark A.4. One has to be careful with coherence when pushing forward (reflexive) sheaves, see [Har77, Caution II.5.8.1] in the algebraic setting or [Ser66, p. 372] in the analytic setting.

Remark A.5. Let $\mathcal{F}$ and $\mathcal{G}$ be two reflexive sheaves on an integral normal locally noetherian scheme. Then $\mathcal{F} \cong \mathcal{G}$ if and only if there exists a big open subset $V \subseteq X$ such that $\mathcal{F}|_V \cong \mathcal{G}|_V$. This follows from Item 5 in Fact A.2 by taking $U := X$ and $Y = X \setminus V$.

APPENDIX B

Ramified covers

The main reference for this appendix is [GW10, §12.4].

Definition B.1 (Quasi-finite morphism)**.** A scheme morphism $f\colon X \to Y$ is called *quasi-finite* if it is of finite type and if for all $y \in Y$ the fiber $f^{-1}(y)$ consists of only finitely many points.

In the case of finite type schemes over the complex numbers, a morphism $f\colon X \to Y$ is quasi-finite if and only if the fiber over every closed point of Y is finite [GW10, Remark 12.16]. See also [GW10, Proposition 12.17] for some permanence properties of quasi-finite morphisms.

Grothendieck's version of Zariski's main theorem relates quasi-finite and finite morphisms, justifying the name "quasi-finite":

Fact B.2 ([SP, 05K0])**.** *Let $f\colon X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & T \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

where j is a quasi-compact open immersion and π is finite.

A special case of this statement is the following:

Corollary B.3 ([GW10, Corollary 12.87])**.** *Let $f\colon X \to Y$ be a morphism of separated schemes of finite type over $\mathbb{C}$. If f has finite fibers over closed points in Y, then we can find an open immersion $j\colon X \to \bar{X}$ and a finite morphism $g\colon \bar{X} \to Y$ such that $f = g \circ j$.*

From Grothendieck's version of Zariski's main theorem we can also deduce that being finite is the same as being quasi-finite and proper [GW10, Corollary 12.89].

We will be particularly interested in the following setting:

Lemma B.4. *Let $f\colon X \to Y$ be a quasi-finite morphism between normal varieties of the same dimension. Then f is dominant and open.*

Proof. Since both varieties have the same dimension and f is quasi-finite, the morphism is dominant. Indeed, if it was not dominant, then there would be a proper closed subset $Z \subset Y$ with $\dim(Z) < \dim(X)$ containing the image of f, which contradicts quasi-finiteness.

A finite dominant morphism into a normal variety is open [Dan94, §2.8], so Grothendieck's version of Zariski's main theorem allows us to conclude the same about f. $\square$

Lemma B.5. *Let $f\colon X \to Y$ be a quasi-finite morphism between finite dimensional separated schemes of the same dimension and let $V \subseteq Y$ be a big open subset. Then $f^{-1}(V) \subseteq X$ is a big open subset.*

Proof. Since V is big, the closed subset $Z := Y \setminus V$ has codimension $c \geq 2$ in Y. If $U := f^{-1}(V)$, then $X \setminus U = f^{-1}(Z)$. The morphism f is separated as well [GW10, Proposition 9.13]. Since being quasi-finite and separated is stable under pull-back, the induced morphism $f^{-1}(Z) \to Z$ is also a quasi-finite and separated morphism between finite dimensional separated schemes. So $\dim(f^{-1}(Z)) \leq \dim(Z)$ as a consequence of Grothendieck's version of Zariski's main theorem, finite morphisms being integral and integral morphisms $g\colon W_1 \to W_2$ having the property that $\dim(W_1) \leq \dim(W_2)$ [SP, 0ECG]. The codimension of $f^{-1}(Z)$ in X is then given by

$$\dim(X) - \dim(f^{-1}(Z)) \geq \dim(X) - \dim(Z) = \dim(Y) - \dim(Z) = c \geq 2.$$

$\square$

Remark B.6. In Lemma B.5, the assumption that X and Y have the same dimension is necessary. For example, consider a closed point $P \in \mathbb{A}^2$, the big open subset $V := \mathbb{A}^2 \setminus \{P\}$ and consider the closed immersion of a line $f\colon L \hookrightarrow \mathbb{A}^2$ with $P \in L$, which is a quasi-finite morphism. Then we have $f^{-1}(V) = L \setminus \{P\}$, which is not a big open subset.

Lemma B.7. *Let $f\colon X \to Y$ be a quasi-finite morphism between normal varieties of the same dimension. Then f induces a pull-back group homomorphism for Weil divisors which respects linear equivalence, which we denote by $f^*\colon \operatorname{Div}(Y) \to \operatorname{Div}(X)$. We also denote by f^* its $\mathbb{Q}$-linear extension. Moreover, the pull-back of a Cartier divisor D is again Cartier, and*

$$f^*\mathcal{O}_Y(D) \cong \mathcal{O}_X(f^*D).$$

More generally, for any Weil divisor D on Y we have

$$f^{[*]}\mathcal{O}_Y(D) \cong \mathcal{O}_X(f^*D).$$

Proof. Let $V \subseteq Y$ be the regular locus, which is a big open subset. Let U be the intersection of $f^{-1}(V)$ and the regular locus of X. Since X is normal, the regular locus is a big open subset, and $f^{-1}(V)$ is a big open subset by Lemma B.5. So U is also a big open subset. Since U and V are big open subsets, a Weil divisor on X (resp. on Y) is the same as a Weil divisor on U (resp. on V). And since U (resp. V) are smooth, this is in turn the same as a Cartier divisor on U (resp. V). The induced morphism $f|_U\colon U \to V$ is still a quasi-finite morphism between normal varieties of the same dimension, so it is a dominant morphism by Lemma B.4. This implies that we have a well-defined pull-back of Cartier divisors which by construction preserves principal divisors [GW10, Proposition 11.48], and the claim follows.

The second to last assertions regarding Cartier divisors follow also from the discussion in [GW10, (11.16)], in particular from [GW10, Proposition 11.48] and [GW10, p. 312] respectively. And the last assertion follows in turn from the second to last assertion, Lemma B.5 and Remark A.5. □

For explicit computations we can use the following:

Lemma B.8. *Let $f\colon X \to Y$ be a quasi-finite morphism between normal varieties of the same dimension and let D be a prime Weil divisor on Y. Let $V \subseteq Y$ be an open subset such that $D \cap V \neq \emptyset$ and $D \cap V$ is a Cartier divisor on V, and let $U := f^{-1}(V)$. Then $f^*(D) = \overline{(f|_U)^{-1}(D|_V)}$ as topological spaces. Moreover, if $(f|_U)^*(D|_V) = \sum_i n_i D_i$, then*

$$f^*(D) = \sum_i n_i \overline{D_i}.$$

Proof. Assume first that D is Cartier to begin with. If $V = Y$, then we have $f^*(D) = f^{-1}(D)$ as closed subschemes of X [GW10, Corollary 11.49], hence also as topological spaces. If $V \subseteq Y$ is any other open subset such that $D \cap V \neq \emptyset$, then we have $\overline{D \cap V} = D$ for dimensional reasons, because D is irreducible. The first claim follows now from Lemma B.4, because if $f\colon X \to Y$ is an open continuous function then $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$ for all $A \subseteq Y$. The last assertion follows from the fact that the coefficient of each prime Weil divisor can be computed at its generic point.

If D is not Cartier, then $f^*(D) = \overline{(f|_{U_0})^*(D|_{V_0})}$ for suitable big open subsets $V_0 \subseteq Y$ and $U_0 \subseteq f^{-1}(V_0)$ such that $D \cap V_0$ is Cartier, cf. proof of Lemma B.7. Now $D \cap V$ and $D \cap V_0$ are two dense open subsets in the irreducible topological space D, so $D \cap (V \cap V_0) \neq \emptyset$. Applying the Cartier case shown above to the morphism $f|_{U_0}\colon U_0 \to V_0$ and the divisor $D \cap V_0$ we deduce that $(f|_{U_0})^*(D|_{V_0})$ is the Zariski closure of $(f|_{U_0 \cap U})^*(D|_{V_0 \cap V})$ inside U_0, hence $f^*(D)$ is the Zariski closure of $(f|_{U_0 \cap U})^*(D|_{V_0 \cap V})$ inside X. By construction, $(f|_U)^*(D|_V)$ is the Zariski closure of $(f|_{U_0 \cap U})^*(D|_{V_0 \cap V})$ inside U, cf. proof of Lemma B.7. Therefore, $f^*(D)$ is the Zariski closure of $(f|_U)^*(D|_V)$ inside X. □

Remark B.9. The quasi-finiteness assumption in Lemma B.8 is necessary. Otherwise there could be some prime Weil divisor on X whose image is contained in $D \cap (Y \setminus V)$, e.g., this could happen on a blow-up.

Remark B.10. We can talk about ramification along arbitrary quasi-finite morphisms [GW10, §12.5], but Lemma B.7 allows us to simplify the discussion for quasi-finite morphisms between normal varieties of the same dimension. We will be interested in the *ramification divisor* of such a morphism $f\colon X \longrightarrow Y$, which we will denote by $\mathrm{Ram}(f)$. Over the complex numbers we can write

$$\mathrm{Ram}(f) = \sum_{D \in W(Y)} \left(f^*D - (f^*D)_{\mathrm{red}}\right), \tag{B.1}$$

where $W(Y)$ denotes the set of prime Weil divisors on Y. Indeed, the equivalence between the two notions of ramification is given by [Liu02, Exercise 7.1.8]. See also Definition 4.14 in Vladimir Lazić's notes on Foliations.

In the remaining of this appendix we will consider a particular case of quasi-finite morphisms between varieties of the same dimension, namely finite surjective morphisms.

Definition B.11 (Cover)**.** A scheme morphism $f\colon X \longrightarrow Y$ is called a *cover* if X and Y are normal [SP, 033I] and f is finite and surjective.

Defining covers for schemes in general has the following advantage:

Remark B.12. Covers are local on the target [GW10, Appendix C].

But we will be mostly interested in covers of varieties. Hence we note:

Remark B.13. Let $f\colon X \longrightarrow Y$ be a cover in which Y is a variety. Then X is a separated reduced scheme of finite type over $\mathbb{C}$. Moreover, if Y is affine (resp. projective, quasi-projective), then X is affine (resp. projective, quasi-projective).

Lemma B.14. *Let $f\colon X \longrightarrow Y$ be a cover in which X and Y are integral schemes. Then f induces a finite extension of function fields $K(Y) \subseteq K(X)$ and X is the normalization of Y inside $K(X)$ [GW10, Definition 12.42].*

Proof. Since f is finite it induces a finite extension of function fields. Let us show that X is then the normalization of Y inside $K(X)$.

The question is local on Y, so we may assume that $Y = \mathrm{Spec}(A)$. Moreover, since the morphism f is finite, it is also affine, so $f^{-1}(\mathrm{Spec}(A))$ is affine and we may also assume that $X = \mathrm{Spec}(B)$ is affine as well.

Since $f\colon X \longrightarrow Y$ is a finite and surjective morphism between normal affine varieties, it corresponds to a finite (integral) extension $A \subseteq B$ of integrally closed domains with fraction fields $A_{(0)} = K(Y) \subseteq K(X) = B_{(0)}$. We want to show that

$$B = \{b \in B_{(0)} \mid b \text{ integral over } A\} =: \overline{A}^{B_{(0)}}.$$

So let $b \in B$. Since $A \subseteq B$ is integral, we can find a monic polynomial $P \in A[T]$ such that $P(b) = 0$. Therefore $b \in \overline{A}^{B_{(0)}}$. Conversely, let $b \in B_{(0)}$ such that there exists a monic polynomial $P \in A[T]$ with $P(b) = 0$. Then we may regard P as a monic polynomial in $B[T]$ using the ring extension

$A \subseteq B$. This shows that $b \in B_{(0)}$ is integral over B, and thus $b \in B$ because B is an integrally closed domain. □

In the case of varieties, the converse is also true:

Lemma B.15. *Let $f: X \to Y$ be a morphism of integral normal schemes of finite type over a field. Then f is a cover if and only if f induces a finite field extension of function fields $K(Y) \subseteq K(X)$ and X is the normalization of Y inside $K(X)$.*

Proof. Let us prove the direction which is not a particular case of Lemma B.14. Suppose that the field extension $K(Y) \subseteq K(X)$ induced by f is finite, hence algebraic. Then [GW10, Proposition 12.43] ensures that f is an integral surjective morphism, so it only remains to show that f is a morphism of finite type. But this follows from X and Y being schemes of finite type over a field [GW10, Proposition 10.7]. □

Remark B.16. The converse is true more generally for integral Japanese schemes, see [SP, 033R] and [SP, 035R].

Covers are quasi-finite and surjective, but the converse is not true:

Example B.17. A quasi-finite and surjective morphism need not be finite. For example, consider the restriction of the ramified cover $\mathbb{A}^1 \to \mathbb{A}^1, z \mapsto z^2$ to the complement of a closed point different from the origin. This restriction is not proper, as can be seen from the valuative criterion for properness or from passing to the analytic topology, so it cannot be finite.

Finally, let us study a particular case of covers, namely, those which correspond to the quotient by a finite group.

Definition B.18 (Galois cover)**.** A cover $f: X \to Y$ is called *Galois* if there exists a finite group G acting algebraically on X with geometric quotient $q: X \to X/G$ in such a way that there exists an isomorphism $\varphi: X/G \to Y$ making the following triangle commute:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f"'] & X/G \arrow[d, "\varphi"] \\ & Y. \end{tikzcd}$$

Remark B.19. Unless we explicitly say otherwise, a group action will mean a left action. If a group G acts on a scheme X (from the left) and $f: U \to \mathbb{A}^1$ is a regular function defined on some G-invariant open subset U, then we can define a new function gf given by $x \mapsto f(g^{-1}x)$, and this defines a (left) G-action on $\mathcal{O}_X(U)$. This agrees with the conventions in [Mum70, §II.7] but not with the conventions in [SGA1, Exposé V, §1].

Remark B.20. We want to allow ramification, so we do not require that Galois covers be étale. In particular, our notion differs from other notions of Galois cover which can be found in the literature, see [SP, 03SF].

Lemma B.21. *Let $f: X \to Y$ be a Galois cover in which X and Y are integral schemes. Then f induces a finite Galois extension of function fields $K(Y) \subseteq K(X)$ and X is the normalization of Y inside $K(X)$.*

Proof. After Lemma B.14, it remains only to show that in this case the field extension $K(Y) \subseteq K(X)$ is Galois. The statement is local on Y and f is a finite morphism, so we may assume that both X and Y are affine, say $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$. We use f to identify A with a subring of B. Since $f: X \to Y$ is isomorphic to a geometric quotient of X by the action of a finite group, there exists a finite group G acting via ring homomorphisms on B such that $A = B^G$. This induces an action of G on $B_{(0)}$ such that $B_{(0)}^G = A_{(0)}$ [AM69, Exercise 5.12], i.e., a G-action on $K(X)$ such that $K(Y) = K(X)^G$ when we identify $K(Y)$ with a subfield of $K(X)$ using f. It follows now from [Bos18, Proposition 4.1/4] that $K(Y) \subseteq K(X)$ is a Galois extension with Galois group G. □

In the case of varieties we have again the converse:

Lemma B.22. *Let $f: X \to Y$ be a morphism of integral normal schemes of finite type over a field. Then f is a Galois cover if and only if f induces a finite Galois field extension of function fields $K(Y) \subseteq K(X)$ and X is the normalization of Y inside $K(X)$.*

Proof. After Lemma B.15, it remains only to express Y as a quotient of X by a finite group, and Lemma B.21 shows that a good candidate is $G = \operatorname{Gal}(K(X)/K(Y))$. We show locally on Y that G acts on X over Y. Assume again that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, and regard $A \subseteq B$ as a subring and $A_{(0)} \subseteq B_{(0)}$ as a subfield using f. We want to show that the G-action on $B_{(0)}$ induces a G-action on B such that $A = B^G$. Let $b \in B$ and $g \in G$. We want to show that $gb \in B$. Since $A \subseteq B$ is an integral extension, we can find a monic polynomial $P \in A[T]$ such that $P(b) = 0$. Since $A \subseteq A_{(0)} = B_{(0)}^G$, letting g act on P coefficientwise we have $gP = P$. Since G acts by ring homomorphisms we have

$$0 = g0 = g(P(b)) = (gP)(gb) = P(gb),$$

which shows that $gb \in B_{(0)}$ is integral over A and hence also over B. Since B is integrally closed we deduce that $gb \in B$, as we wanted to show. Hence we obtain a G-action on B such that $A = B^G$, and since there was no choice involved in the process we obtain a global G-action on X over Y expressing f as a geometric quotient of X by G. □

Remark B.23. In Lemma B.22 it is necessary to require that X be the normalization of Y inside $K(X)$. Otherwise we could replace X by a non-trivial blow-up of X. This would induce the same extension of function fields, but the corresponding morphism of varieties would not be finite anymore, so it would not be a cover.

APPENDIX C

$\mathbb{Q}$-factorial varieties

We will frequently work with $\mathbb{Q}$-factorial varieties, so in this appendix we follow the advice in [KM98, Example 2.17] and compare the following notions carefully:

- (Algebraic) $\mathbb{Q}$-factorial.
- Analytic $\mathbb{Q}$-factorial (or globally analytically $\mathbb{Q}$-factorial).
- Locally analytically $\mathbb{Q}$-factorial, i.e., every non-empty analytic-open subset is analytic $\mathbb{Q}$-factorial.

We first make the following observation, which explains why we have not explicitly mentioned Zariski-locally $\mathbb{Q}$-factorial:

Lemma C.1. *Let X be a normal variety. Then the following are equivalent:*

(1) *The variety X is $\mathbb{Q}$-factorial.*
(2) *Every non-empty Zariski-open subset of X is $\mathbb{Q}$-factorial.*
(3) *Every closed point of X has a Zariski-open neighborhood which is $\mathbb{Q}$-factorial.*

Proof. Suppose that Item 1 holds and let $U \subseteq X$ be a non-empty Zariski-open subset. Let D be a Weil divisor on U. Then $\overline{D}$ is a Weil divisor on X, hence there exists an $N \in \mathbb{N}_{>0}$ such that $N\overline{D}$ is Cartier. In particular, $(N\overline{D})|_U = ND$ is also Cartier, so U is $\mathbb{Q}$-factorial and Item 2 holds.

Appendix C follows immediately from Item 2, so it remains to show that Item 3 implies Item 1. Since X is quasi-compact for the Zariski topology, we can find a finite open cover $\{U_1, \dots, U_r\}$ of X such that U_i is $\mathbb{Q}$-factorial for all $i \in \{1, \dots, r\}$. Let D be a Weil divisor on X. For each $i \in \{1, \dots, r\}$ we can find some $N_i \in \mathbb{N}_{>0}$ such that $N_i D|_{U_i}$ is Cartier. In particular, if we take $N \in \mathbb{N}_{>0}$ to be the least common multiple of the N_i, then $N(D|_{U_i})$ is a Cartier divisor on U_i for each $i \in \{1, \dots, r\}$. The Weil divisor ND is then a Cartier divisor on X, because each of the restrictions $(ND)|_{U_i}$ is a Cartier divisor. □

Since every algebraic subvariety is also an analytic subvariety, every analytic $\mathbb{Q}$-factorial variety is also $\mathbb{Q}$-factorial. By definition, locally analytically $\mathbb{Q}$-factorial varieties are also globally analytically $\mathbb{Q}$-factorial. Therefore we always have the following implications:

Locally analytically $\mathbb{Q}$-factorial $\Rightarrow$ Analytic $\mathbb{Q}$-factorial $\Rightarrow$ $\mathbb{Q}$-factorial.

A $\mathbb{Q}$-factorial variety does not need to be analytic $\mathbb{Q}$-factorial. However, if X is a normal projective variety, then Chow's Theorem implies that $\mathbb{Q}$-factoriality and analytic $\mathbb{Q}$-factoriality are equivalent. See [KM98, Example 2.17].

We check next that $\mathbb{Q}$-factorial or analytic $\mathbb{Q}$-factorial do not imply locally analytically $\mathbb{Q}$-factorial. To this end, the following is an important example to keep in mind:

Example C.2. We consider the irreducible hypersurface

$$X = \operatorname{Spec}(\mathbb{C}[x, y, z, w]/(xy - wz)) \subseteq \mathbb{A}^4.$$

This is a normal variety, because hypersurfaces satisfy Serre's condition (S_2) and this hypersurface is moreover smooth in codimension 1. Let D be the prime Weil divisor defined by the equations $\{x = z = 0\}$, i.e.,

$$D = \operatorname{Spec}(\mathbb{C}[x, y, z, w]/(xy - wz, x, z)) \cong \operatorname{Spec}(\mathbb{C}[y, w]).$$

Let D' be the prime Weil divisor defined by the equations $\{y = w = 0\}$, i.e.,

$$D' = \operatorname{Spec}(\mathbb{C}[x, y, z, w]/(xy - wz, y, w)) \cong \operatorname{Spec}(\mathbb{C}[x, z]).$$

Now $D \cap D' \cong \operatorname{Spec}(\mathbb{C})$, so the intersection of these two prime Weil divisors has more codimension than it would if they were $\mathbb{Q}$-Cartier divisors. This shows that none of them are $\mathbb{Q}$-Cartier, so X is not $\mathbb{Q}$-factorial. Note also that X is the analytic-local model of a node, because we can perform the change of coordinates

$$\begin{cases} x_1 = \frac{1}{2}(x + y), \\ x_2 = \frac{i}{2}(x - y), \\ x_3 = \frac{1}{2}(w - z), \\ x_4 = \frac{i}{2}(w + z) \end{cases}$$

to rewrite X as

$$X \cong \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1^2 + x_2^2 + x_3^2 + x_4^2)).$$

From this example we can deduce that local analytic $\mathbb{Q}$-factoriality does not follow from $\mathbb{Q}$-factoriality nor from analytic $\mathbb{Q}$-factoriality:

Example C.3. Let $X \subseteq \mathbb{P}^4$ be a nodal quartic 3-fold with at most 8 nodes. Then X is $\mathbb{Q}$-factorial [Che06, Theorem 2], hence also analytic $\mathbb{Q}$-factorial because it is projective. But X is not locally analytically $\mathbb{Q}$-factorial, because we have seen in Example C.2 that 3-dimensional nodes are note $\mathbb{Q}$-factorial.

We finally note the following:

Fact C.4 ([KM98, Proposition 5.15])**.** *Let X be an algebraic or analytic variety over $\mathbb{C}$ with quotient singularities only. Then X has rational singularities and X is $\mathbb{Q}$-factorial.*

See also [SP, 0AG8] for more details in the algebraic case. More results on $\mathbb{Q}$-factoriality can be found in [BGS11].

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