Derived categories of Fano fibrations

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Born 10th February 1995 in Vigo, Spain

18th July 2019

Master's Thesis Mathematics

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Abstract

Fano fibrations arise naturally in the birational classification of algebraic varieties. We show that these morphisms always induce a semiorthogonal decomposition on the derived category of the domain, even if we do not assume them to be flat. This decomposition will be given by a collection of relative exceptional objects, and in contrast with the Fano case, we will see that Calabi-Yau fibrations do not admit any relative exceptional object. Finally, we will make some remarks about the singular case, which is of central importance in higher dimensional birational geometry.

Acknowledgements

First I would like to thank my supervisors Luca Tasin and Pieter Belmans for all their advice and for many useful and interesting discussions. I would also like to thank Daniel Huybrechts for all his mathematical and academic advice during my Master's.

The atmosphere in Bonn is really stimulating and welcoming. I would like to thank all the professors, students and organisational staff at Bonn for this. I feel particularly grateful to Karen Bingel for her help at the beginning of the Master's and to my classmates for working together throughout the Master's.

Finally, I would like to thank all the people working to make LaTeX better every day. I believe that knowledge should be widely shared, and their work makes it easier for the rest of us to achieve this goal.

Notation and conventions

- We will follow the notation and conventions in [Har77] and in [KM98]. In particular, we will always consider varieties to be irreducible.
- We will use bold fonts for categories (e.g. $\mathbf{A}$) and calligraphy for sheaves (e.g. $\mathcal{F}$). All rings are commutative and with unit and we denote the category of modules (resp. finitely generated modules) over a ring R by $\mathbf{Mod}(R)$ (resp. $\mathbf{mod}(R)$). We denote the category of sheaves of modules (resp. coherent and quasi-coherent sheaves of modules) over a scheme X by $\mathbf{Mod}(X)$ (resp. $\mathbf{Coh}(X)$ and $\mathbf{Qcoh}(X)$).
- The right (resp. left) derived functor of a functor F will be denoted $\mathbb{R}F$ (resp. $\mathbb{L}F$). The higher derived functors will be denoted R^iF (resp. L^iF).
- The i-th cohomology of a complex of sheaves $\mathcal{F}^\bullet$ will be denoted by $\mathcal{H}^i(\mathcal{F}^\bullet)$ to distinguish it from sheaf cohomology.
- The dual of a vector space V (resp. of a locally free sheaf $\mathcal{E}$) will be denoted by $V^\vee$ (resp. $\mathcal{E}^\vee$) to avoid overloading the symbol $(-)^*$.
- We will use the terms distinguished triangles and triangulated functors instead of exact triangles and exact functors. Again, the reason is to avoid overloading the word exact.
- All triangulated subcategories are assumed to be strictly full. Most explicitly defined subcategories are defined in terms of isomorphism invariant properties, so strictly fullness will often be automatic and we will not check it every time separately.
- If $\mathcal{E}$ is a locally free sheaf of finite rank on a projective variety Y, we denote by $\mathbb{V}(\mathcal{E}) = \mathrm{Spec}_Y(\mathrm{Sym}^\bullet \mathcal{E})$ (resp. $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_Y(\mathrm{Sym}^\bullet \mathcal{E}))$) the associated vector bundle (resp. projective bundle). When we deal with these geometric objects, we usually restrict our attention to closed points without further remarks about it (cf. [Har77, Proposition II.2.6]). The fibre of $\mathbb{V}(\mathcal{E})$ over some closed point $y \in Y$ will be denoted by $\mathbb{V}(\mathcal{E})_y$.

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Introduction

Our main source of motivation is the Minimal Model Program (MMP) for the birational classification of algebriac varieties, so let us start by recalling its main ideas. During this introduction, a *variety* will mean a normal projective variety over $\mathbb{C}$ with terminal singularities. In particular, curves and surfaces will always be smooth.

Let S be an algebraic surface. If we blow up a point $p \in S$, we obtain a new surface $\tilde{S}$ which is birational but not isomorphic to S. In particular, each birational equivalence class of surfaces contains infinitely many smooth representatives, so a natural first step towards the birational classification of algebraic surfaces is to restrict our attention to a smaller class of nice representatives. The purpose of the MMP is to produce such nice representatives after a sequence of well-understood birational transformations.

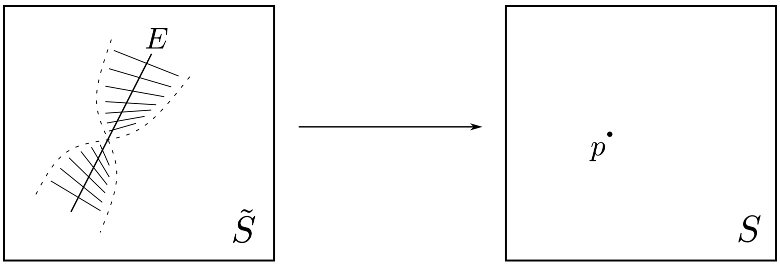

Figure 1: The blow-up of a surface at a point.

When we blow up a surface at a point, we replace this point by a line $E \cong \mathbb{P}^1$ in such a way that the normal bundle to E is the tautological line bundle on $\mathbb{P}^1$, which implies that it has negative self-intersection $E^2 = -1$. Conversely, Castelnuovo's contractibility criterion allows us to contract all curves $E \cong \mathbb{P}^1$ with $E^2 = -1$ to a point. So the idea of the MMP in dimension 2 consists of contracting all such curves until we obtain a surface which is not the blow-up of any other surface at a point. This was already known to the Italian School at the beginning of the 20th century, but it was not until the '80s when Mori found a way to generalise this to higher dimensions. His idea was based on the notion of *extremal ray*, which we now briefly describe.

Let X be a variety as above. The codimension of its singular locus is at least 2, so we can extend the canonical divisor on its smooth locus to a Weil divisor K_X. Let $N_1(X)$ be the real vector space of 1-cycles with $\mathbb{R}$ coefficients up to numerical equivalence, where two cycles are numerically equivalent if and only if they have the same intersection with every

Cartier divisor. Let $N^1(X)$ be the real vector space of Cartier divisors with $\mathbb{R}$ coefficients up to numerical equivalence, which is finite dimensional by the Theorem of the Base of Severi (see [Laz04, Proposition 1.1.16]). By definition of numerical equivalence we obtain a perfect pairing

$$N_1(X) \times N^1(X) \to \mathbb{R}$$

and in particular $N_1(X)$ is also a finite dimensional $\mathbb{R}$-vector space. The closure of the convex cone generated by all effective 1-cycles in the Euclidean topology is called the *cone of curves* of X, denoted $\overline{\mathrm{NE}}(X)$. Since X has only terminal singularities, K_X is $\mathbb{Q}$-cariter, so we can intersect it with 1-cycles obtaining a linear form on $N_1(X)$. To help visualization, we can picutre the cone of curves on a hyperplane section which we also divide into K_X positive and K_X negative side:

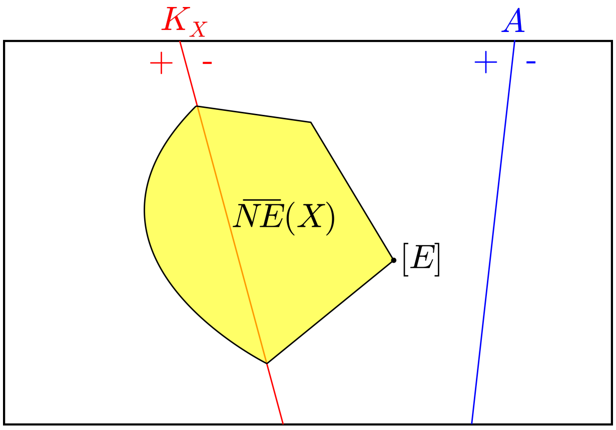

Figure 2: The cone of curves on X.

Since X is projective, we can find some very ample divisor A which is also drawn in the picture. By Mori's Cone Theorem (see [Mat02, Theorem 7-2-1]), the part of the cone of curves which lies inside the K_X-negative half-space has the polyhedral shape that we can see in figure 2. Moreover, its edges (seen as points in the hyperplane section) always contain some curve $E \subseteq X$. We should introduce some terminology now:

Definition 0.1. We say that X is...

i) *Fano*, if $\overline{\mathrm{NE}}(X)$ is contained in the interior of the K_X negative half-space.

ii) *Minimal*, if $\overline{\mathrm{NE}}(X)$ is contained in the K_X non-negative half-space.

The idea of MMP is then to contract all K_X-negative extremal rays in an attempt to make the canonical divisor more positive and hopefully obtain a minimal model of X, i.e. a minimal variety birational of X. To find such a contraction, we modify our ample divisor A slightly by adding multiples of K_X. This will bring it closer to our cone of curves until intersecting it eventually. Since ampleness is an open condition, we can slightly perturb A and assume that this intersection with the cone of curves will consist of a single point in the hyperplane section, say $[E]$. Some multiple mL of the resulting nef divisor L will then yield a base point free linear system $|mL|$ contracting a curve precisely when its class lies in the ray generated by $[E]$ (cf. [Mat02, Theorem 8-1-3]), so if there are only a few such curves, then the corresponding morphism $f \colon X \to Y$ will be birational. In this case, we substitute X by Y or by a flip X^+ and then we repeat the process on the corresponding variety. Otherwise, the morphism $f \colon X \to Y$ will be a Fano fibration, with $\dim X > \dim Y$. Since the dimension drops, this morphism is not birational anymore and we cannot iterate the process any further. So in this case it is not possible to find the desired minimal model. But Fano fibrations still have a nice structure, so we gladly accept this as a second possible outcome of our MMP.

Therefore, assuming that the MMP terminates, we can regard minimal varieties and Fano fibrations as the building blocks in birational geometry. More precisely, we can obtain any variety by performing a finite sequence of well-understood birational transformations on a minimal variety or on a Fano fibration.

Suppose that we are interested now in computing the bounded derived category of coherent sheaves of a variety X. We can understand X in terms of its output under the MMP and the correspondent sequence of well-understood birational transformations. So if we knew how the derived categories of these building blocks look like and how these birational transformations affect the derived category, then we could compute $\mathbf{D}^{\mathrm{b}}(X)$.

A main example of birational transformations are blow-ups. If $f \colon X = \mathrm{Bl}_Z(Y) \to Y$ is the blow-up of a smooth projective variety along a smooth center of codimension $c \geqslant 2$, then Orlov's blow-up formula gives us a *semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Z)_1, \ldots, \mathbf{D}^{\mathrm{b}}(Z)_{c-1} \rangle.$$

The precise meaning of this decomposition will be given in the first chapter, but for now let us just point out that this allows us to clarify the structure of $\mathbf{D}^{\mathrm{b}}(X)$ in terms of the structures of $\mathbf{D}^{\mathrm{b}}(Y)$ and $\mathbf{D}^{\mathrm{b}}(Z)$. Hence this already gives us some motivation to investigate the approach described above a bit further.

One of the building blocks of algebraic varieties are Fano varieties, which are the absolute case of Fano fibrations. Among them, the first exmaple are projective spaces. Beilinson established in [Bei78] the existence of a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathbf{D}^{\mathrm{b}}(\mathbb{C})_0, \ldots, \mathbf{D}^{\mathrm{b}}(\mathbb{C})_n \rangle.$$

This result was generalised by Orlov to projective bundles in [Orl93], paper in which he also proves his blow-up formula. The result, known as Orlov's projective bundle formula, states that for every $\mathbb{P}^n$-bundle $f \colon X = \mathbb{P}(\mathcal{E}) \to Y$ over a smooth projective variety we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(Y)_0, \ldots, \mathbf{D}^{\mathrm{b}}(Y)_n \rangle.$$

Beilinson's semiorthogonal decomposition was extended by Kuznetsov to smooth Fano varieties (see for example [Kuz09a]). As pointed out by several authors (e.g. Auel and Bernardara in [AB17, Proposition 2.3.6]) this generalises to the relative case of flat

Fano fibrations between smooth projective varieties. In the second chapter we will explain this result in some detail, extending it also to the non-flat case:

Theorem (See theorem 2.28)**.** *Let $f\colon X \to Y$ be a Fano fibration between smooth projective varieties. Let H be an f-ample Cartier divisor such that $-K_X \equiv_f rH$ for some rational number $r \geqslant 1$ and let $\mathcal{L}$ denote $\mathcal{O}_X(H)$. Then we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_f, \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes 2}, \ldots, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes \lceil r-1 \rceil} \rangle,$$

where $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}$ denotes the essential image of the fully faithful functor

$$\mathcal{F}^\bullet \mapsto \mathbb{L}f^*(\mathcal{F}^\bullet) \otimes \mathcal{L}.$$

At the end of the second chapter we will make some remarks about the singular case, which we have to consider in order to run a MMP in higher dimensions. Ironically, most problems arise from the category theoretic part, rather than from the geometric part. The kind of singularities that we want to consider are well-behaved with respect to differentials, so all the Kodaira-type vanishings needed in the proofs will still hold. But the bounded derived category loses some key properties such as saturatedness. Roughly speaking, when X is a smooth projective variety, Bondal and Kapranov showed in the '90s that any embedding $\mathbf{D}^{\mathrm{b}}(X) \subseteq \mathbf{T}$ into a triangulated category has a right adjoint, which implies the existence of a semiorthogonal decomposition. But in the singular setting the existence of adjoints is not for automatic anymore, so from the beginning we have tried to avoid using the machinery from [BK90] and produce semiorthogonal decompositions by hand instead.

Finally, since we are interested in the triangulated structure of $\mathbf{D}^{\mathrm{b}}(X)$, we have included an appendix on triangulated categories. We will focus on admissible subcategories and all results related to this notion, since it is closely related to semiorthogonal decompositions.

Chapter 1

Derived categories of varieties

Our goal in this first chapter is to recall the basics of derived categories and study some aspects of the natural triangulated structure on the derived category $\mathbf{D}^{\mathrm{b}}(X)$ of coherent sheaves on a variety X.

▲. We will recall the necessary definitions on triangulated categories like this whenever they come up. For details we refer to appendix A.

1.1 Recollections on derived categories

In this section we will give a quick overview of the necessary preliminaries in homological algebra. We refer to the literature for the missing details, e.g. [GM03, Chapter III].

Construction of the derived category

Let us briefly recall the main ideas of the construction of the derived category of an abelian category.

Definition 1.1. Let $\mathbf{A}$ be an abelian category. The *derived category* of $\mathbf{A}$ is the localisation of its category of cochain complexes $\mathbf{Ch}(\mathbf{A})$ at all quasi-isomorphisms. Explicitly, this localisation would consist of a category $\mathbf{D}(\mathbf{A})$ together with a functor $Q\colon \mathbf{Ch}(\mathbf{A}) \to \mathbf{D}(\mathbf{A})$ such that:

(i) If f is a quasi-isomorphism, then $Q(f)$ is an isomorphism.

(ii) If $F\colon \mathbf{Ch}(\mathbf{A}) \to \mathbf{C}$ is another functor with the previous property, then there exists a unique functor $G\colon \mathbf{D}(\mathbf{A}) \to \mathbf{C}$ such that $F = G \circ Q$.

The usual construction has two steps. They affect only morphisms:

$$\begin{tikzcd} \boxed{\begin{array}{c}\mathbf{Ch}(\mathbf{A})\\ \text{Cochain complexes}\end{array}} \arrow[r, rightsquigarrow, "\begin{array}{c}\text{Quotient out}\\ \text{null homotopics}\end{array}"] & \boxed{\begin{array}{c}\mathbf{K}(\mathbf{A})\\ \text{Homotopy category}\end{array}} \arrow[r, rightsquigarrow, "\begin{array}{c}\text{Localise at}\\ \text{quasi-isomorphisms}\end{array}"] & \boxed{\begin{array}{c}\mathbf{D}(\mathbf{A})\\ \text{Derived category}\end{array}} \end{tikzcd}$$

The reason for the intermediate step $\mathbf{K}(\mathbf{A})$ is that we want to construct $\mathbf{D}(\mathbf{A})$ using the calculus of fractions described in [GM03, Lemma III.2.8], and quasi-isomorphisms form a multiplicative system only after we identify chain homotopic morphisms (cf. [GM03, Remark III.2.7.b)]).

The quotient functor $\mathbf{Ch}(\mathbf{A}) \to \mathbf{K}(\mathbf{A})$ is the localisation of $\mathbf{Ch}(\mathbf{A})$ at all chain homotopy equivalences. Quotients and localisations are not the same in general, but in this case they are. The proof can be summarised in the following picture:

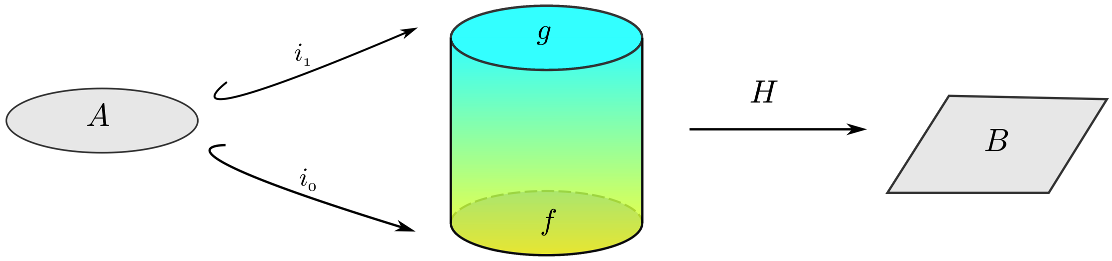

Figure 1.1: A functor that sends chain homotopy equivalences to isomorphisms takes the same value at chain homotopic morphisms.

Indeed, a chain homotopy s between f and g is the same as a morphism $H = (f, s, g)\colon \mathrm{Cyl}(A) \to B$ as in figure 1.1. Since i_1 and i_0 are chain homotopy inverses of the same morphism and F sends chain homotopy equivalences to isomorphisms, we have $F(i_0) = F(i_1)$ and thus $F(f) = F(H) \circ F(i_0) = F(g)$.

Chain homotopy equivalences are quasi-isomorphisms, so this brings us closer to our goal. We should make sure now that there is still some non trivial step required to construct $\mathbf{D}(\mathbf{A})$.

Example 1.2. $\mathbf{K}(\mathbf{Ab})$ and $\mathbf{D}(\mathbf{Ab})$ are not equivalent. For example, the complex $A^\bullet$ corresponding to the short exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a zero object in $\mathbf{D}(\mathbf{Ab})$ but not in $\mathbf{K}(\mathbf{Ab})$. This happens because $A^\bullet \xrightarrow{\sim} 0$ is a quasi-isomorhpism but not a chain homotopy equivalence. Indeed, the only candidate for homotopy inverse is $0 \to A^\bullet$, and the identity on $A^\bullet$ is not null homotopic.

The next step is then to check that quasi-isomorphisms form a multiplicative system on $\mathbf{K}(\mathbf{A})$ and use the aforementioned calculus of fractions to construct $\mathbf{D}(\mathbf{A})$. In most cases $\mathbf{K}(\mathbf{A})$ and $\mathbf{D}(\mathbf{A})$ are not abelian categories, but they always carry natural trianuglated structues. We will use this triangulated structure to check that quasi-isomorphisms form a multiplicative system which is compatible with the triangulation (cf. appendix A.2).

▲ (A.1)**.** A *triangulated category* is an additive category $\mathbf{T}$ together with an additive automorphism $\Sigma\colon \mathbf{T} \to \mathbf{T}$ and a specified collection of *distinguished triangles*, which are sextuples (X, Y, Z, f, g, h) of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ verifying certain axioms (TR1) to (TR4).

We refer to Σ as *suspension*[1]. By the (TR1) axiom we can always complete a morphism $f\colon X \to Y$ into a distinguished triangle $X \to Y \to C \to \Sigma X$. We call C a *cone* of f.

A *triangulated functor* between triangulated categories is one that commutes with the suspension and preserves distinguished triangles.

To describe the triangulated structure on $\mathbf{K}(\mathbf{A})$ we need:

Definition 1.3. Let $A^\bullet$ be a cochain complex. Define its *shift* as the complex $A[p]^\bullet$ with $A[p]^i = A^{i+p}$ and with differential $d_{A[p]} = (-1)^p d_A$.

Define the *mapping cone* of a morphism of cochain complexes $f\colon A^\bullet \to B^\bullet$ as the complex $C(f)^\bullet$ with $C(f)^i = A^{i+1} \oplus B^i$ and with differential $d_{C(f)} = \left(\begin{smallmatrix} d_{A[1]} & 0 \\ f & d_B \end{smallmatrix}\right)$.

We take Σ to be the shift [1] and we take distinguished triangles to be all triangles isomorphic to a *strict triangle*, which is one of the form

$$A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet.$$

This turns $\mathbf{K}(\mathbf{A})$ into a triangulated category. The zero cohomology functor $H^0\colon \mathbf{Ch}(\mathbf{A}) \to \mathbf{A}$ factors through $\mathbf{K}(\mathbf{A})$. Because of the cone short exact sequence $0 \to B^\bullet \to C(f)^\bullet \to A[1]^\bullet \to 0$ and its long exact sequence in cohomology, in which the coboundary morphism is the morphism induced by f in cohomology, applying H^0 to a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ yields an exact sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$. By rotating our triangle and swapping some signs we obtain a long exact sequence

$$\cdots \to H^0(A[i]^\bullet) \to H^0(B[i]^\bullet) \to H^0(C[i]^\bullet) \to H^0(A[i+1]^\bullet) \to \cdots$$

which is nothing but the original long exact sequence of the cone when the triangle is strict.

▲ (A.5)**.** Let $\mathbf{T}$ be a triangulated category and $\mathbf{A}$ an abelian category. An additive functor $H\colon \mathbf{T} \to \mathbf{A}$ is called a *cohomological functor* if it induces an exact sequence $H(X) \to H(Y) \to H(Z)$ for every distinguished triangle $X \to Y \to Z \to \Sigma X$.

By (TR2) the previous exact sequence can be extended to a long exact sequence

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}h)} H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

We have then our triangulated category $\mathbf{K}(\mathbf{A})$ with a cohomological functor $H^0\colon \mathbf{K}(\mathbf{A}) \to \mathbf{A}$. Whenever we have a cohomological functor $H\colon \mathbf{T} \to \mathbf{A}$ we can consider its kernel $\mathrm{Ker}(H) = \{X \in \mathbf{T} \mid H(\Sigma^n X) = 0 \text{ for all } n \in \mathbb{Z}\}$, which is a thick triangulated subcategory.

[1]We will deliberately use topological terminology when we talk about triangulated categories in general. Many constructions in homological algebra are inspired in their topological analogues, making them behave in a similar way (cf. figure 1.1). So thinking of these constructions topologically can help find the right formulas. See also [Wei94, Section 1.5].

▲ (A.16). Let $\mathbf{T}$ be a triangulated category and $\mathbf{S}$ a strictly full and non-empty subcategory. We say that $\mathbf{S}$ is a *triangulated subcategory* of $\mathbf{T}$, denoted $\mathbf{S} \leqslant \mathbf{T}$, if it is closed under suspension and under taking cones. We say that $\mathbf{S}$ is a *thick* triangulated subcategory of $\mathbf{T}$, denoted $\mathbf{S} \trianglelefteq \mathbf{T}$, if it is moreover closed under direct summands.

If $\mathbf{S} \trianglelefteq \mathbf{T}$, then the collection of morphisms in $\mathbf{T}$ whose cone lies in $\mathbf{S}$ is a multiplicative system compatible with the triangulation (see example A.32). By [GM03, Theorem IV.2.2], the localisation of $\mathbf{T}$ at this family of morphisms carries a natural triangulated structure making the localisation functor triangulated. The result is called the *Verdier quotient* of $\mathbf{T}$ by $\mathbf{S}$, denoted $\mathbf{T}/\mathbf{S}$.

So back to our homotopy category, we can consider

$$\mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{A})/\operatorname{Ker} H^0.$$

By the cone long exact sequence, a morphism $A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if its cone is exact, which means precisely that it is in $\operatorname{Ker} H^0$. Hence the collection of morphisms in $\mathbf{K}(\mathbf{A})$ whose cone lies in $\operatorname{Ker} H^0$ is the collection of quasi-isomorphisms, so the Verdier quotient $\mathbf{K}(\mathbf{A})/\operatorname{Ker} H^0$ is the localisation $\mathbf{D}(\mathbf{A})$. The axioms of multiplicative system compatible with the triangulation ensure that the suspension is well defined in the localisation, and distinguished triangles in $\mathbf{D}(\mathbf{A})$ are then those isomorphic to the image of a distinguished triangle in $\mathbf{K}(\mathbf{A})$.

The conclusion is that a morphism $A^\bullet \to B^\bullet$ in $\mathbf{D}(\mathbf{A})$ is represented by a *roof*

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "\sim"'] \arrow[dr, "f"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

where $s\colon C^\bullet \xrightarrow{\sim} A^\bullet$ denotes a quasi-isomorphism. The upper and bottom roofs below are equivalent if and only if we can find such a dashed roof making the diagram commute up to homotopy:

$$\begin{tikzcd}
& C_2^\bullet \arrow[dl, "\sim"'] \arrow[dr] & \\
A^\bullet & C^\bullet \arrow[u, dashed] \arrow[d, dashed] & B^\bullet \\
& C_2^\bullet \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

Note that if the diagram commutes up to homotopy, both dashed arrows are quasi-isomorphisms as soon as one of them is.

The composition of morphisms represetned by roofs (s, f) and (t, g) can be represented by the roof (st', gf') as shown in the following diagram commutative up to homotopy:

$$\begin{tikzcd}
 & & C \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f'"] & & \\
 & X \arrow[dl, "s"'] \arrow[dr, "f"] & & Z \arrow[dl, "t"'] \arrow[dr, "g"] & \\
A & & Y & & B
\end{tikzcd}$$

So in order to compose two such roofs we need to find a third roof (t', f') making the previous diagram commute up to homotopy. In our case we can use the mapping cone construction and set $C = C(i \circ f)$, where $i\colon Y \to C(t)$ denotes the inclusion. See [GM03, Theorem III.4.4] for the details.

Bounded variants of the derived category

We may repeat the previous construction but starting from $\mathbf{Ch}^\star(\mathbf{A})$, where $\star \in \{\mathrm{b}, +, -\}$ stands for bounded, bounded below and bounded above complexes respectively. We have then $\mathbf{K}^\star(\mathbf{A}) \leqslant \mathbf{K}(\mathbf{A})$, and the restriction of H^0 is still a cohomological functor on this triangulated subcategory. Hence we can quotient again by its kernel. The resulting category $\mathbf{D}^\star(\mathbf{A})$ is then equivalent under the natural inclusion to the full subcategory of $\mathbf{D}(\mathbf{A})$ with bounded, bounded below and bounded above cohomologies respectively (see [Huy06, Proposition 2.30]).

Let now $\mathbf{I_A} \subseteq \mathbf{A}$ be the full subcategory of injective objects and let $\mathbf{K}^+(\mathbf{I_A})$ be the full subcategory $\mathbf{K}^+(\mathbf{A})$ of bounded below cochain complexes of injective objects. Then $\mathbf{K}^+(\mathbf{I_A}) \leqslant \mathbf{K}^+(\mathbf{A})$, because the direct sum of injective objects is injective and thus this full subcategory is closed under cones. Again, H^0 restricts to a cohomological functor on this triangulated subcategory, so we can localise $\mathbf{K}^+(\mathbf{I_A})$ at all quasi-isomorphisms with roofs as before.

Lemma 1.4 ([Har66, Lemma I.4.5])**.** *Let $s\colon I^\bullet \to A^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded below complex of injective objects. Then s has a retraction up to homotopy, i.e. we can find $t\colon A^\bullet \to I^\bullet$ such that $ts \sim \mathrm{id}_I$.*

Proof. Consider the natural projection $(1,0)\colon C(s)^\bullet \to I[1]^\bullet$ from the mapping cone. This is a morphism from an acyclic complex to a bounded below complex of injective objects. An inductive argument shows that any such morphism must be null homotopic, so let $((k^i, t^i))_{i\in\mathbb{Z}}$ be a null homotopy of $(1,0)$. For all $i \in \mathbb{Z}$ we have

$$(\mathrm{id}_{I^{i+1}}, 0) = (k^{i+1}, t^{i+1}) \begin{pmatrix} -d_I^{i+1} & 0 \\ s^{i+1} & d_A^i \end{pmatrix} - d_I^i(k^i, t^i).$$

We see from the second component that $t\colon A^\bullet \to I^\bullet$ is a morphism of cochain complexes. And from the first component we obtain a null homotopy of $\mathrm{id}_I - ts$. □

Note that such a retraction t must also be a quasi-isomorphism by the two out of three rule. An important formal consequence of this lemma is that every quasi-isomorphism s in $\mathbf{K}^+(\mathbf{I_A})$ is already an isomorphism. Indeed, we can find t and r such that $ts = \mathrm{id}$ and

$rt = \mathrm{id}$ in the homotopy category. Since t is a retraction, it is an epimorphism. So from $st = rtst = r(ts)t = rt$ we conclude that $s = r$ in the homotopy category.

This implies in turn that $\mathbf{K}^+(\mathbf{I_A})$ is already the localisation of $\mathbf{Ch}^+(\mathbf{I_A})$ at all quasi-isomorphisms. So we can describe this category equivalently with roofs instead of usual morphisms. This has the following consequence:

Proposition 1.5 ([GM03, Theorem III.5.21]). *The canonical functor* $\mathbf{K}^+(\mathbf{I_A}) \to \mathbf{D}^+(\mathbf{A})$ *is fully faithful and triangulated. In particular, if* $\mathbf{A}$ *has enough injectives, then we have an equivalence of categories*

$$K^+(\mathbf{I_A}) \simeq \mathbf{D}^+(\mathbf{A})$$

Proof. This functor is the composition of the triangulated inclusion $\mathbf{K}^+(\mathbf{I_A}) \leqslant \mathbf{K}^+(\mathbf{A})$ and the triangulated localisation $\mathbf{K}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{A})$, so it is indeed triangulated.

To check fully faithfulness, we may work with left roofs instead of with right roofs as we have done up until now[2]. We only need to flip all the arrows involved in the definition of roofs and their equivalence relation, putting quasi-isomorphisms on the side of the codomain. Then both injectivity and surjectivity on roofs follow directly from the previous lemma.

If $\mathbf{A}$ has enough injectives, then every bounded below complex has an injective resolution. See [GM03, Section III.5.25] for an explicit description of this injective resolution. $\square$

Example 1.6. Let $\Bbbk$ be a field and let $\mathbf{D}^{\mathrm{b}}(\Bbbk)$ (resp. $\mathbf{K}^{\mathrm{b}}(\Bbbk)$) denote the bounded derived (resp. homotopy) category of finite dimensional $\Bbbk$-vector spaces. Every $\Bbbk$-vector space is both injective and projective, so $\mathbf{D}^{\mathrm{b}}(\Bbbk) \cong \mathbf{K}^{\mathrm{b}}(\Bbbk)$. Moreover, every $V^\bullet \in \mathbf{D}^{\mathrm{b}}(\Bbbk)$ is isomorphic to the complex of its cohomology groups with trivial differentials:

$$V^\bullet \cong \bigoplus_{p\in\mathbb{Z}} H^p(V^\bullet)[-p]$$

Indeed, we can write $V^p \cong \mathrm{Ker}(d^p) \oplus \mathrm{Im}(d^p)$ and then define the quasi-isomorphism given by the commutative squares

$$\begin{tikzcd}
\mathrm{Ker}(d^p) \oplus \mathrm{Im}(d^p) \arrow[r] \arrow[d] & \mathrm{Ker}(d^{p+1}) \oplus \mathrm{Im}(d^{p+1}) \arrow[d] \\
\mathrm{Ker}(d^p)/\,\mathrm{Im}(d^{p-1}) \arrow[r, "0"] & \mathrm{Ker}(d^{p+1})/\,\mathrm{Im}(d^p)
\end{tikzcd}$$

If a morphism between two complexes with trivial differentials is null homotopic, then it must be the zero morphism. Moreover, commutativity of all necessary squares is trivially satisfied. Therefore we have

$$\mathrm{Hom}_{\mathbf{D}^b(\Bbbk)}(V^\bullet, W^\bullet) \cong \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\Bbbk}(H^i(V^\bullet), H^i(W^\bullet))$$

[2]The resulting category is canonically isomorphic to the category constructed from right roofs, because both of them satisfy the same universal property.

Remark 1.7 ([GM03, Section III.6.3]). We can generalise proposition 1.5 as follows. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a left exact functor. We say that a class $\mathbf{E} \subseteq \mathrm{Ob}(\mathbf{A})$ of objects is *adapted* to F if the following conditions are satisfied:

i) $\mathbf{E}$ is closed under direct sums.

ii) F maps acyclic complexes in $\mathbf{Ch}^+(\mathbf{E})$ to acyclic complexes.

iii) Any object $A \in \mathrm{Ob}(\mathbf{A})$ is a subobject of an object from $\mathbf{E}$.

If $\mathbf{E}$ is adapted for a left exact functor F, then $\mathbf{K}^+(\mathbf{E}) \leqslant \mathbf{K}^+(\mathbf{A})$ as before, and the canonical triangulated functor

$$\mathbf{K}^+(\mathbf{E})/\operatorname{Ker} H^0 \to \mathbf{D}^+(\mathbf{A})$$

induced by the universal property of Verdier quotients is an equivalence of categories.

The dual statements with bounded above complexes of projective objects and classes of objects adapted to right derived functors are analogous.

Derived functors.

We come finally to derived functors. Let us discuss for example the case of right derived functors. The case of left derived functors is dual of this one, so there is no loss of generality in doing so.

Definition 1.8 ([GM03, Definition III.6.6]). Let $F\colon \mathbf{A} \to \mathbf{B}$ be a left exact functor between abelian categories. Its *right derived functor* is a triangulated functor $\mathbb{R}F\colon \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{B})$ together with a natural transformation $\eta_0\colon Q_{\mathbf{B}} \circ \mathbf{K}^+(F) \to \mathbb{R}F \circ Q_{\mathbf{A}}$ satisfying the following universal property: any triangulated functor $G\colon \mathbf{D}^+(\mathbf{A}) \to D^+(\mathbf{B})$ together with a natural transformation $\eta\colon Q_{\mathbf{B}} \circ \mathbf{K}^+(F) \to G \circ Q_{\mathbf{A}}$ factors through a unique natural transformation $\mathbb{R}F \to G$.

In a diagram, using $\Rightarrow$ for natural transformations:

$$\begin{tikzcd}
\mathbf{K}^+(\mathbf{A}) \arrow[r, "\mathbf{K}^+(F)"] \arrow[d, "Q_{\mathbf{A}}"'] & \mathbf{K}^+(\mathbf{B}) \arrow[d, "Q_{\mathbf{B}}"] \arrow[dl, Rightarrow, "\eta_0"'] \arrow[ddl, Rightarrow, "\eta"] \\
\mathbf{D}^+(\mathbf{A}) \arrow[r, "\mathbb{R}F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\exists!"] & \mathbf{D}^+(\mathbf{B}) \arrow[d, equal] \\
\mathbf{D}^+(\mathbf{A}) \arrow[r, "G"'] & \mathbf{D}^+(\mathbf{B})
\end{tikzcd}$$

If F is left exact but not exact, then we can find some short exact sequence $0 \to A^1 \to A^2 \to A^3 \to 0$ such that $0 \to F(A^1) \to F(A^2) \to F(A^3) \to 0$ is not exact. The first sequence is acyclic as a complex, hence quasi-isomorphic to zero, whereas the second one is not. Therefore the naive candidate for $\mathbb{R}F$ is not even well-defined, because it does not preserve isomorphisms.

Instead, if we assume that $\mathbf{A}$ has enough injectives, we may define $\mathbb{R}F$ to be the composition

$$\mathbf{D}^+(\mathbf{A}) \simeq \mathbf{K}^+(\mathbf{I_A}) \leqslant \mathbf{K}^+(\mathbf{A}) \xrightarrow{\mathbf{K}^+(F)} \mathbf{K}^+(\mathbf{B}) \to \mathbf{D}^+(\mathbf{B}).$$

That is, $\mathbb{R}F(A^\bullet)$ is $F(I^\bullet)$ seen as an object in the derived category $\mathbf{D}^+(\mathbf{B})$, where $I^\bullet$ is an injective resolution of $A^\bullet$. There is some ambiguity involved in this construction, e.g. the choice of a triangulated inverse to the equivalence $\mathbf{K}^+(\mathbf{I_A}) \to \mathbf{D}^+(\mathbf{A})$ is not canonical. But since $\mathbb{R}F$ is indeed the right derived functor of F in the sense of the previous universal property, these choices do not affect the final result. See [GM03, Sections III.6.8-11] for the proof of the universal property.

From derived functors between the derived categories we obtain the classical derived functors by taking the cohomology of the resulting complex. That is, for $A^\bullet \in \mathbf{D}^+(\mathbf{A})$ we define

$$\mathrm{R}^i F(A^\bullet) = H^i(\mathbb{R}F(A^\bullet)) \in \mathbf{B}.$$

If we regard $A \in \mathbf{A}$ as a complex concentrated on degree zero, then the previous construction yields indeed the usual way to compute classical derived functors, namely, we replace A by an injective resolution, apply the functor and take cohomology of the result.

Example 1.9 ([GM03, Section III.6.14])**.** Let A be an object in an abelian category $\mathbf{A}$ with enough injectives. The functor $\mathrm{Hom}_{\mathbf{A}}(A,-)\colon \mathbf{A} \to \mathbf{Ab}$ is left exact, so we can consider its right derived functor $\mathbb{R}\mathrm{Hom}_{\mathbf{A}}(A,-)$. Denote by $\mathrm{Ext}^i_{\mathbf{A}}(A,-)$ the corresponding classical derived functors. Then there are natural isomorphisms

$$\mathrm{Ext}^i_{\mathbf{A}}(A,B) \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(A, B[i])$$

where we regard A and B as complexes concentrated on degree 0.

Proof. First note that for $I^\bullet$ a bounded below complex of injective objects we have isomorphisms

$$\mathrm{Hom}_{\mathbf{K}(\mathbf{A})}(A^\bullet, I^\bullet) \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(A^\bullet, I^\bullet)$$

for all $A^\bullet \in \mathbf{D}(\mathbf{A})$. This is an immediate consequence of lemma 1.4 if we use left roofs instead of right roofs.

Let now

$$0 \to B \to I^0 \to I^1 \to \cdots$$

be an injective resolution of B. Then $\mathbb{R}\mathrm{Hom}_{\mathbf{A}}(A,B)$ is by definition a complex isomorphic to $\mathrm{Hom}_{\mathbf{A}}(A, I^\bullet)$ in the derived category $\mathbf{D}^+(\mathbf{Ab})$. So by definition we have

$$\mathrm{Ext}^i_{\mathbf{A}}(A,B) = H^i(\mathrm{Hom}_{\mathbf{A}}(A, I^\bullet)).$$

An element $f \in \mathrm{Hom}_{\mathbf{A}}(A, I^i)$ is a cocycle if and only if f defines a morphism of complexes $f\colon A \to I[i]^\bullet$. Moreover, this morphism is null homotopic if and only if f is a coboundary. Hence

$$\mathrm{Ext}^i_{\mathbf{A}}(A,B) \cong \mathrm{Hom}_{\mathbf{K}(\mathbf{A})}(A, I[i]^\bullet) \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(A, I[i]^\bullet).$$

Using that $B \cong I^\bullet$ in $\mathbf{D}^+(\mathbf{A})$ we get the result. □

Remark 1.10 ([Har66, Theorem I.5.1]). We can generalise remark 1.7 still a bit more. Instead of starting from a functor between the abelian categories, we may directly start from a triangulalted functor between triangulated categories $F\colon \mathbf{K}^+(\mathbf{A}) \to \mathbf{K}(\mathbf{B})$ such that there is a triangulated subcategory $\mathbf{K}_F \leqslant \mathbf{K}^+(\mathbf{A})$ which is *adapted* to F. This means that F preserves exactness of complexes in $\mathbf{K}_F$ and that every complex in $\mathbf{K}^+(\mathbf{A})$ is quasi-isomorphic to a complex in $\mathbf{K}_F$. Note that if $\mathbf{E}_F$ is an F-adapted class of objects, then $\mathbf{K}^+(\mathbf{E}_F) \leqslant \mathbf{K}^+(\mathbf{A})$ is a triangulated subcategory adapted to F.

Using this remark we can extend the previous example to

$$\operatorname{Ext}^i(A^\bullet, B^\bullet) = H^i(\mathbb{R}\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)),$$

yielding isomorphisms functorial in $A^\bullet$ (resp. in $B^\bullet$)

$$\operatorname{Ext}^i(A^\bullet, B^\bullet) \cong \operatorname{Hom}_{\mathbf{D}(\mathbf{A})}(A^\bullet, B[i]^\bullet)$$

whenever $\mathbf{A}$ has enough projectives (resp. injectives), as explained in [Har66, Theorem 6.4]. Motivated by these examples we further extend the meaning of Ext on arbitrary triangulated categories as $\operatorname{Ext}^i_{\mathbf{T}}(X, Y) = \operatorname{Hom}_{\mathbf{T}}(X, \Sigma^i Y)$.

One of the advantages of derived categories is that we can compute the composition of right derived functors in a nice way. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three abelian categories. Let $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ be left exact functors. Let $\mathbf{A}_F$ be an F-adapted class of objects and $\mathbf{B}_G$ be a G-adapted class of objects such that $F(\mathbf{A}_F) \subseteq \mathbf{B}_G$. Then the derived functors $\mathbb{R}F$, $\mathbb{R}G$ and $\mathbb{R}(G \circ F)$ exist. Moreover, the canonical natural transformation

$$\mathbb{R}(G \circ F) \to \mathbb{R}G \circ \mathbb{R}F$$

is a natural isomorphism. See [GM03, Theorem III.7.1].

But the composition of higher derived functors is not the higher derived functor of the composition. Instead, these two are related through the *Grothendieck spectral sequence.*

Theorem 1.11 ([Huy06, Proposition 2.66]). *Let* $F\colon \mathbf{K}^+(\mathbf{A}) \to \mathbf{K}^+(\mathbf{B})$ *and* $G\colon \mathbf{K}^+(\mathbf{B}) \to \mathbf{K}(\mathbf{C})$ *be two triangulated functors (cf. remark 1.7), where* $\mathbf{A}$ *and* $\mathbf{B}$ *are abelian categories with enough injective objects. Suppose moreover that* $\mathbf{B}_G \leqslant \mathbf{B}$ *is a* G*-adapted triangulated subcategory such that* $F(\mathbf{I}_{\mathbf{A}}) \subseteq \mathbf{B}_G$*. Then for every* $A^\bullet \in \mathbf{D}^+(\mathbf{A})$ *we have a bounded spectral sequence*

$$E_2^{p,q} = \mathrm{R}^p G(\mathrm{R}^q F(A^\bullet)) \Rightarrow \mathrm{R}^{p+q}(G \circ F)(A^\bullet).$$

Remark 1.12. This result follows from the standard spectral sequence of a double complex after reducing to the particular case of $F = \mathrm{id}_{\mathbf{A}}$, which yields a spectral sequence

$$E_2^{p,q} = \mathrm{R}^p F(H^q(A^\bullet)) \Rightarrow \mathrm{R}^{p+q} F(A^\bullet)$$

renaming G as F. In fact, in most of the applications we will only need this particular case.

1.2 Triangulated structure

In this section we will study some particularities of the triangulated structure on the derived category of a variety over a field $\Bbbk$. To avoid complications we will assume that $\Bbbk$ is an algebraically closed field of characteristic zero, but most of this first chapter can be generalised to arbitrary fields.

Definition 1.13. A $\Bbbk$-linear category is an additive category $\mathbf{A}$ such that the hom-groups are endowed with a $\Bbbk$-vector space structure making composition $\Bbbk$-bilinear.

A functor $F\colon \mathbf{A} \to \mathbf{B}$ between $\Bbbk$-linear categories is called $\Bbbk$-linear if the induced maps $\mathrm{Hom}_{\mathbf{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{B}}(FX, FY)$ are $\Bbbk$-linear for all $X, Y \in \mathbf{A}$.

All the implicit or explicit instances of the word *additive* in the previous section can be replaced by $\Bbbk$-*linear* if we are working with $\Bbbk$-linear abelian categories. This is the case when we deal with coherent sheaves on a scheme X over $\Bbbk$.

Different derived categories associated to a scheme

There are several derived categories one can associate to a variety X, so let us briefly recall what they are and how they relate to each other.

Definition 1.14. Let X be a scheme over $\Bbbk$.

i) The *unbounded derived category* of X is defined as the $\Bbbk$-linear triangulated category $\mathbf{D}(X) = \mathbf{D}(\mathbf{Qcoh}(X))$.

ii) The *bounded derived category* or simply *derived category* of X is defined as the $\Bbbk$-linear triangulated category $\mathbf{D}^{\mathrm{b}}(X) = \mathbf{D}^{\mathrm{b}}(\mathbf{Coh}(X))$.

Proposition 1.15 ([Har66, Propositions II.1.1 and II.1.2])**.** *The categories* $\mathbf{Mod}(X)$ *and* $\mathbf{Qcoh}(X)$ *have enough injectives and every sheaf of* $\mathcal{O}_X$ *modules is a quotient of a flat* $\mathcal{O}_X$*-module.*

Proposition 1.16 ([Huy06, Proposition 3.5] and [BN93, Corollary 5.5])**.** *Let X be a quasi-projective variety. The inclusions induce triangulated equivalences*

$$\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}_{coh}(\mathbf{Qcoh}(X)) \textit{ and } \mathbf{D}(X) \simeq \mathbf{D}_{qcoh}(\mathbf{Mod}(X)).$$

So in order to define a right derived functor on $\mathbf{D}^{\mathrm{b}}(X)$ one would pass to the subcategory of $\mathbf{D}(X)$ of complexes of quasi-coherent sheaves with coherent and bounded cohomology. There we can find an injective resolution and construct the derived functor as usual.

A complex $\mathcal{E}^\bullet \in \mathbf{D}(X)$ is called *perfect* if it is locally[3] isomorphic in $\mathbf{D}(X)$ to a bounded complex of locally free sheaves. Keeping proposition 1.16 in mind we can regard

[3]On a quasi-projective variety we do not need the word locally here (cf. [TT90, Theorem 2.4.3]).

the full subcategory of perfect complexes $\mathbf{D}^{\mathrm{perf}}(X) \leqslant \mathbf{D}(X)$ as a subcategory of $\mathbf{D}^{\mathrm{b}}(X)$. Hence on any quasi-projective variety X we have the following situation:

$$\mathbf{D}^{\mathrm{perf}}(X) \leqslant \mathbf{D}^{\mathrm{b}}(X) \leqslant \mathbf{D}(X).$$

If X is moreover regular, then we have $\mathbf{D}^{\mathrm{perf}}(X) \cong \mathbf{D}^{\mathrm{b}}(X)$, because every coherent sheaf has a locally free resolution of length at most the dimension of X (see [Huy06, Proposition 3.26]).

▲ (A.41). We say that a triangulated category $\mathbf{T}$ is *compactly generated* if it has all coproducts and there is a collection $\mathbf{E}$ of compact objects which generates $\mathbf{T}$. Explicitly:

i) For all $E \in \mathbf{E}$, the functor $\mathrm{Hom}_{\mathbf{T}}(E, -)$ preserves coproducts.

ii) For all $X \in \mathbf{T}$, if $\mathrm{Ext}^{\bullet}(E, X) = 0$ for all $E \in \mathbf{E}$, then $X = 0$.

Example 1.17 ([Nee96, Example 1.10])**.** Let X be a quasi-projective variety and let $\mathcal{L}$ be an ample line bundle on X. Every $\mathcal{L}^{\otimes j}[i]$ is a compact object, and the collection $\{\mathcal{L}^{\otimes j}[i] \mid i, j \in \mathbb{Z}\}$ generates $\mathbf{D}(X)$.

In particular $\mathbf{D}(X)$ is compactly generated and the triangulated subcategory $\mathbf{D}^{\mathrm{perf}}(X) \leqslant \mathbf{D}(X)$ of perfect complexes coincides with the triangulated subcategory of compact objects by [Nee96, Corollary 2.3]. Compact generation has other strong consequences such as the Brown representability theorem (see [Nee96, Theorem 3.1]). Neeman uses this to give a non-constructive proof of Grothendieck duality in [Nee96].

Another useful notion of generation in triangulated categories is the following:

▲ (A.33). We say that an object $E \in \mathbf{T}$ is a *classical generator* if the smallest thick subcategory of $\mathbf{T}$ containing E is $\mathbf{T}$ itself.

Proposition 1.18 ([Orl09, Theorem 4])**.** *Let $\mathcal{L}$ be a very ample line bundle on a smooth quasi-projective variety X of dimension n. Then $\bigoplus_{i=0}^{n} \mathcal{L}^i$ is a classical generator of $\mathbf{D}^{\mathrm{b}}(X)$.*

Proof. Let $X \subseteq \mathbb{P} := \mathbb{P}^N$ be a projective embedding of X such that $\mathcal{L} = \mathcal{O}_{\mathbb{P}}(1)|_X$.

The first claim is that all powers $\mathcal{L}^i$ are in the smallest thick triangulated subcategory of $\mathbf{D}^b(X)$ containing $\oplus_{i=0}^{n} \mathcal{L}^i$, call it $\mathbf{T}$. Recall that this means that $\mathbf{T}$ is a full subcategory closed under isomorphisms, direct summands, shifts and cones.

Let $H_0, \dots, H_N$ be $N{+}1$ hyperplanes in general position in $\mathbb{P}$, and let $s = (s_0, \dots, s_N) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(N+1)})$ be the corresponding section. General position means that the hyperplanes do not all meet at a single point. In particular, for every $x \in \mathbb{P}$ there is some hyperplane H_i with $x \notin H_i$. This means that s_i does not vanish at x, and therefore the zero locus of the section s is empty. The coresponding Koszul complex is exact away from the zero locus of the section (see [Ful84, Appendix B.3.4]), so we get the following exact sequence of locally free sheaves:

$$0 \to \Lambda^{N+1}(\mathcal{O}_{\mathbb{P}}(-1)^{\oplus(N+1)}) \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1)^{\oplus(N+1)} \to \mathcal{O}_{\mathbb{P}} \to 0$$

Restrict it to obtain a new exact sequence on X and truncate it to obtain a complex in $\mathbf{D}^b(X)$:

$$\mathcal{F}^\bullet := (\Lambda^{n+1}(\mathcal{O}_X(-1)^{\oplus(N+1)}) \to \cdots \to \Lambda^2(\mathcal{O}_X(-1)^{\oplus(N+1)}) \to \mathcal{O}_X(-1)^{\oplus(N+1)})$$

This complex has only two non-trivial cohomologies. On the right end it is just $\mathcal{O}_X$. On the left end, we have some term $\mathcal{G} := \mathcal{H}^{-(n+1)}(\mathcal{F}^\bullet)$. Consider the morphism $\mathcal{F}^\bullet \to \mathcal{O}_X$, which induces isomorphisms in cohomology except on degree $-n$. Its cone is then quasi-isomorphic to a coherent sheaf concentrated on degree $-n-1$, and there are no morphisms in $\mathbf{D}^{\mathrm{b}}(X)$ from $\mathcal{O}_X$ to such a complex since $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{O}_X, \mathcal{F}[n+1]) = H^{n+1}(X, \mathcal{F}) = 0$. Hence the corresponding distinguished triangle splits by proposition A.14, which shows that $\mathcal{O}_X$ is a direct summand of $\mathcal{F}^\bullet$. The formula

$$\Lambda^n(M \oplus N) \cong \bigoplus_{p+q=n} (\Lambda^p M) \otimes (\Lambda^q N)$$

shows that $\mathcal{F}^\bullet$ is made up of the powers $\mathcal{L}^{-1}, \ldots, \mathcal{L}^{-(n+1)}$, because all higher exterior powers of line bundles vanish. Tensoring $\mathcal{F}^\bullet$ with $\mathcal{L}^{n+1}$, which is again an exact operation, we get a complex made up of line bundles $\mathcal{L}^0, \ldots, \mathcal{L}^n$, which is hence in $\mathbf{T}$ by definition of $\mathbf{T}$. Since $\mathcal{L}^{n+1}$ is a direct summand of this complex, we must have $\mathcal{L}^{n+1} \in \mathbf{T}$ by thickness.

Now tensor again with $\mathcal{L}$ to obtain $\mathcal{L}^{n+2} \in \mathbf{T}$, and so on. This shows that all non-negative powers of $\mathcal{L}$ are in $\mathbf{T}$. For the negative powers, dualise $\mathcal{F}^\bullet$ and tensor by $\mathcal{L}^{-1}$, then by $\mathcal{L}^{-2}$, and so on. This shows that all powers of $\mathcal{L}$ are in $\mathbf{T}$.

Next we see that $\mathbf{T}$ contains every coherent sheaf $\mathcal{F} \in \mathbf{Coh}(X)$. By ampleness of $\mathcal{L}$ we may find $k \gg 0$ such that $\mathcal{F} \otimes \mathcal{L}^k$ is generated by global sections, which means that we have a surjection

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^k) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^k.$$

Tensor with $\mathcal{L}^{-k}$ we get a surjection

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^k) \otimes \mathcal{L}^{-k} \twoheadrightarrow \mathcal{F}.$$

This map has a kernel in $\mathbf{Coh}(X)$. We may find a surjection onto this kernel in the same way and continue resolving like this to get

$$0 \to \mathcal{F}_{n+1} \to (\mathcal{L}^{-k_{n+1}})^{\oplus i_{n+1}} \to \cdots \to (\mathcal{L}^{-k_1})^{\oplus i_1} \to \mathcal{F} \to 0.$$

Again, the cone of the canonical morphism from this complex to $\mathcal{F}$ is concentrated up to quasi-isomorphism on degree $-n-1$. Since X is regular, the corresponding triangle splits as before, which again implies that $\mathcal{F} \in \mathbf{T}$ because $\mathbf{T}$ is by assumption closed under direct summands.

By example A.38, since all $\mathcal{F} \in \mathbf{Coh}(X)$ are in $\mathbf{T}$, it follows that $\mathbf{D}^b(X) = \mathbf{T}$. □

Remark 1.19. The argument in the previous proof can be slightly modified to show that $\oplus_{i=0}^n \mathcal{L}^i$ is a classical generator of $\mathbf{D}^{\mathrm{perf}}(X)$ for any singular quasi-projective variety X (see [Orl09, Theorem 4]).

Properness over the base field

In what follows it will be convenient to work with triangulated categories which have finite dimensional hom-vector spaces, so let us check that this is the case in our situation.

Definition 1.20. A $\Bbbk$-linear triangulated category $\mathbf{T}$ is called *proper* over $\Bbbk$ if

$$\dim_{\Bbbk}(\oplus_{i\in\mathbb{Z}} \operatorname{Ext}^i(X,Y)) < \infty$$

for all $X, Y \in \mathbf{T}$.

Proposition 1.21 ([Huy06, Corollary 3.14])**.** *Let X be a smooth projective variety over $\Bbbk$. Then $\mathbf{D}^{\mathrm{b}}(X)$ is a proper triangulated category.*

Proof. We show first that $\dim_{\Bbbk}(\operatorname{Ext}^p(\mathcal{F},\mathcal{G})) < \infty$ for all $\mathcal{F},\mathcal{G} \in \mathbf{Coh}(X)$.

If $\mathcal{F},\mathcal{G} \in \mathbf{Coh}(X)$, then $\mathcal{H}om(\mathcal{F},\mathcal{G}) \in \mathbf{Coh}(X)$. Since X is proper, by Serre's theorem (see [EGA III_1, Theorem 2.2.1]) we have that $\operatorname{Hom}(\mathcal{F},\mathcal{G}) = H^0(X, \mathcal{H}om(\mathcal{F},\mathcal{G}))$ is a finite dimensional $\Bbbk$-vector space.

Suppose now that $\mathcal{F} \cong \oplus_i \mathcal{L}_i$ is a direct sum of line bundles. Then $\operatorname{Ext}^p(\mathcal{F},\mathcal{G}) \cong \oplus_i \operatorname{Ext}^p(\mathcal{L}_i,\mathcal{G}) \cong \oplus_i H^p(X, \mathcal{G}\otimes\mathcal{L}_i^{-1})$, which is again a finite dimensional $\Bbbk$-vector space by Serre's theorem. For a general $\mathcal{F} \in \mathbf{Coh}(X)$ consider an exact sequence

$$0 \to \mathcal{M} \to \oplus_i \mathcal{L}_i \to \mathcal{F} \to 0.$$

Taking the long exact sequence of the derived functor we have

$$\cdots \to \operatorname{Ext}^{p-1}(\mathcal{M},\mathcal{G}) \to \operatorname{Ext}^p(\mathcal{F},\mathcal{G}) \to \operatorname{Ext}^p(\oplus_i\mathcal{L}_i,\mathcal{G}) \to \cdots$$

By induction and the case of direct sum of line bundles, the two sides are finite dimensional vector spaces. Hence the vector space in the middle is also finite dimensional.

We check next that almost all $\operatorname{Ext}^p(\mathcal{F},\mathcal{G})$ vanish for $\mathcal{F},\mathcal{G} \in \mathbf{Coh}(X)$. If $p < 0$, then it is 0 by definition. Let $n = \dim X$. Then by Serre duality we have

$$\operatorname{Ext}^p(\mathcal{F},\mathcal{G}) \cong \operatorname{Ext}^{p-n}(\mathcal{G},\mathcal{F}\otimes\omega_X)^\vee$$

which vanishes whenever $p > n$.

Hence $\dim_{\Bbbk}(\oplus_p \operatorname{Ext}^p(\mathcal{F},\mathcal{G})) < \infty$ for all $\mathcal{F},\mathcal{G} \in \mathbf{Coh}(X)$.

Let now $\mathcal{F} \in \mathbf{Coh}(X)$ and $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$. A particular case of the Grothendieck spectral sequence yields

$$E_2^{p,q} = \operatorname{Ext}^p(\mathcal{F}, \mathcal{H}^q(\mathcal{G}^\bullet)) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F},\mathcal{G}^\bullet).$$

Since the second page consists of finite dimensional vector spaces and is concentrated on a vertical strip of width n and finite height, all $\operatorname{Ext}^{p+q}(\mathcal{F},\mathcal{G}^\bullet)$ are finite dimensional and almost all of them vanish.

Finally, let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$. Replace $\mathcal{G}^\bullet$ by a complex $\mathcal{I}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$ of injectives (recall proposition 1.15 and proposition 1.16). We can write $\operatorname{Hom}^\bullet(\mathcal{F}^\bullet,\mathcal{I}^\bullet)$ as the total complex of a double complex $K^{i,j} = \operatorname{Hom}(\mathcal{F}^{-i},\mathcal{I}^j)$ with horizontal differential $(-1)^{j-i+1}d_{\mathcal{F}}$ and vertical differential $d_{\mathcal{I}}$ (cf. [Huy06, Example 2.62]):

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{F}^1,\mathcal{I}^1) & \xrightarrow{-d^*} & \mathrm{Hom}(\mathcal{F}^0,\mathcal{I}^1) & \xrightarrow{d^*} & \mathrm{Hom}(\mathcal{F}^{-1},\mathcal{I}^1) \\
{\scriptstyle d_*}\big\uparrow & & {\scriptstyle d_*}\big\uparrow & & {\scriptstyle d_*}\big\uparrow \\
\mathrm{Hom}(\mathcal{F}^1,\mathcal{I}^0) & \xrightarrow{d^*} & \mathrm{Hom}(\mathcal{F}^0,\mathcal{I}^0) & \xrightarrow{-d^*} & \mathrm{Hom}(\mathcal{F}^{-1},\mathcal{I}^0) \\
{\scriptstyle d_*}\big\uparrow & & {\scriptstyle d_*}\big\uparrow & & {\scriptstyle d_*}\big\uparrow \\
\mathrm{Hom}(\mathcal{F}^1,\mathcal{I}^{-1}) & \xrightarrow{-d^*} & \mathrm{Hom}(\mathcal{F}^0,\mathcal{I}^{-1}) & \xrightarrow{d^*} & \mathrm{Hom}(\mathcal{F}^{-1},\mathcal{I}^{-1})
\end{array}$$

Since the $\mathcal{I}^j$ are injective, the functors $\mathrm{Hom}(-,\mathcal{I}^j)$ are exact for all $j \in \mathbb{Z}$. In particular, they commute with cohomology, so that

$$H^q(\mathrm{Hom}(\mathcal{F}^\bullet,\mathcal{I}^j)) \cong \mathrm{Hom}(\mathcal{H}^{-q}(\mathcal{F}^\bullet),\mathcal{I}^j).$$

Hence the spectral sequence of the double complex $K^{\bullet,\bullet}$ reads

$$E_2^{p,q} = H^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet),\mathcal{I}^\bullet) = \mathrm{Ext}^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet),\mathcal{G}^\bullet) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}^\bullet,\mathcal{G}^\bullet).$$

By the previous case, the second page contains only finitely many non-zero vector spaces, all of them finite dimensional. Therefore we have

$$\dim_{\Bbbk}(\oplus_p \mathrm{Ext}^p(\mathcal{F}^\bullet,\mathcal{G}^\bullet)) < \infty.$$

□

Definition 1.22. Let $\mathbf{T}$ be a $\Bbbk$-linear triangulated category. A contravariant cohomological functor $H\colon \mathbf{T}^{\mathrm{op}} \to \mathbf{mod}(\Bbbk)$ is called of *finite type* if for all $E \in \mathbf{T}$ all but finitely many $H(\Sigma^i E)$ are zero.

Note that the category of such functors is equivalent to the category of $\Bbbk$-linear triangulated functors $\mathbf{T}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\Bbbk)$ (see [BK90, Lemma 2.2]).

Definition 1.23. Let $\mathbf{T}$ be a proper triangulated category over $\Bbbk$. We say that $\mathbf{T}$ is right (resp. left) *saturated* if every contravariant (resp. covariant) functor of finite type is representable.

Theorem 1.24 ([BK90, Theorem 2.14])**.** *Let X be a smooth projective variety. Then $\mathbf{D}^{\mathrm{b}}(X)$ is both left and right saturated.*

Serre functors

One of the main tools in the study of cohomology on a smooth projective variety is Serre duality and, more generally, Grothendieck-Verdier duality. In the context of $\Bbbk$-linear triangulated categories, this is conveniently encoded in the following notion:

Definition 1.25. Let $\mathbf{T}$ be a proper triangulated category over $\Bbbk$. A *Serre functor* on $\mathbf{T}$ is a $\Bbbk$-linear equivalence $\mathbb{S}\colon \mathbf{T} \to \mathbf{T}$ such that for every $X, Y \in \mathbf{T}$ we have an isomorphism

$$\eta_{X,Y}\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,\mathbb{S}(X))^\vee$$

which is functorial in X and in Y.

Remark 1.26. The definition of Serre functor also makes sense in an arbitrary $\Bbbk$-linear category, but we will always consider them in the context of proper triangulated categories over $\Bbbk$.

Example 1.27. The identity is a Serre functor on $\mathbf{D}^{\mathrm{b}}(\Bbbk)$.

We check first that Serre functors commute with $\Bbbk$-linear equivalences (not necessarily triangulated).

Lemma 1.28 ([Huy06, Lemma 1.30])**.** *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a $\Bbbk$-linear equivalence between proper triangulated categories over $\Bbbk$. Then we have a natural isomorphism*

$$F \circ \mathbb{S}_{\mathbf{A}} \cong \mathbb{S}_{\mathbf{B}} \circ F$$

Proof. Using the defining property of Serre functors, fully faithfulness of F and the canonical isomorphisms $V \to V^{\vee\vee}$ sending $v \mapsto \mathrm{ev}_v$ for finite dimensional $\Bbbk$-vector spaces, we have functorial isomorphisms

$$\begin{aligned} &\mathrm{Hom}(F(X), F\mathbb{S}(Y)) \cong \mathrm{Hom}(X, \mathbb{S}(Y)) \cong \mathrm{Hom}(X, \mathbb{S}(Y))^{\vee\vee} \cong \\ &\quad\cong \mathrm{Hom}(Y, X)^{\vee} \cong \mathrm{Hom}(F(Y), F(X))^{\vee} \cong \mathrm{Hom}(F(X), \mathbb{S}F(Y))^{\vee} \cong \\ &\qquad\cong \mathrm{Hom}(F(X), \mathbb{S}F(Y)) \end{aligned}$$

for all $X, Y \in \mathbf{A}$. By Yoneda and essential surjectivity of F we get the natural isomorphism that we wanted. $\square$

In particular, for $F = \mathrm{id}_{\mathbf{A}}$ the previous lemma shows that Serre functors are unique up to natural isomorphism.

We show next how to use Serre functors to define a left (resp. right) adjoint in the presence of a right (resp. left) adjoint.

Lemma 1.29 ([Huy06, Remark 1.31])**.** *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor between proper triangulated categories over $\Bbbk$ endowed with Serre functors.*

i) If F has a left adjoint $G \dashv F$, then $F \dashv \mathbb{S}_{\mathbf{A}} \circ G \circ \mathbb{S}_{\mathbf{B}}^{-1}$.

ii) If F has a right adjoin $F \dashv H$, then $\mathbb{S}_{\mathbf{A}}^{-1} \circ H \circ \mathbb{S}_{\mathbf{B}} \dashv F$.

Proof. In the first case we have functorial isomorphisms

$$\begin{aligned} &\mathrm{Hom}(X, \mathbb{S}_{\mathbf{A}} G \mathbb{S}_{\mathbf{B}}^{-1}(Y)) \cong \mathrm{Hom}(G\mathbb{S}_{\mathbf{B}}^{-1}(Y), X)^{\vee} \cong \\ &\quad\cong \mathrm{Hom}(\mathbb{S}_{\mathbf{B}}^{-1}(Y), F(X))^{\vee} \cong \mathrm{Hom}(F(X), \mathbb{S}_{\mathbf{B}}\mathbb{S}_{\mathbf{B}}^{-1}(Y)) \cong \mathrm{Hom}(F(X), Y) \end{aligned}$$

for all $X \in \mathbf{A}$ and all $Y \in \mathbf{B}$.

For the second case we have functorial isomorphisms

$$\begin{aligned} &\mathrm{Hom}(\mathbb{S}_{\mathbf{A}}^{-1} H \mathbb{S}_{\mathbf{B}}(X), Y) \cong \mathrm{Hom}(Y, H\mathbb{S}_{\mathbf{B}}(X))^{\vee} \cong \\ &\qquad\cong \mathrm{Hom}(F(Y), \mathbb{S}_{\mathbf{B}}(X))^{\vee} \cong \mathrm{Hom}(X, F(Y)) \end{aligned}$$

for all $X \in \mathbf{A}$ and all $Y \in \mathbf{B}$. $\square$

Remark 1.30. Recall that if F is triangulated, so are its adjoints (see proposition A.21).

We can now reformulate Serre duality as the existence of a certain Serre functor:

Theorem 1.31 ([Huy06, Theorem 3.12])**.** *Let X be an n-dimensional smooth projective variety over $\Bbbk$. We define a $\Bbbk$-linear triangulated autoequivalence $\mathbb{S}_X \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X)$ as the composition*

$$\mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\omega_X \otimes (-)} \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{[n]} \mathbf{D}^{\mathrm{b}}(X).$$

Then $\mathbb{S}_X$ is a Serre functor on $\mathbf{D}^{\mathrm{b}}(X)$.

Admissible subcategories

Admissible subcategories will play an important role for us, since they can be thought of as semiorthogonal components (cf. lemma 1.42 and lemma A.26).

▲ (A.24)**.** A triangulated subcategory $\mathbf{A} \leqslant \mathbf{T}$ is called *right admissible* (resp. *left admissible*) if the inclusion functor admits a right (resp. left) adjoint. If it is both left and right admissible we simply call it *admissible*.

Let $\mathbf{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be a right admissible subcategory. Then $\mathbf{A}^{\perp} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ is automatically left admissible by lemma A.26. But when is $\mathbf{A}^{\perp}$ also right admissible?

Proposition 1.32. *Let X be a smooth projective variety over $\Bbbk$. Let $\mathbf{A} \subseteq \mathbf{D}^{\mathrm{b}}(X)$ be a right (resp. left) admissible subcategory. Then $\mathbf{A}$ is also left (resp. right) admissible.*

Proof. We will check that the Serre functor on $\mathbf{D}^{\mathrm{b}}(X)$ induces a Serre functor on $\mathbf{A}$. This implies the result by lemma 1.29.

Let $i_* \colon \mathbf{A} \to \mathbf{D}^{\mathrm{b}}(X)$ denote the inclusion functor. Suppose first that $i^! \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{A}$ is a right adjoint of this inclusion. We claim that

$$\mathbb{S}_{\mathbf{A}} = i^! \circ \mathbb{S}_X \circ i_*$$

is a Serre functor on $\mathbf{A}$. We check first that it is a $\Bbbk$-linear equivalence. Being $\Bbbk$-linear follows from being a composition of $\Bbbk$-linear functors. To check fully faithfulness, let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{A}$. Using that i_* is just the inclusion and that $i^! \circ i_* \cong \mathrm{id}_{\mathbf{A}}$ we get

$$\begin{aligned}\mathrm{Hom}(i^!\mathbb{S}_X i_*(\mathcal{F}^\bullet), i^!\mathbb{S}_X i_*(\mathcal{G}^\bullet)) &\cong \mathrm{Hom}(i^! i_*\mathbb{S}_X i_*(\mathcal{F}^\bullet), i^! i_*\mathbb{S}_X i_*(\mathcal{G}^\bullet)) \cong \\ &\cong \mathrm{Hom}(\mathbb{S}_X i_*(\mathcal{F}^\bullet), \mathbb{S}_X i_*(\mathcal{G}^\bullet)) \cong \mathrm{Hom}(\mathbb{S}_X(\mathcal{F}^\bullet), \mathbb{S}_X(\mathcal{G}^\bullet))\end{aligned}$$

Now $\mathbb{S}_X$ is an autoequivalence, hence we can continue the chain of isomorphisms to conclude

$$\mathrm{Hom}(\mathbb{S}_X(\mathcal{F}^\bullet), \mathbb{S}_X i_*(\mathcal{G}^\bullet)) \cong \mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

Again because $\mathbb{S}_X$ is an autoequivalence, for all $\mathcal{F}^\bullet \in \mathbf{A}$ we can find some $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$ such that $\mathcal{F}^\bullet \cong \mathbb{S}_X(\mathcal{G}^\bullet)$. Hence $\mathcal{F} \cong i^! i_*(\mathcal{F}^\bullet) = i^!(\mathcal{F}^\bullet)$ is isomorphic to $i^!\mathbb{S}_X(\mathcal{G}^\bullet)$ and $\mathbb{S}_{\mathbf{A}}$ is essentially surjective. For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{A}$ we have functorial isomorphisms

$$\mathrm{Hom}(\mathcal{F}^\bullet, i^!\mathbb{S}_X i_*(\mathcal{G}^\bullet)) \cong \mathrm{Hom}(i_*(\mathcal{F}^\bullet), \mathbb{S}_X i_*(\mathcal{G}^\bullet)) \cong \mathrm{Hom}(i_*(\mathcal{G}^\bullet), i_*(\mathcal{F}^\bullet))^\vee$$

Since i_* is just the inclusion, the last hom-space is just $\mathrm{Hom}(\mathcal{G}^\bullet, \mathcal{F}^\bullet)^\vee$. Hence $\mathbb{S}_{\mathbf{A}}$ is a Serre functor on $\mathbf{A}$.

Suppose now that $i^* \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{A}$ is a left adjoint to the inclusion. Then

$$\mathbb{S}_{\mathbf{A}} = i^* \circ \mathbb{S}_X^{-1} \circ i_*$$

is a Serre functor on $\mathbf{A}$. To check that it is a $\Bbbk$-linear equivalence we argue as before. And for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{A}$ we have functorial isomorphisms

$$\mathrm{Hom}(i^*\mathbb{S}_X^{-1} i_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \cong \mathrm{Hom}(\mathbb{S}_X^{-1} i_*(\mathcal{F}^\bullet), i_*(\mathcal{G}^\bullet)) \cong \mathrm{Hom}(i_*(\mathcal{G}^\bullet), i_*(\mathcal{F}^\bullet))^\vee$$

which imply the claim as in the previous case. □

So right and left admissibility are equivalent for triangulated subcategories of the derived category $\mathbf{D}^{\mathrm{b}}(X)$ of a smooth projective variety.

Remark 1.33. The proof works the same way replacing $\mathbf{D}^{\mathrm{b}}(X)$ by any proper triangulated category endowed with a Serre functor. In particular it works for admissible subcategories of $\mathbf{D}^{\mathrm{b}}(X)$, since the proof shows that they also have a Serre functor.

We will rely on the previous proposition when we discuss admissibility in order to reduce the machinery we use, but using saturatedness of $\mathbf{D}^{\mathrm{b}}(X)$ and the results from [BK90] would make our life much easier. For example, if X is smooth and we see $\mathbf{D}^{\mathrm{b}}(X)$ as a full subcategory of some $\mathbf{D}^{\mathrm{b}}(Y)$ via some fully faithful triangulated functor, then $\mathbf{D}^{\mathrm{b}}(X)$ is automatically admissible inside $\mathbf{D}^{\mathrm{b}}(Y)$.

Proposition 1.34 ([BK90, Proposition 2.6])**.** *Let* $\mathbf{A}$ *be a right (resp. left) saturated triangulated category and suppose* $\mathbf{A} \leqslant \mathbf{T}$ *is embedded as a triangulated subcategory in a proper triangulated category* $\mathbf{T}$*. Then* $\mathbf{A}$ *is right (resp. left) admissible in* $\mathbf{T}$*.*

Proposition 1.35 ([BK90, Proposition 2.8])**.** *Let* $\mathbf{A} \leqslant \mathbf{T}$ *be a left (resp. right) admissible subcategory of a proper and right (resp. left) saturated triangulated category* $\mathbf{T}$*. Then* $\mathbf{A}$ *is right (resp. left) saturated.*

These two propositions combined give another proof of proposition 1.32, but also show the announced claim:

Corollary 1.36 ([BK90])**.** *Let* X *be a smooth projective variety and let* $\mathbf{D}^{\mathrm{b}}(X) \leqslant \mathbf{T}$ *for some proper triangulated category* $\mathbf{T}$*. Then* $\mathbf{D}^{\mathrm{b}}(X)$ *is admissible in* $\mathbf{T}$*.*

Remark 1.37. In the appendix we study some stability properties of admissible subcategories. For example, if $\mathbf{A} \leqslant \mathbf{B}$ is admissible and $\mathbf{B} \leqslant \mathbf{C}$ is admissible, then $\mathbf{A} \leqslant \mathbf{C}$ is admissible. Note also that if $\mathbf{A} \leqslant \mathbf{C}$ is admissible, then $\mathbf{A} \leqslant \mathbf{B}$ is admissible as well, because we can restrict the adjoint of the inclusion to $\mathbf{B}$.

1.3 Semiorthogonal decompositions

In order to study $\mathbf{D}^{\mathrm{b}}(X)$ we want to decompose it into simpler pieces. A natural first attempt would be to look for orthogonal decompositions:

Definition 1.38. Two triangulated subcategories $\mathbf{A}$ and $\mathbf{B}$ of a triangulated category $\mathbf{T}$ are called an *orthogonal decomposition* of $\mathbf{T}$ if the following two conditions hold:

1) $\operatorname{Hom}(A, B) = \operatorname{Hom}(B, A) = 0$ for all $A \in \mathbf{A}$ and all $B \in \mathbf{B}$.

2) For all $X \in \mathbf{T}$ we can find a distinguished triangle

$$A \to X \to B \to \Sigma A.$$

The orthogonality condition implies that the dashed arrow is zero in the previous triangle, hence $X \cong A \oplus B$ (see proposition A.14). If $\mathbf{T}$ does not admit any non-trivial orthogonal decomposition, then we say that it is *indecomposable*.

Proposition 1.39 ([Bri99, Example 3.2])**.** *If X is a quasi-projective variety, then $\mathbf{D}^{\mathrm{b}}(X)$ is indecomposable if and only if X is connected.*

Proof. This proof is taken from [Huy06, Proposition 3.10]. We show only the direction that we are interested in, namely, that the derived category of a connected scheme is indecomposable. Suppose $\mathbf{D}^{\mathrm{b}}(X) = \mathbf{A} \oplus \mathbf{B}$. Then we have a distinguished triangle

$$\mathcal{A} \to \mathcal{O}_X \to \mathcal{B} \to \mathcal{A}[1].$$

By proposition A.14, we have $\mathcal{O}_X = \mathcal{A} \oplus \mathcal{B}$. In particular, both complexes $\mathcal{A}$ and $\mathcal{B}$ have cohomologies concentrated on degree zero, so we may assume that they are coherent sheaves. Moreover, they are $\mathcal{O}_X$-submodules of the structure sheaf, so $\mathcal{A}$ (resp. $\mathcal{B}$) is the ideal sheaf of a closed subscheme X_A (resp. X_B). The sum of ideal sheaves (which in our case is the whole $\mathcal{O}_X$) is contained in the ideal sheaf of the intersection, and therefore we have $X_A \cap X_B = \varnothing$. On the other hand, the ideal sheaf of the union is contained in the intersection of the ideal sheaves (which in our case is zero), so $X_A \cup X_B = X$. By connectedness of X, one of the two closed subsets must be empty, say $X_B = \varnothing$. But then $\mathcal{B} = 0$ and

$$\mathcal{O}_X \cong \mathcal{A} \in \mathbf{A}.$$

Let now $\kappa(x)$ be the skyscraper sheaf of the residue field at a closed point $x \in X$. This coherent sheaf cannot be written as a non-trivial direct sum, so the decomposition given by proposition A.14 shows that either $\kappa(x) \in \mathbf{A}$ or $\kappa(x) \in \mathbf{B}$. This coherent sheaf has non-zero global sections, so there is a non-zero morphism $\mathcal{O}_X \to \kappa(x)$ and thus $\kappa(x) \in \mathbf{A}$.

Suppose now that we had a complex $0 \neq \mathcal{F}^\bullet \in \mathbf{B}$. Since it is non-zero, we can define $m \in \mathbb{Z}$ as the largest integer for which $\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. Let then $x \in X$ be a closed point in the support of the coherent sheaf $\mathcal{H}^m(\mathcal{F}^\bullet)$. The claim is that there is a non-zero morphism

$$\mathcal{F}^\bullet \to \kappa(x)[-m]$$

which implies that $\mathcal{F}^\bullet \in \mathbf{A}$, contradicting our assumption. To see this, we may replace $\mathcal{F}^\bullet$ by its canonical upper truncation $\tau_{\leqslant m}\mathcal{F}^\bullet$, which is isomorphic to $\mathcal{F}^\bullet$ in $\mathbf{D}^{\mathrm{b}}(X)$. The quotient $\mathrm{Ker}(d^m) \to \mathcal{H}^m(\mathcal{F}^\bullet)$ corresponds then to a morphism $\mathcal{F}^\bullet \to \mathcal{H}^m(\mathcal{F}^\bullet)[-m]$. Since x is in the support of $\mathcal{H}^m(\mathcal{F}^\bullet)$, we also have a morphism $\mathcal{H}^m(\mathcal{F}^\bullet) \to \kappa(x)$. The composition $\mathcal{F}^\bullet \to \kappa(x)[-m]$ induces a non-trivial morphism in cohomology on degree m, so this morphism cannot be trivial in the derived category. □

Hence orthogonal decompositions are too restrictive for our purposes. Instead we introduce the following:

Definition 1.40. Two triangulated subcategories $\mathbf{A}$ and $\mathbf{B}$ of a triangulated category $\mathbf{T}$ are called a *semiorthogonal decomposition* of $\mathbf{T}$ if the following two conditions hold:

1) $\mathrm{Hom}(B, A) = 0$ for all $A \in \mathbf{A}$ and all $B \in \mathbf{B}$.
2) For all $X \in \mathbf{T}$ we can find a distinguished triangle

$$B \to X \to A \to \Sigma B.$$

We denote such a decomposition by $\mathbf{T} = \langle \mathbf{A}, \mathbf{B} \rangle$.

Remark 1.41. Unlike in the case of orthogonal decompositions, the order in the previous triangle matters. Indeed, if we swap A and B in the triangle, then it would always split (see proposition A.14) and X would again be the direct sum of A and B.

This kind of decomposition is closely related to admissible subcategories. If $\mathbf{A} \leqslant \mathbf{T}$ is a right admissible subcategory, it follows directly from lemma A.26 that $\mathbf{T} = \langle \mathbf{A}^\perp, \mathbf{A} \rangle$. In fact, the converse is also true:

Lemma 1.42 ([Kuz11, Lemma 2.5])**.** *If* $\mathbf{T} = \langle \mathbf{A}, \mathbf{B} \rangle$*, then* $\mathbf{B}$ *is right admissible and the inclusion* $\mathbf{A} \subseteq \mathbf{B}^\perp$ *is an equivalence. In particular,* $\mathbf{A}$ *is left admissible.*

Proof. We use the characterization of admissibility given in lemma A.26.

We need to show that the inclusion $\mathbf{A} \subseteq \mathbf{B}^\perp$ is essentially surjective, so let $X \in \mathbf{B}^\perp$. Since $\mathbf{T} = \langle \mathbf{A}, \mathbf{B} \rangle$, we have a distinguished triangle

$$B \to X \to A \to \Sigma B.$$

The morphism $B \to X$ must be trivial, so the triangle splits by proposition A.14. In particular $A \cong X \oplus \Sigma B$. But then the inclusion $\Sigma B \to X \oplus \Sigma B \cong A$ must vanish, hence the identity on ΣB factors through zero. This implies that $B \cong 0$ and thus $A \cong X$. □

We can iterate these kind of decompositions by further decomposing each semiorthogonal component. Ideally, we would start from an indecomposable triangulated subcategory $\mathbf{A}$ and then find further semiorthogonal decompositions of its complement $\mathbf{A}^\perp$, until we eventually obtain an expression of the form

$$\mathbf{T} = \langle \cdots \langle \mathbf{A}_1, \mathbf{A}_2 \rangle, \mathbf{A}_3 \rangle, \cdots \rangle, \mathbf{A}_m \rangle.$$

Lemma 1.43. *Let* $\mathbf{T}$ *be a triangulated category and* $\mathbf{A}_1, \ldots, \mathbf{A}_m \leqslant \mathbf{T}$ *be a semiorthogonal collection of triangulated subcategories, i.e.* $\mathbf{A}_i \subseteq \mathbf{A}_j^{\perp}$ *whenever* $j > i$*. The following are equivalent:*

a) $\mathbf{T} = \langle \cdots \langle \mathbf{A}_1, \mathbf{A}_2 \rangle, \mathbf{A}_3 \rangle, \cdots \rangle, \mathbf{A}_m \rangle$.

b) There is a filtration $0 = \mathbf{T}_0 \leqslant \mathbf{T}_1 \leqslant \cdots \leqslant \mathbf{T}_m = \mathbf{T}$ *such that each* $\mathbf{T}_{i-1}$ *is left admissible in* $\mathbf{T}_i$ *and such that the left orthogonal complement of* $\mathbf{T}_{i-1}$ *in* $\mathbf{T}_i$ *is* $\mathbf{A}_i$*.*

c) For all $X \in \mathbf{T}$ *we can find a sequence of morphisms* $0 = T_m \to T_{m-1} \to \cdots \to T_0 = X$ *such that the cone of the morphism* $T_i \to T_{i-1}$ *is in* $\mathbf{A}_i$*.*

Proof. Suppose we have a). To show b) we take

$$\mathbf{T}_i = \langle \cdots \langle \mathbf{A}_1, \mathbf{A}_2 \rangle, \cdots \rangle, \mathbf{A}_i \rangle.$$

By lemma 1.42 the inclusions $\mathbf{T}_{i-1} \leqslant \mathbf{T}_i$ are left admissible. Moreover, by corollary A.27 ${}^{\perp}\mathbf{T}_{i-1} \cong \mathbf{A}_i$, so we are done.

Suppose now that b) holds and let $X \in \mathbf{T}$. We show that c) holds by induction on the length m of the decomposition. If $m = 2$, then from lemma A.26 we obtain a distinguished triangle

$$B \to X \to A \to \Sigma B$$

with $B \in {}^{\perp}\mathbf{T}_1 \cong \mathbf{A}_2$ and $A \in \mathbf{T}_1 = \mathbf{A}_1$. Hence we can set $T_1 = B$ and obtain the desired sequence of morphisms. If the result is true for $m-1$, then we can consider the sequence of morphisms

$$0 \to T_{m-2} \to \cdots \to T_1 \to X$$

with the desired property, obtained from the filtration $0 \leqslant \mathbf{T}_1 \leqslant \cdots \leqslant \mathbf{T}_{m-1}$. Since $\mathbf{T}_{m-1} \leqslant \mathbf{T}$ is left admissible, we can again find a distinguished triangle $B \to T_{m-2} \to A \to \Sigma B$ with $B \in {}^{\perp}\mathbf{T}_{m-1} \cong \mathbf{A}_m$ and with $A \in \mathbf{T}_{m-1}$. We need to show that $A \in \mathbf{A}_{m-1}$, which is the orthogonal complement of $\mathbf{T}_{m-2}$ in $\mathbf{T}_{m-1}$. Since T_{m-2} is already in it and B has no non-zero morphisms to any object $Y \in \mathbf{T}_{m-1} \geqslant \mathbf{T}_{m-2}$, this follows from the long exact sequence of hom-groups. We can thus complete the previous sequence to a sequence

$$0 \to B \to T_{m-2} \to \cdots \to T_1 \to X$$

in which $B \in \mathbf{A}_m$ and the cone of $B \to T_{m-2}$ is in $\mathbf{A}_{m-1}$.

Assume now that c) is true and let again $X \in \mathbf{T}$. We will again proceed by induction on the length m of the decomposition. If $m = 2$, then we have a sequence $0 \to T_1 \to X$ such that $T_1 \in \mathbf{A}_2$ and the cone of $T_1 \to X$ is in A_1. Hence we have the distinguished triangle that we need and $\mathbf{T} = \langle \mathbf{A}_1, \mathbf{A}_2 \rangle$. Suppose that the implication holds for decompositions of length $m-1$. Given a sequence of morphisms

$$0 \to T_{m-1} \to \cdots \to T_2 \to T_1 \to X$$

such that the cone of $T_i \to T_{i-1}$ is in $\mathbf{A}_i$, we will show that the cone of the composition $T_2 \to T_1 \to X$ is in $\langle \mathbf{A}_1, \mathbf{A}_2 \rangle$. This implies the result by induction.

By (TR4) we can find a commutative diagram

$$\begin{tikzcd}
T_2 \arrow[r] \arrow[d, equal] & T_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d, dashed] & \Sigma T_2 \arrow[d, equal] \\
T_2 \arrow[r] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma T_2 \\
& A_1 \arrow[r, equal] \arrow[d] & A_1 \arrow[d] & \\
& \Sigma T_1 \arrow[r] & \Sigma A_2 &
\end{tikzcd}$$

in which all horizontal and vertical lines are distinguished triangles. Hence $C \in \langle \mathbf{A}_1, \mathbf{A}_2 \rangle$ as we wanted. □

Definition 1.44. Let $\mathbf{T}$ be a triangulated category and let $\mathbf{A}_1, \dots, \mathbf{A}_m \leqslant \mathbf{T}$ be a semiorthogonal sequence of triangulated subcategories. We say that they form a *semiorthogonal decomposition* of $\mathbf{T}$ if they satisfy any of the equivalent conditions in lemma 1.43. We denote this by

$$\mathbf{T} = \langle \mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m \rangle.$$

Remark 1.45. Semiorthogonal decompositions clarify the structure on the triangulated category to some extent. But even if we understand all pieces in the decomposition very well, there is still some information missing when we write $\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_m \rangle$, namely the morphisms going to the right[4].

A particularly nice case of semiorthogonal decomposition occurs when all pieces are right admissible (e.g. on $\mathbf{D}^{\mathrm{b}}(X)$ for X a smooth projective variety). This has the following nice consequence:

Lemma 1.46. *Let $\mathbf{A}_1, \dots, \mathbf{A}_m$ be a semiorthogonal sequence of right (resp. left) admissible triangulated subcategories of $\mathbf{T}$. Then the following are equivalent:*

a) The smallest triangulated subcategory containing all of them is $\mathbf{T}$.

b) The smallest thick triangulated subcategory containing all of them is $\mathbf{T}$.

c) They generate $\mathbf{T}$, i.e. $\bigcap \mathbf{A}_i^{\perp} = \{0\}$.

d) $\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_m \rangle$.

Proof. The triangulated envelope is always contained in the thick envelope, so $a) \Rightarrow b)$ is automatic.

For $b) \Rightarrow c)$, let $X \in \bigcap \mathbf{A}_i^{\perp}$. Then $\mathrm{Hom}(T, X) = 0$ for all $T \in \bigcup \mathbf{A}_i$. But this property is closed under taking shifts, cones and direct summands. Hence $\mathrm{Hom}(T, X) = 0$ for all T in the thick envelope of the $\mathbf{A}_i$, so $\mathrm{Hom}(T, X) = 0$ for all $T \in \mathbf{T}$ and in particular $\mathrm{Hom}(X, X) = 0$. Thus $X = 0$.

Let us first show $c) \Rightarrow d)$ for $m = 2$. Let $X \in \mathbf{T}$. Since $\mathbf{A}_2$ is right admissible, we can find a distinguished triangle $A_2 \to X \to A_0 \to \Sigma A_2$, with $A_0 \in \mathbf{A}_2^{\perp}$. Since $\mathbf{A}_1$ is

[4] These are called *gluing data*.

right admissible we can find another distinguished triangle $A_1 \to A_0 \to A_0' \to \Sigma A_1$, with $A_0' \in \mathbf{A}_1^\perp$. Since $A_1, A_0 \in \mathbf{A}_2^\perp$, we also have $A_0' \in \mathbf{A}_2^\perp$. Hence $A_0' \in \mathbf{A}_1^\perp \cap \mathbf{A}_2^\perp = \{0\}$ and $A_1 \cong A_0 \in \mathbf{A}_1$, finishing this case. The general case follows from this one by induction, for if $\cap_{i=1}^{m+1} \mathbf{A}_i^\perp = \{0\}$, then $\cap_{i=1}^{m} \mathbf{A}_i^\perp = \{0\}$ inside $\mathbf{A}_{m+1}^\perp$. Hence $\mathbf{A}_{m+1}^\perp = \langle \mathbf{A}_1, \dots, \mathbf{A}_m \rangle$ and thus we obtain a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_m, \mathbf{A}_{m+1} \rangle.$$

The last implication $d) \Rightarrow a)$ follows directly from the definition of semiorthogonal decomposition. Indeed, suppose we have $0 \to T_{m-1} \to \cdots \to T_1 \to X$. Then we first have that T_{m-2} is in the triangulated envelope, because $T_{m-1} \in \mathbf{A}_m$ and the cone of $T_{m-1} \to T_{m-2}$ is in $\mathbf{A}_{m-1}$. Repeating this process we see that each T_i is in the triangulated envelope, including $T_0 = X$. □

If $\mathbf{T}$ does not admit any non-trivial semiorthogonal decomposition we call it *semi-orthogonally indecomposable.*

Exceptional objects and exceptional collections

The easiest example of semiorthogonally indecomposable triangulated category is given by the ground field:

Example 1.47. The bounded derived category of finite dimensional vector spaces over $\Bbbk$ is semiorthogonally indecomposable.

Proof. By example 1.6 every bounded complex of vector spaces is isomorphic to the complex of its cohomology groups with trivial differentials.

Suppose then that $\mathbf{D}^{\mathrm{b}}(\Bbbk) = \langle \mathbf{A}, \mathbf{B} \rangle$ with $\mathbf{A} \subseteq \mathbf{B}^\perp$. We claim first that $\mathbf{B} \subseteq \mathbf{A}^\perp$ as well. Indeed, let $\oplus_p V^p[p] \in \mathbf{B}$ and $\oplus_p W^p[p] \in \mathbf{A}$. Then $\mathrm{Hom}(\oplus_p W^p[p], \oplus_q V^q[q]) = \oplus_{p,q} \mathrm{Hom}(W^p, V^q[q-p]) = \oplus_p \mathrm{Hom}(W^p, V^p)$ has to be zero, because $\mathbf{A} \subseteq \mathbf{B}^\perp$ implies that $\oplus_p \mathrm{Hom}(V^p, W^p)$ is.

Hence every semiorthogonal decomposition is in fact an orthogonal decomposition. This already implies the claim by proposition 1.39, because a point is connected. But we can also see it directly in this case. Indeed, if there is some $0 \neq \oplus_p V^p[p] \in \mathbf{A}$, then all $\oplus_p W^p[p] \in \mathbf{B}$ are zero, because there are always some non-zero morphisms between two non-zero vector spaces. □

Definition 1.48. An object E in a $\Bbbk$-linear triangulated category $\mathbf{T}$ is called *exceptional* if $\mathrm{Ext}^\bullet(E, E) = \Bbbk$.

In this case, the triangulated functor $\mathbf{D}^{\mathrm{b}}(\mathbb{C}) \to \mathbf{T}$ sending $V \mapsto V \otimes E$ is fully faithful. Its essential image is the triangulated envelope of E, denoted $\langle E \rangle$. It is a thick subcategory of $\mathbf{T}$. Moreover:

Proposition 1.49 ([Huy06, Lemma 1.58])**.** *Let $E \in \mathbf{T}$ be an exceptional such that $\dim_{\Bbbk}(\oplus_{i \in \mathbb{Z}}(\mathrm{Ext}^i(E, X)) < \infty$ for all $X \in \mathbf{T}$. Then $\langle E \rangle \leqslant \mathbf{T}$ is right admissible.*

Proof. We will use lemma A.26. For $X \in \mathbf{T}$, we consider the evaluation morphism

$$Y = \oplus_{i\in\mathbb{Z}} \operatorname{Ext}^i(E,X) \otimes \Sigma^{-i}E \to X.$$

The finiteness assumption guarantees that $Y \in \langle E\rangle$, so it remains to show that the cone of this morphism C lies in $\langle E\rangle^{\perp}$. To check this we use the long exact sequence induced by the cohomological functor $\operatorname{Hom}(E,-)$. Since E is exceptional, a morphism in $\operatorname{Hom}(E,Y) \cong \operatorname{Hom}(E, \operatorname{Hom}(E,X)\otimes E)$ corresponds precisely to the choice of a morphism in $\operatorname{Hom}(E,X)$. Hence the corresponding morphism in the long exact sequencce

$$\operatorname{Hom}(E,Y) \to \operatorname{Hom}(E,X)$$

is an isomorphism. Similarly, all other morphisms $\operatorname{Ext}^i(E,Y) \to \operatorname{Ext}^i(E,X)$ are isomorphisms as well. Thus $\operatorname{Ext}^i(E,C) = 0$ for all $i \in \mathbb{Z}$, which is what we needed to show. □

In particular, exceptional objects induce semiorthogonal decompositions

$$\mathbf{T} = \langle E^{\perp}, E\rangle$$

where we denote the thick envelope $\langle E\rangle$ simply by E. These kind of semiorthogonal decompositions are particularly nice, because we understand each piece very well and we know that each piece is already semiorthogonally indecomposable. A standard technique to find semiorthogonal decompositions will be then given by exceptional collections.

Definition 1.50. Let $\mathbf{T}$ be a $\Bbbk$-linear triangulated category. A sequence $E_1, \ldots, E_m$ of exceptional objects is called an *exceptional collection* if $\operatorname{Ext}^\bullet(E_i, E_j) = 0$ whenever $i > j$. Moreover:

i) If the E_i generate $\mathbf{T}$ (i.e. if $\bigcap E_i^{\perp} = \{0\}$), then we say that the exceptional collection is *full.*

ii) If $\operatorname{Ext}^l(E_i, E_j) = 0$ for all $i, j \in \{1, \ldots, m\}$ and all $l \neq 0$, then we say that the exceptional collection is *strong.*

Proposition 1.51. *Every exceptional collection $E_1, \ldots, E_m$ induces a semiorthogonal decomposition*

$$\mathbf{T} = \langle \mathbf{A}, E_1, \ldots, E_m\rangle$$

where we denote $\langle E_i\rangle$ simply by E_i and where $\mathbf{A} = \bigcap E_i^{\perp}$.

Proof. Note that $\{E\}^{\perp} = \langle E\rangle^{\perp}$, because if $X \in \{E\}^{\perp}$, then $E \in {}^{\perp}\{X\}$. And since ${}^{\perp}\{X\}$ is a triangulated subcategory and $\langle E\rangle$ is the triangulated envelope, we must also have $\langle E\rangle \leqslant {}^{\perp}\{X\}$, i.e. $X \in \langle E\rangle^{\perp}$. So from $\operatorname{Ext}^\bullet(E_i, E_j) = 0$ we can deduce that that $E_j \in \langle E_i\rangle^{\perp}$. Again, since $\langle E_i\rangle^{\perp} \leqslant \mathbf{T}$, the whole triangulated envelope $\langle E_j\rangle$ must be in $\langle E_i\rangle^{\perp}$. The claimed semiorthogonal decomposition is then the result of iteratively decomposing each orthogonal component starting from $\langle E_m\rangle$. □

This previous proposition will be a standard way to find semiorthogonal decompositions.

Remark 1.52. Semiorthogonal decompositions obtained from exceptional collections clarify the structure of the triangulated category already much more than arbitrary semiorthogonal decompositions, because the pieces on the right are well understood. But besides of the information already missing in arbitrary semiorthogonal decompositions (cf. remark 1.45), there may also be a lot of information hidden in the orthogonal complement $\mathbf{A}$. So the structure of the whole triangulated category may still be very complicated.

In the second chapter we will generalise some results from the absolute setting to the relative setting, so we will need the notion of relative exceptional object.

Suppose that $f\colon X \to Y$ is a proper morphism of regular quasi-projective varieties over $\Bbbk$. By properness and the standard spectral sequence argument, the derived pushforward preserves the bounded derived category of coherent sheaves. By regularity, the bounded derived category of coherent sheaves can be regarded as the full subcategory of perfect complexes, so it is also preserved under derived pullback (see [Stacks, Tag 09UA]). Hence we have an adjunction $\mathbb{L}f^* \dashv \mathbb{R}f_*$ between the bounded derived categories of coherent sheaves of X and Y. Again by regularity of X we have an adjunction $\mathcal{E}^\bullet \otimes^{\mathbb{L}} (-) \dashv \mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, -)$ on the bounded derived category of coherent sheaves (see [Stacks, Tag 09J5], [Stacks, Tag 08DQ] and [Stacks, Tag 08DJ]). Composing these two adjunctions we obtain the adjunction

$$\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}f^*(-)\colon \mathbf{D}^{\mathrm{b}}(Y) \leftrightarrows \mathbf{D}^{\mathrm{b}}(X)\colon \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, -).$$

Definition 1.53. In the previous situation, if the unit of the adjunction $\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}f^*(-) \dashv \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, -)$ is an isomorphism at $\mathcal{O}_Y$, then we say that $\mathcal{E}^\bullet$ is an *f-exceptional object.*

Example 1.54. As a reality check, consider the contraction of a variety to a point $f\colon X \to \operatorname{Spec}\mathbb{C}$. The notions of f-exceptional and exceptional objects should be equivalent in this case. Indeed, in this case we have $\mathbb{R}f_* \cong \mathbb{R}\Gamma(X, -)$ and $\mathbb{R}\Gamma \circ \mathbb{R}\mathcal{H}om(\mathcal{F}^\bullet, -) \cong \mathbb{R}\operatorname{Hom}(\mathcal{F}^\bullet, -)$. Hence $\mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \cong \operatorname{Ext}^\bullet(\mathcal{E}^\bullet, \mathcal{E}^\bullet)$, which is what we needed.

Proposition 1.55 ([Kuz16, Lemma 3.1])**.** *Let $f\colon X \to Y$ be a proper morphism between smooth quasi-projective varieties and let $\mathcal{E}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$ be an f-exceptional object. Then the composition $\mathcal{F}^\bullet \mapsto \mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}f^*(\mathcal{F}^\bullet)$ is a right admissible functor, yielding therefore a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}, \mathcal{E}^\bullet \boxtimes \mathbf{D}^{\mathrm{b}}(Y) \rangle$$

where $\mathcal{E}^\bullet \boxtimes \mathbf{D}^{\mathrm{b}}(Y)$ stands for the essential image of the previous functor and $\mathbf{A}$ is its right orthogonal complement.

Proof. We need to check that the unit of this adjunction is a natural isomorphism, i.e. that it is an isomorphism at all complexes $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(Y)$. This follows from the sequence of canonical isomorphisms

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}f^*(\mathcal{F}^\bullet)) \cong \mathbb{R}f_*(\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \otimes^{\mathbb{L}} \mathbb{L}f^*(\mathcal{F}^\bullet)) \cong$$
$$\cong \mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{E}^\bullet) \otimes^{\mathbb{L}} \mathcal{F}^\bullet \cong \mathcal{O}_Y \otimes \mathcal{F}^\bullet \cong \mathcal{F}^\bullet.$$

These follow from [Stacks, Tag 08DQ] and [Stacks, Tag 0B54] respectively. □

Semiorthogonally indecomposable triangulated categories

We will soon look for semiorthogonal decompositions on $\mathbf{D}^{\mathrm{b}}(X)$, so it is good to know in which cases this is already known to be hopeless. This is also useful when $\mathbf{D}^{\mathrm{b}}(X)$ admits a semiorthogonal decomposition in order to conclude that the components are semiorthogonally indecomposable.

We have already seen the example of a point $\mathbf{D}^{\mathrm{b}}(\mathbb{k})$. Generalising this to higher dimensions, we can look at those smooth projective varieties with trivial canonical bundle.

Definition 1.56. A smooth projective variety X is called a *Calabi-Yau* manifold if its canonical sheaf is trivial.

Examples of Calabi-Yau manifolds are abelian varieties and smooth hypersurfaces of dimension n and degree $d = n + 2$. By Riemann-Roch, the only Calabi-Yau manifolds of dimension 1 are elliptic curves. K3 surfaces are a particular case of Calabi-Yau manifold in dimension 2, but some Calabi-Yau surfaces are not K3, e.g. the product of two elliptic curves.

Proposition 1.57 ([Bri99]). *Let X be a smooth projective Calabi-Yau variety of dimension n. Then $\mathbf{D}^{\mathrm{b}}(X)$ is semiorthogonally indecomposable.*

Proof. The Serre functor on $\mathbf{D}^{\mathrm{b}}(X)$ is given by the shift $\mathcal{F}^\bullet \mapsto \mathcal{F}[n]^\bullet$. Suppose $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}, \mathbf{B} \rangle$. Then $\mathbf{A} \subseteq \mathbf{B}^\perp$ by definition. Let now $\mathcal{F}^\bullet \in \mathbf{B}$ and $\mathcal{G}^\bullet \in \mathbf{A}$. Then $\mathrm{Hom}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}[n]^\bullet)^\vee = 0$, so we also have $\mathbf{B} \subseteq \mathbf{A}^\perp$.

Hence every semiorthogonal decomposition of $\mathbf{D}^{\mathrm{b}}(X)$ is in fact an orthogonal decomposition. But then it must be trivial by proposition 1.39. □

From the previous proof we see that the result also generalises to *Calabi-Yau* categories, which are triangulated categories with a Serre functor given by a shift.

Finally, let us mention the following result:

Theorem 1.58 ([KO15, Theorem 1.4]). *Let X be a smooth projective variety such that the canonical linear system $|K_X|$ has at most finitely many base points. Then $\mathbf{D}^{\mathrm{b}}(X)$ does not admit any semiorthogonal decomposition.*

The canonical linear system $|K_C|$ on a curve C of genus $g \geqslant 2$ contains at least one non-zero effective divisor, so its base locus is at least a finite amount of points. Together with the Calabi-Yau case of elliptic curves, this shows that curves of genus $g \geqslant 1$ are semiorthogonally indecomposable. Hence this result generalises [Oka11, Theorem 1.1].

Hochschild homology

At certain points will use as a blackbox the machinery of Hochschild homology. We refer to [Kuz09b] for all details.

Associated to a $\Bbbk$-linear triangulated category $\mathbf{T}$ we can define a graded $\Bbbk$-vector space $\mathrm{HH}_\bullet(\mathbf{T})$ with the following properties:

i) If $\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_m \rangle$, then $\mathrm{HH}_\bullet(\mathbf{T}) \cong \bigoplus_{i=1}^m \mathrm{HH}_\bullet(\mathbf{A}_i)$.

ii) For any smooth projective variety X we have $\mathrm{HH}_i(\mathbf{D}^{\mathrm{b}}(X)) \cong \bigoplus_{q-p=i} H^q(X, \Omega^p)$.

Remark 1.59. Hochschild homology is usually defined for algebras and for varieties, so whatever definition we take for triangulated categories, it should verify that $\mathrm{HH}_\bullet(X) = \mathrm{HH}_\bullet(\mathbf{D}^{\mathrm{b}}(X))$. Condition *ii)* corresponds then to the Hochschild-Kostant-Rosenberg decomposition.

Condition *ii)* implies that the dimension of $\mathrm{HH}_i(X)$ is the sum of the numbers in the i-th column of the Hodge diamond of X for all $i \in \{-\dim X, \dots, \dim X\}$. A nice consequence of this machinery is the following:

Example 1.60 ([Kuz16, Corollary 2.16])**.** Suppose that $\mathbf{T}$ admits a full exceptional sequence of length $l \in \mathbb{N}$. Then $\mathrm{HH}_i(\mathbf{T}) = 0$ for all $i \neq 0$ and $\mathrm{HH}_0(\mathbf{T}) = \Bbbk^l$. In particular, if X is a smooth projective variety over $\mathbb{C}$ such that $\mathbf{D}^{\mathrm{b}}(X)$ has a full exceptional collection, then its Hodge diamond must have zeros everywhere except on the vertical line $h^{p,p}$. Moreover, the sum of this vertical line is precisely the length of the full exceptional collection.

Chapter 2

The case of Fano fibrations

Recall that by *variety* we mean an integral separated scheme of finite type over a field $\Bbbk$. In this second chapter we will always work over the complex numbers $\mathbb{C}$.

2.1 Fano manifolds

Let us start applying the general machinery developed in the first chapter to Fano varieties, which are varieties with ample anticanonical divisor. These are the absolute case of Fano fibrations, in which we collapse the whole variety to a single point. In this section we will only consider smooth Fano varieties.

Definition 2.1. A smooth projective variety X is called a *Fano manifold* if its anticanonical divisor $-K_X$ is ample.

To avoid trivial cases, we will always assume that Fano varieties are at least one dimensional. Some examples of Fano manifolds include projective spaces, Grassmannians and smooth hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree $d \leqslant n+1$. By Riemann-Roch, the only Fano manifold of dimension 1 is $\mathbb{P}^1$. Fano manifolds of dimension 2 are also called del Pezzo surfaces.

Proposition 2.2 ([PS99, Proposition 2.1.2])**.** *If X is a Fano manifold, then $H^\bullet(X, \mathcal{O}_X) = \mathbb{C}$.*

Proof. Since $\mathbb{C}$ is algebraically closed and X is integral and proper over $\mathbb{C}$, we have $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. So we only need to check the vanishings on degrees $p > 0$. This is a direct consequence of the Kodaira vanishing theorem (see [Mat02, Theorem 5-1-1]):

$$H^p(X, \mathcal{O}_X) = H^p(X, \omega_X \otimes \omega_X^{-1}) = 0$$

□

From these vanishings in cohomology we already obtain some information about derived categories of Fano manifolds, namely:

Corollary 2.3. *If X is a Fano manifold, then every line bundle $\mathcal{L}$ is an exceptional object in $\mathbf{D}^{\mathrm{b}}(X)$.*

Proof. We need to show that $\operatorname{Ext}^\bullet(\mathcal{L},\mathcal{L}) = \mathbb{C}$. Since tensoring with a line bundle induces an autoequivalence of $\mathbf{D}^{\mathrm{b}}(X)$, we have $\operatorname{Ext}^\bullet(\mathcal{L},\mathcal{L}) = \operatorname{Ext}^\bullet(\mathcal{O}_X,\mathcal{O}_X)$. The result follows then directly from the previous lemma. □

The main idea now is to exploit these Kodaira type vanishings as much as possible in order to obtain an exceptional collection on $\mathbf{D}^{\mathrm{b}}(X)$.

Proposition 2.4 ([PS99, Proposition 2.1.2.])**.** *Let X be a Fano manifold. Then* $\operatorname{Pic}(X) \cong H^2(X_h,\mathbb{Z})$ *is a finitely generated torsion free abelian group. Moreover, numerical and linear equivalence of divisors on X agree.*

Definition 2.5. The *index* of a Fano manifold X is the largest integer $r > 0$ which divides $-K_X$ in $\operatorname{Pic}(X)$.

Since $-K_X$ is ample, this means that we can write

$$-K_X \sim rH$$

with $r \in \mathbb{N}$ maximal and H ample, or equivalently $-K_X \equiv rH$ with $r \in \mathbb{N}$ maximal and H ample.

Example 2.6. Let $\mathbb{P}^n$ be the n-dimensional complex projective space and let $H \subseteq \mathbb{P}^n$ be a hyperplane. We have $-K \sim (n+1)H$, and therefore the index of $\mathbb{P}^n$ is $n+1$.

Example 2.7. Let $Q_n \subseteq \mathbb{P}^{n+1}$ be a smooth n-dimensional quadric and let $H \subseteq Q_n$ be a hyperplane section. By the adjunction formula we have $-K_{Q_n} \sim nH$, hence the index of Q_n is n.

In fact, the index of X is bounded above by $\dim X + 1$ and that high index Fano manifolds are characterised as:

i) Projective spaces, which are the Fano manifolds of index $\dim X + 1$.

ii) Smooth quadrics, which are the Fano manifolds of index $\dim X$.

See for example [Kuz09a, Theorem 2.1].

Proposition 2.8 ([Kuz09a, Lemma 3.4])**.** *Let X be a Fano manifold of index r. Then $\mathbf{D}^{\mathrm{b}}(X)$ has an exceptional collection*

$$\mathcal{O}_X,\ \mathcal{O}_X(H),\ \ldots,\ \mathcal{O}_X((r-1)H).$$

Proof. We have already seen in corollary 2.3 that all line bundles are exceptional, so it remains only to check the semiorthogonality condition.

We want to show that $\operatorname{Ext}^p(\mathcal{O}_X(lH),\mathcal{O}_X(kH)) = \operatorname{Ext}^p(\mathcal{O}_X,\mathcal{O}_X((k-l)H)) = H^p(X,\mathcal{O}_X((k-l)H)) = 0$ for all $0 \leqslant k < l \leqslant r-1$ and all $p \in \mathbb{Z}$.

The case $p = 0$ corresponds to the general fact that negative multiples of ample line bundles on positive dimensional projective varieties have no global sections. Indeed, if $s \in H^0(X, \mathcal{O}_X(-mH))$ is a non zero global section, then we can find an effective divisor $D \sim -mH$. But $\mathcal{O}_X(D) \cong \mathcal{O}_X(-mH)$, so $\mathcal{O}_X(-D)$ is an ample line bundle. This is a contradiction, because the global sections of $\mathcal{O}_X(-D)$ must all have zeros along the closed subscheme D, so this sheaf is not generated by its global sections (and neither are its powers for the same reason).

The cases $p > 0$ follow again from Kodaira vanishing. To apply this theorem, we want to find an ample divisor D such that $-mH \sim D + K_X$ for certain values of $m \in \mathbb{Z}$. Let then $D = -mH - K_X \sim -mH + rH = (r-m)H$. This divisor D is ample when $(r-m)H$ is ample, which is the case when $0 \leqslant m \leqslant r-1$. Therefore $H^p(X, \mathcal{O}_X(-mH)) = 0$ for

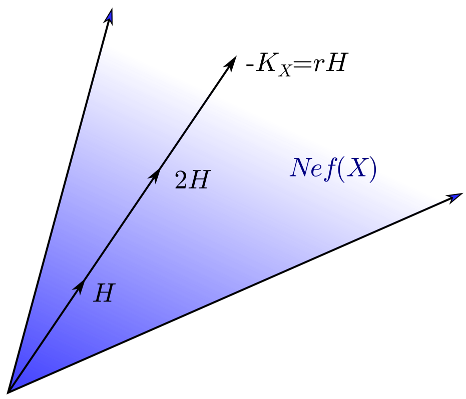

Figure 2.1: $-K_X - mH$ is still ample for $0 \leqslant m \leqslant r-1$.

all $p > 0$ and all $0 \leqslant m \leqslant r-1$. In particular for all $0 \leqslant k < l \leqslant r-1$ and all $p > 0$ we have $\mathrm{Ext}^p(\mathcal{O}_X(lH), \mathcal{O}_X(kH)) = 0$, as we wanted to show. □

Remark 2.9. In figure 2.1 we see that in order to apply Kodaira vanishing we need to restrict to the collection in the statement, and this collection cannot be enlarged from above with the same argument. In fact, projective spaces already give an example in which this collection cannot be enlarged at all. On the other hand, we will see later that in the case of quadrics we can enlarge this collection with one or two more vector bundles on the left.

Corollary 2.10 ([Kuz09a, Corollary 3.5])**.** *Let X be a Fano manifold of index r. Then we have a semiorthogonal decomposition*

$$\mathbf{D}^b(X) = \langle \mathbf{A}_X, \mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((r-1)H) \rangle.$$

Proof. The result follows directly from proposition 2.8 and proposition 1.51. □

In the remaining of this section we will study some particular cases of Fano manifolds in which the result of corollary 2.10 can be strengthened.

Projective spaces

The study of exceptional collections and more generally of semiorthogonal decompositions started in 1978 with Beilinson's work on projective space (see [Bei78]). The upshot is that the category $\mathbf{A}$ appearing in the corresponding Fano semiorthogonal decomposition is trivial, so that we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle.$$

We will give two proofs of this result. The first one is just a combination of previous results discussed here. To show that these line bundles form such a decomposition, it suffices to show that their thick envelope is the whole $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$ (see lemma 1.46). This is true because if $\mathcal{O}(1)$ is a very ample line bundle on an n-dimensional smooth projective variety X, then $\oplus_{i=0}^n \mathcal{O}(i)$ classically generates $\mathbf{D}^{\mathrm{b}}(X)$ (see proposition 1.18).

The second proof is perhaps more interesting, as it deals with some useful general machinery which can be adapted to other situations.

Lemma 2.11 (Beilinson)**.** *Consider the product $p, q\colon \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ and let $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal. There is a locally free resolution of $\mathcal{O}_\Delta$ as follows:*

$$0 \to \Lambda^n(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \Lambda^{n-1}(\mathcal{O}(-1) \boxtimes \Omega(1)) \to \cdots$$
$$\cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

where $\mathcal{F} \boxtimes \mathcal{G}$ means $p^*\mathcal{F} \otimes q^*\mathcal{G}$.

Proof. This proof is based on [EH16, Section 3.2.4] and [Huy06, Lemma 8.27]. We think of all locally free sheaves involved as vector bundles instead.

The sheaf $\mathcal{O}(-1)$ corresponds to the tautological line bundle, as can be seen from its transition functions. To understand the sheaf Ω we first describe its dual, the tangent sheaf $\mathcal{T}$. Let $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map defining $\mathbb{P}^n$ and let $v \in \mathbb{C}^{n+1} \setminus \{0\}$. We want to understand $T_{\pi(v)}\mathbb{P}^n$ in terms of $T_v\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$ and the differential $d\pi_v\colon T_v\mathbb{C}^{n+1} \to T_{\pi(v)}\mathbb{P}^n$. This differential is a surjective $\mathbb{C}$-linear homomorphism from a complex vector space of dimension $n+1$ to a complex vector space of dimension n. Its kernel has therefore dimension 1, and it corresponds precisely to $l = \mathrm{span}(v) \subseteq \mathbb{C}^{n+1}$. This already gives us an isomorphism $T_{\pi(v)}\mathbb{P}^n \cong \mathbb{C}^{n+1}/l$, but it is not canonical, as it depends on a choice of preimage of $\pi(v)$. Hence it will not glue to an isomorphism of vector bundles, which is what we are looking for.

The solution proposed in [EH16, Section 3.2.4] consists on realising that making the preimage of $\pi(v)$ λ times bigger will shrink our isomorphism by a factor of λ. This can be pictured locally as in figure 2.2. The further away from the origin a curve is, the slower its projection onto projective space goes.

To compensate this shrinking we may fix a linear form $\alpha\colon l \to \mathbb{C}$ and multiply the previous isomorphism by $\alpha(v)$. Then $\alpha(\lambda v) d\pi_{\lambda v}(w) = \lambda \frac{1}{\lambda} \alpha(v) d\pi_v(w) = \alpha(v) d\pi_v(w)$, as we wanted. So we have a canonical isomorphism $l^\vee \otimes \mathbb{C}^{n+1}/l \cong T_{\pi(v)}\mathbb{P}^n$ sending $\alpha \otimes (w \bmod l) \mapsto \alpha(v) d\pi_v(w)$, or equivalently $\mathbb{C}^{n+1}/l \cong T_{\pi(v)}\mathbb{P}^n \otimes l$. We have therefore

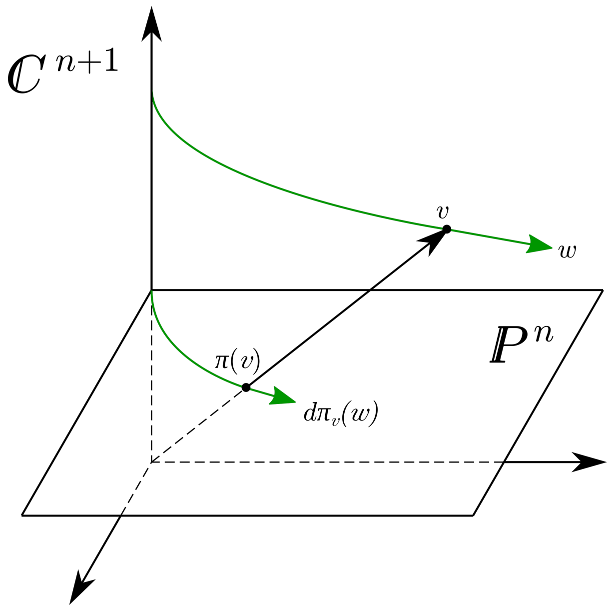

Figure 2.2: Tangent space of complex projective space.

a canonical short exact sequence $0 \to l \to \mathbb{C}^{n+1} \to T_{\pi(v)}\mathbb{P}^n \otimes l \to 0$ which after dualising becomes

$$0 \to \mathbb{V}(\Omega(1))_{\pi(v)} \to (\mathbb{C}^{n+1})^\vee \to l^\vee \to 0.$$

This shows that the fibre of the twisted sheaf of differentials $\Omega(1)$ over a point $\pi(v) \in \mathbb{P}^n$ is canonically identified with the space of linear forms $\alpha\colon \mathbb{C}^{n+1} \to \mathbb{C}$ whose restriction to $l = \operatorname{span}(v) \subseteq \mathbb{C}^{n+1}$ is trivial.

The fibre over $(l_1, l_2) \in \mathbb{P}^n \times \mathbb{P}^n$ of the vector bundle $\mathcal{O}(-1) \boxtimes \Omega(1)$ consists of points $x \in l_1 \subseteq \mathbb{C}^{n+1}$ and linear forms $\alpha\colon \mathbb{C}^{n+1} \to \mathbb{C}$ whose restriction to $l_2 \subseteq \mathbb{C}^{n+1}$ is trivial. Therefore we have a canonical evaluation morphism of vector bundles

$$\mathcal{E} = \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$$

which corresponds to a global section of the dual $\mathcal{E}^\vee$ whose zero locus is precisely the diagonal $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$. The Koszul complex yields then the desired locally free resolution. $\square$

Denote this locally free resolution by $\mathcal{E}^\bullet \to \mathcal{O}_\Delta$. For a line bundle $\mathcal{L}$ and a vector bundle $\mathcal{F}$ we have a canonical isomorphism $\Lambda^i(\mathcal{L} \otimes \mathcal{F}) \cong \mathcal{L}^{\otimes i} \otimes \Lambda^i \mathcal{F}$, so we can write

$$\mathcal{E}^{-i} \cong \mathcal{O}(-i) \boxtimes \Omega^i(i).$$

To simplify the notation, we will use *Fourier-Mukai* functors.

Definition 2.12. Let X_1 and X_2 be smooth projective varieties. Consider their product $X_1 \times X_2$ with the two projections $p_i\colon X_1 \times X_2 \to X_i$ and let $\mathcal{P}^\bullet \in \mathbf{D}^{\mathrm{b}}(X_1 \times X_2)$. The

Fourier-Mukai functor with kernel $\mathcal{P}^\bullet$, denoted $\Phi_{\mathcal{P}^\bullet}\colon \mathbf{D}^{\mathrm{b}}(X_1) \to \mathbf{D}^{\mathrm{b}}(X_2)$, is defined as

$$\mathcal{F}^\bullet \mapsto \mathbb{R}p_{2,*}(\mathcal{P}^\bullet \otimes^L p_1^*(\mathcal{F}^\bullet)).$$

Remark 2.13. The pullback is not derived because the projection from the product is flat. Usually we will take $\mathcal{P}^\bullet$ to be a complex of locally free sheaves, so that the tensor product need not be derived either.

Hence Fourier-Mukai functors are by definition triangulated functors between the derived categories. This class of functors is very useful, because it is closed under composition and adjoints. In fact, Mukai has found explicit formulas for the kernels of the composition or of the adjoints (see [Huy06, Propositions 5.9 and 5.10]). But even more interestingly, it turns out that a partial converse is true. Orlov has shown that every fully faithful triangulated functor $\mathbf{D}^{\mathrm{b}}(X_1) \to \mathbf{D}^{\mathrm{b}}(X_2)$ with right and left adjoints is a Fourier-Mukai functor (see [Huy06, Theorem 5.14]).

Example 2.14 ([Huy06, Examples 5.4])**.** Using the projection formula and some commutative diagrams one can show the following:

i) $\mathrm{id}_{\mathbf{D}^{\mathrm{b}}(X)} \cong \Phi_{\mathcal{O}_\Delta}$.

ii) $\mathbb{R}f_* \cong \Phi_{\mathcal{O}_{\Gamma_f}}$. In particular $H^\bullet(X, -) \cong \Phi_{\mathcal{O}_X}$.

iii) $(-) \otimes \mathcal{L} \cong \Phi_{\Delta_*\mathcal{L}}$.

iv) $[1] \cong \Phi_{\mathcal{O}_\Delta[1]}$.

Back to our locally free resolution of the diagonal $\mathcal{E}^\bullet \to \mathcal{O}_\Delta$, we had written $\mathcal{E}^{-i}$ as $\mathcal{O}(-i) \boxtimes \Omega^i(i)$. For $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$ we have then by the projection formula

$$\Phi_{\mathcal{E}^i}(\mathcal{F}^\bullet) \cong \mathbb{R}q_*p^*(\mathcal{F}^\bullet \otimes \Omega^{-i}(-i)) \otimes \mathcal{O}(i).$$

By base change, we may bring $\mathcal{F}^\bullet$ to the derived category of the other factor passing by $\operatorname{Spec}\mathbb{C}$. Hence

$$\Phi_{\mathcal{E}^i}(\mathcal{F}^\bullet) \cong H^\bullet(\mathbb{P}^n, \mathcal{F}^\bullet \otimes \Omega^{-i}(-i)) \otimes \mathcal{O}(i)$$

which is what we should keep in mind for the upcoming proof.

Proposition 2.15 (Beilinson)**.** *The line bundles $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ form a full and strong exceptional collection on $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$. In particular, we have the claimed semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)\rangle.$$

Proof. The collection is strong because

$$\mathrm{Ext}^l(\mathcal{O}(i), \mathcal{O}(j)) \cong H^l(\mathbb{P}^n, \mathcal{O}(j-i)) = 0$$

for all $i, j \in \{0, \dots, n\}$ and all $l \neq 0$. Indeed, the only possibly conflictive case is $l = n$, for which $H^n(\mathbb{P}^n, \mathcal{O}(j-i)) \cong H^0(\mathbb{P}^n, \mathcal{O}(i-j-n-1))^\vee = 0$ by Serre duality.

Now let us see that the smallest triangulated subcategory containing the line bundles $\mathcal{O}, \dots, \mathcal{O}(n)$ is the whole $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$ following the argument in [Huy06, Corollary 8.29]. Since tensoring with a line bundle induces an autoequivalence of the derived category, we may equivalently show that the smallest triangulated subcategory containing the line bundles $\mathcal{O}(-n), \dots, \mathcal{O}$ is $\mathbf{D}^{\mathrm{b}}(X)$.

So let $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$. Split the resolution $\mathcal{E}^\bullet \to \mathcal{O}_\Delta$ into n short exact sequences $0 \to \mathcal{M}^{-i-1} \to \mathcal{E}^{-i} \to \mathcal{M}^{-i} \to 0$ with $\mathcal{M}^{-n} = \mathcal{E}^{-n}$ and $\mathcal{M}^0 = \mathcal{O}_\Delta$. Each of them induces a distinguished triangle in the derived category. First tensor these triangles with $p^*\mathcal{F}^\bullet$ and then apply $\mathbb{R}q_*$ to obtain distinguished triangles

$$\Phi_{\mathcal{M}^{-i-1}}(\mathcal{F}^\bullet) \to \Phi_{\mathcal{E}^{-i}}(\mathcal{F}^\bullet) \to \Phi_{\mathcal{M}^{-i}}(\mathcal{F}^\bullet) \to \Phi_{\mathcal{M}^{-i-1}}(\mathcal{F}^\bullet)[1].$$

We have seen before that $\Phi_{\mathcal{E}^{-i}}(\mathcal{F}^\bullet) \cong H^\bullet(\mathbb{P}^n, \mathcal{F}^\bullet \otimes \Omega^{-i}(-i)) \otimes \mathcal{O}(i)$, and thus this complex lies in the smallest triangulated subcategory containing $\mathcal{O}(-i)$. Since this subcategory must be closed under cones, by induction we eventually obtain that $\Phi_{\mathcal{O}_\Delta}(\mathcal{F}^\bullet) \cong \mathcal{F}^\bullet$ is in the smallest triangulated subcategory containing these line bundles, as we wanted to show. □

Remark 2.16 ([Huy06, Proposition 8.28]). Using the isomorphisms discussed in this section one can deduce the *Beilinson spectral sequences* from the Grothendieck spectral sequence. For $\mathcal{F} \in \mathbf{Coh}(\mathbb{P}^n)$, the first Beilinson spectral sequence reads

$$E_1^{r,s} = H^s(\mathbb{P}^n, \mathcal{F} \otimes \Omega^{-r}(-r)) \otimes \mathcal{O}(r) \Rightarrow \mathcal{F}$$

where by $\mathcal{F}$ we mean the complex with $\mathcal{F}$ concentrated on degree zero as usual. The second Beilinson spectral sequence is

$$E_1^{r,s} = H^s(\mathbb{P}^n, \mathcal{F}(r)) \otimes \Omega^{-r}(-r) \Rightarrow \mathcal{F}.$$

del Pezzo surfaces

Smooth Fano surfaces are called *del Pezzo* surfaces. There are only ten families of such surfaces (see [Bea96, Exercises V.21.1]):

i) The projective plane $\mathbb{P}^2$.

ii) The quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

iii) The blow-up of $\mathbb{P}^2$ at $m \in \{1, \dots, 8\}$ distinct points in general position.

We have already seen that

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

For the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ we can also find a full exceptional collection. In this case it will follow from Orlov's projective bundle formula, which we will discuss in proposition 2.32. The upshot is that we have a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1) \rangle$$

where $\mathcal{O}(a,b)$ denotes $\mathcal{O}(a) \boxtimes \mathcal{O}(b) = p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)$.

It remains to discuss blow-ups of $\mathbb{P}^2$. In this case, we also have a full exceptional sequence given by Orlov's blow-up formula (see [Orl93, Theorem 4.3]):

Proposition 2.17 (Orlov's blow-up formula)**.** *Let X be a smooth projective variety and Y a smooth subvariety of codimension*[1] *$c > 1$. Let $f\colon \tilde{X} \to X$ be the blow-up of X along Y. Then we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\tilde{X}) = \langle \mathbf{D}^{\mathrm{b}}(X), \mathbf{D}^{\mathrm{b}}(Y)_1, \ldots, \mathbf{D}^{\mathrm{b}}(Y)_{c-1} \rangle$$

where $\mathbf{D}^{\mathrm{b}}(X)$ denotes the essential image of the functor $\mathbb{L}f^*$ and $\mathbf{D}^{\mathrm{b}}(Y)_m$ denotes the essential image of the functor $\mathbb{R}i_*(\mathcal{O}_E(m) \otimes^{\mathbb{L}} \mathbb{L}p^*(-))$. Here i denotes the inclusion of the exceptional divisor $i\colon E \to \tilde{X}$ and p denotes the projection onto the centre $p\colon E \to Y$.

This proposition yields particularly nice results applied to blow-ups of $\mathbb{P}^2$, because $Y = \operatorname{Spec} \mathbb{C}$ and $\mathbb{P}^2$ has a full exceptional collection.

Let $f\colon S \to \mathbb{P}^2$ be the blow-up of the plane at one point $P \in \mathbb{P}^2$. Then Orlov's blow-up formula tells us that

$$\mathbf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}_P \rangle.$$

Iterating this, for the blow-up S of $\mathbb{P}^2$ at $m \in \{1, \ldots, 8\}$ distinct points in general position we obtain a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(S) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}_{P_1}, \ldots, \mathcal{O}_{P_m} \rangle.$$

We can combine these semiorthogonal decompositions with example 1.60 to compute the Hodge diamond of a del Pezzo surface.

Corollary 2.18. *Let S be a del Pezzo surface. Then its Hodge diamond is*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & m & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where $m = 2$ if $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $m \in \{0, \ldots, 8\}$ is the number of points at which we have blown-up the plane to obtain S plus 1.

Proof. The existence of a full exceptional collection implies that the only column with non-zero entries is the 0-th column. We also know that the dimension of $\mathrm{HH}_0(S)$ must be the sum of the entries of this column, which is $2 + m$, so we only need to compute $\mathrm{HH}_0(S)$. But by example 1.60 we have $\mathrm{HH}_0(S) = \mathbb{C}^l$, where l is the length of the full exceptional collection. If $S = \mathbb{P}^1 \times \mathbb{P}^1$, then $l = 4$ as claimed. If S is a blow-up of $\mathbb{P}^2$ at (possibly zero) points, then the length of the full exceptional collection is 3 plus the number of points, also as claimed. □

[1] If we blow-up an effective Cartier divisor, nothing happens. In fact, this can be taken as a defining property of blow-ups: they replace a potentially ugly centre by a nice effective Cartier divisor.

Grassmannians and quadrics

To close this section let us briefly mention two classical results concerning Grassmannians and quadrics, which were historically the sequel of Beilinson's work on projective space. They were studied by Kapranov in the mid-eighties.

Proposition 2.19 ([Kap83]). *Let $G = G(k, \mathbb{C}^n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$. Then from the universal rank k vector bundle we can construct a full and strong exceptional collection on $\mathbf{D}^{\mathrm{b}}(G)$ indexed by Young diagrams with at most k rows and at most $n-k$ columns, or equivalently indexed by tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $0 \leqslant \alpha_k \leqslant \cdots \leqslant \alpha_1 \leqslant n-k$, each α_i representing the number of boxes in the i-th row of the corresponding Young diagram.*

Proposition 2.20 ([Kap86]). *Let Q_n be an n-dimensional smooth quadric. From the spinor bundles we may construct a full and strong exceptional collection on $\mathbf{D}^{\mathrm{b}}(Q_n)$ as follows:*

i) $\mathcal{S}, \mathcal{O}, \ldots, \mathcal{O}(n-1)$ if n is odd.

ii) $\mathcal{S}_-, \mathcal{S}_+, \mathcal{O}, \ldots, \mathcal{O}(n-1)$ if n is even.

The case of Grassmannians is a generalisation of the argument with the tautological short exact sequence and the locally free resolution of the diagonal, whereas the case of quadrics requires more work.

2.2 Fano fibrations

In this section we will generalise the results obtained for Fano manifolds to the relative setting.

Definition 2.21. A *Fano fibration* is a morphism $f\colon X \to Y$ of smooth projective varieties such that

i) the canonical morphism $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism,

ii) $-K_X$ is f-ample, and

iii) $\dim X > \dim Y$.

Such a morphism f has to be proper, because X is proper over $\mathbb{C}$ and Y is separated over $\mathbb{C}$. And it has to be surjective, since otherwise there would be a dense open subset not contained in the image, making it impossible for the canonical morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ to be an isomorphism. Let us first make a couple more remarks on this definition.

Remark 2.22. The condition $\dim X > \dim Y$ makes sure that all fibres are at least 1-dimensional. This follows from Chevalley's upper semi-continuity theorem (see [EGA IV$_3$, Corollary 13.1.5]).

Remark 2.23. By the theorem on formal functions, a proper morphism $f\colon X \to Y$ between algebraic varieties in which the canonical morphism $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism has connected fibres (see [EGA III_1, Corollary 4.3.2]). Conversely, a proper morphism $f\colon X \to Y$ with connected fibres onto a normal projective variety must verify that the canonical morphism $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. This follows from the Stein factorisation (see [EGA III_1, Theorem 4.3.1]), in which the finite part must be a homeomorphism onto a normal variety, hence an isomorphism (this is only true in characteristic zero, cf. [Deb01, Section 1.13]).

Remark 2.24. We say that a divisor D on X is f-ample if for all $\mathcal{F} \in \mathbf{Coh}(X)$ there exists an integer m_0 such that the counit $f^*(f_*(\mathcal{F}(mD))) \to \mathcal{F}(mD)$ is surjective for all $m \geqslant m_0$. This implies that the restriction to every fibre is ample (see [EGA II, Proposition 4.6.13]). Conversely, if f is proper and the restriction of D to each fibre is ample, then D is f-ample (see [EGA III_1, Theorem 4.7.1]).

If f is a projective morphism, then we also have a relative version of Kleiman's criterion, namely, D is f-ample if and only if it is positive on $\overline{\mathrm{NE}}(f)\setminus\{0\}$, where $\mathrm{NE}(f) \subseteq \mathrm{NE}(X)$ is the subcone generated by the classes of curves contracted by f (cf. [Deb01, Section 7.41]).

Besides of Fano manifolds, examples of Fano fibrations include projectivisations of vector bundles, Grassmann bundles or conic bundles.

Lemma 2.25. *If $f\colon X \to Y$ is a Fano fibration, then $\mathbb{R}f_*\mathcal{O}_X \cong \mathcal{O}_Y$.*

Proof. By definition $\mathcal{O}_Y \cong f_*\mathcal{O}_X$ (cf. remark 2.23), so we only need to check the vanishings on degrees $p > 0$. This is a direct consequence of the relative Kawamata-Viehweg vanishing (see [Mat02, Theorem 5-2-6]):

$$\mathrm{R}^p f_*\mathcal{O}_X = \mathrm{R}^p f_*(\omega_X \otimes \omega_X^{-1}) = 0$$

□

Corollary 2.26. *If $f\colon X \to Y$ is a Fano fibration, then every line bundle $\mathcal{L}$ is an f-exceptional object in $\mathbf{D}^{\mathrm{b}}(X)$.*

Proof. We need to show that $\mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{L},\mathcal{L}) \cong \mathcal{O}_Y$. Since $\mathcal{L}$ is locally free we have $\mathbb{R}\mathcal{H}om(\mathcal{L},\mathcal{L}) \cong \mathcal{H}om(\mathcal{L},\mathcal{L}) \cong \mathcal{O}_X$. The result follows then directly from the previous lemma. □

As before, we would like to use the relative negativity of the canonical divisor to deduce more vanishings, and then deduce certain semiorthogonalities from them.

Definition 2.27. Let $f\colon X \to Y$ be a Fano fibration. The *relative index* of X over Y is defined as the largest rational $r > 0$ such that $-K_X \equiv_f rH$ for some f-ample Cartier divisor H, where $A \equiv_f B$ means that $A - B$ has trivial intersection with every curve contracted by f.

Note that relative ampleness is also a numerical condition by Kleiman's relative ampleness criterion, so by definition of Fano fibration the relative index r is at least 1. By the theorem of the base, the group of divisors modulo numerical equivalence is a finitely generated free abelian group (cf. [Laz04, Proposition 1.1.16]). The group of divisors modulo f-numerical equivalence is a quotient thereof, hence also finitely generated. This implies that r is a well-defined rational number.

Theorem 2.28. *Let $f\colon X \to Y$ be a Fano fibration of relative index r and let $\mathcal{L} = \mathcal{O}_X(H)$, where H is an f-ample Cartier divisor such that $-K_X \equiv_f rH$. Then*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_f, \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes 2}, \dots, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes \lceil r-1 \rceil} \rangle$$

where $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}$ denotes the essential image of $\mathcal{F}^\bullet \mapsto \mathbb{L}f^*(\mathcal{F}^\bullet) \otimes \mathcal{L}$.

Proof. After the previous corollary it only remains to show semiorthogonality. In this case we need to show that

$$\mathrm{Ext}^\bullet(\mathbb{L}f^*\mathcal{F}^\bullet \otimes \mathcal{L}^{\otimes i}, \mathbb{L}f^*\mathcal{G}^\bullet \otimes \mathcal{L}^{\otimes j}) = 0$$

for all $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(Y)$ and all $0 \leqslant j < i \leqslant \lceil r-1 \rceil$. Using first the adjunction $\mathbb{L}f^* \dashv \mathbb{R}f_*$ and then the projection formula we obtain

$$\mathrm{Ext}^\bullet(\mathcal{F}^\bullet, \mathbb{R}f_*(f^*\mathcal{G}^\bullet \otimes \mathcal{L}^{\otimes j-i})) \cong \mathrm{Ext}^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes^L \mathbb{R}f_*\mathcal{L}^{\otimes j-i}).$$

Hence it suffices to show that $\mathbb{R}f_*\mathcal{L}^{\otimes j-i} = 0$ for all $0 \leqslant j < i \leqslant \lceil r-1 \rceil$.

On degrees $p > 0$ we use the relative Kawamata-Viehweg vanishing (see [Mat02, Theorem 5-2-6]). The claim is that the theorem applies for integers $0 \leqslant m = j - i \leqslant \lceil r-1 \rceil$. Indeed, we only need $-K_X - mH$ to be f-ample, which follows from the definition of the relative index as in the absolute case.

For the vanishing on degree $p = 0$, assume first that f is flat. Then we can apply the theorem on formal functions to argue fibrewise. Indeed, since f is flat and $\mathcal{E} = \mathcal{L}^{\otimes j-i}$ is a vector bundle, $\mathcal{E}$ is flat over Y. Each restriction $\mathcal{E}_y$ is antiample on the fibre X_y over the closed point $y \in Y$. This fibre is a positive dimensional proper $\mathbb{C}$-scheme, so we must have $H^0(X_y, \mathcal{E}_y) = 0$. By Grauert's theorem (see [Har77, Corollary III.12.9]), $f_*\mathcal{E}$ is a vector bundle on Y with fibre isomorphic to $H^0(X_y, \mathcal{E}_y) = 0$ over $y \in Y$. Hence $f_*\mathcal{E} = 0$.

The idea for the general case is to use generic flatness (see [EGA IV$_2$, Theorem 6.9.1]). Let $U \subseteq Y$ be a non-empty open subscheme such that $\mathcal{E}|_{f^{-1}(U)}$ is flat over U. The pullback of a projective morphism is projective, so we can still apply Grauert's theorem to conclude that

$$(f|_{f^{-1}(U)})_*(\mathcal{E}|_{f^{-1}(U)}) = (f_*\mathcal{E})|_U = 0.$$

In particular, the stalk of $f_*\mathcal{E}$ at the generic point η_Y of Y vanishes. Since f is dominant, for every non-empty open subset $V \subseteq Y$ we have a commutative diagram

$$\begin{tikzcd}
(f_*\mathcal{E})(V) \arrow[r, equal] \arrow[d] & \mathcal{E}(f^{-1}(V)) \arrow[d] \\
(f_*\mathcal{E})_{\eta_Y} = 0 \arrow[r, dashed] & \mathcal{E}_{\eta_X}
\end{tikzcd}$$

Hence every section $s \in \mathcal{E}(f^{-1}(V))$ maps to 0 in the stalk at the generic point of X. Such a section must be the zero section, because it is an almost everywhere zero section of a line bundle. But to check this formally, consider an affine open cover $V_i = \operatorname{Spec} A_i$ of V such that $\mathcal{E}|_{V_i} \cong \mathcal{O}_{V_i}$. It suffices to check that the restriction of s to each V_i vanishes. But since X is integral, every A_i is an integral domain. The morphism $\mathcal{E}|_{V_i}(V_i) \to \mathcal{E}_{\eta_X}$ is then the localisation of the integral domain A_i at the zero ideal, i.e. the inclusion of A_i in its fraction field. Since the image of $s|_{V_i}$ under this injective homomorphism is 0, the restriction $s|_{V_i}$ must also be zero, as we wanted to show[2]. □

Remark 2.29. If Y is a smooth curve, then every Fano fibration $f\colon X \to Y$ is automatically flat, because a module over a Dedekind domain is flat if and only if it is torsion free (see [Stacks, Tag 0AUW]).

But if the base has higher dimension, then we can find non-flat Fano fibrations.

Example 2.30 (cf. [AW98, Example (3.4.0)])**.** Let $G = G(2, \mathbb{C}^3) \cong \mathbb{P}^2$ be the Grassmannian of planes in $\mathbb{C}^3$ and let $\mathcal{U}$ be its tautological line bundle. Consider the rank 3 vector bundle $\mathcal{U} \oplus \mathcal{O}$ on G, whose total space can be described as

$$\{(v_1, v_2, v_3, v_4; \Pi) \in \mathbb{C}^3 \times G \mid (v_1, v_2, v_3) \in \Pi \subseteq \mathbb{C}^3\}.$$

Its projectivisation $X = \mathbb{P}(\mathcal{U} \oplus \mathcal{O})$ projects onto $\mathbb{P}^3$ via $[v_1, v_2, v_3, v_4; \Pi] \mapsto [v_1, v_2, v_3, v_4]$. The fibre of a point $[v_1, v_2, v_3, v_4] \neq [0, 0, 0, 1]$ consists of the set

$$\{[v_1, v_2, v_3, v_4; \Pi] \in \{[v_1, v_2, v_3, v_4]\} \times G \mid v = (v_1, v_2, v_3) \in \Pi\}.$$

Let us describe this subspace of G. Let $\Pi = \operatorname{span}(a, b) \in G$ with $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Then the corresponding point in the Plücker embedding (which in our case is an isomorphism) is $[p_{12}, p_{13}, p_{23}] \in \mathbb{P}^2$, where

$$p_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \in \mathbb{C}.$$

The condition that $v \in \Pi$ is equivalent to the equation

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0.$$

[2] We have proven everything by hand, but it would suffice to say that the pushforward of a torsion-free sheaf along a dominant morphism is torsion free and every section of $f_*\mathcal{E}$ is torsion (cf. [Stacks, Tag 0AVQ]).

Computing the determinant on the last row we can rewrite the equation as

$$v_1 p_{23} - v_2 p_{13} + v_3 p_{12} = 0.$$

This shows that the set of such planes is a linear subspace of $\mathbb{P}^2$, i.e. the fibres of $f\colon X \to \mathbb{P}^3$ over a point $[v_1, v_2, v_3, v_4] \neq [0,0,0,1]$ is $\mathbb{P}^1$.

Over $[0,0,0,1]$ the condition that $v = (0,0,0) \in \Pi$ becomes trivial. Hence the fibre over this point is just $G \cong \mathbb{P}^2$.

This shows that f is not flat, because flat morphisms between varieties have equidimensional fibres (see [Har77, Corollary III.9.6]).

Projective bundles

Let Y be a smooth projective variety and let $\mathcal{E}$ be a locally free sheaf of rank $r+1$ on Y. Let $f\colon \mathbb{P}(\mathcal{E}) \to Y$ be the corresponding projective bundle. We will continue to think of locally free sheaves as vector bundles and describe the situation in terms of the closed points of the varieties involved.

If we denote by $\mathcal{O}(-1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$, whose fibre over $[e; y] \in \mathbb{P}(\mathcal{E})$ is just the line $\{(\lambda e; y) \mid \lambda \in \mathbb{C}\} \subseteq \mathbb{V}(\mathcal{E})$, then $\mathcal{O}(-1)$ restricts to the tautological line bundle on each fibre $\mathbb{P}^r$ of f. In particular, $\mathcal{O}(1)$ is f-ample, because it restricts to the line bundle associated to a hyperplane on each fibre. Denote by H the divisor on $\mathbb{P}(\mathcal{E})$ associated to $\mathcal{O}(1)$.

Lemma 2.31. *The relative index of $f\colon \mathbb{P}(\mathcal{E}) \to Y$ is $r+1$.*

Proof. By the projective bundle formula (see [EH16, Theorem 9.6]) we can write $K_{\mathbb{P}(\mathcal{E})} = aH + f^*D$ for some divisor D on Y. Since $K_{\mathbb{P}(\mathcal{E})}$ restricts to K_F on each fibre $F \cong \mathbb{P}^r$ and H restricts to a hyperplane section, we have $a = -r-1$. Let C be a curve on X contracted by f. Then by the projection formula (see [Deb01, Formula 1.5]) we have

$$(K_{\mathbb{P}(\mathcal{E})} - (r+1)H).C = (f^*D).C = D.f_*C = 0.$$

This also shows that the numerical classes of divisors relative to f are generated by H, hence the relative index is exactly $r+1$. $\square$

Hence the relative Fano semiorthogonal decomposition yields

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E})) = \langle \mathbf{A}, \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}, \dots, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes r} \rangle$$

for $\mathcal{L} = \mathcal{O}(1)$.

To see that this collection of admissible subcategories generate $\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E}))$, i.e. that $\mathbf{A} = 0$, we will adapt Beilinson's argument with the resolution of the diagonal to this relative setting. This was first done by Orlov in 1993 (see [Orl93, Theorem 2.6]):

Proposition 2.32 (Orlov's projective bundle formula)**.** *The orthogonal complement* $\mathbf{A}$ *in the previous decomposition is trivial. In particular, keeping the notation as above, we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E})) = \langle \mathbf{D}^{\mathrm{b}}(Y), \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}, \dots, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^r \rangle.$$

Proof. Consider the tautological short exact sequence

$$0 \to \mathcal{O}(-1) \to f^*(\mathcal{E}) \to \mathcal{Q} \to 0.$$

The fibre of the pullback $f^*(\mathcal{E})$ over a point $[e; y] \in \mathbb{P}(\mathcal{E})$ is by definition the fibre of $\mathcal{E}$ over its image $f([e; y]) = y$. Hence, the tautological sequence over a point $[e; y] \in \mathbb{P}(\mathcal{E})$ looks as follows:

$$0 \to l_y = \{(\lambda e; y) \mid \lambda \in \mathbb{C}\} \to \mathbb{V}(\mathcal{E})_y \cong \mathbb{C}^{r+1} \to \mathbb{V}(\mathcal{E})_y / l_y \to 0.$$

Dualising this sequence shows that the fibre of $\mathcal{Q}^\vee$ over $[e; y]$ can be canonically identified with linear forms on $\mathbb{V}(\mathcal{E})_y$ which vanish along l_y, so that we have a canonical evaluation morphism

$$s \colon \mathcal{O}(-1) \boxtimes \mathcal{Q}^\vee \to \mathcal{O}_{\mathbb{P}(\mathcal{E}) \times_Y \mathbb{P}(\mathcal{E})}$$

which vanishes precisely along the diagonal $\Delta \subseteq \mathbb{P}(\mathcal{E}) \times_Y \mathbb{P}(\mathcal{E})$. This morphism is a global section of $\mathcal{H}om(\mathcal{O}(-1) \boxtimes \mathcal{Q}^\vee, \mathcal{O}_{\mathbb{P}(\mathcal{E}) \times_Y \mathbb{P}(\mathcal{E})}) \cong \mathcal{O}(1) \boxtimes \mathcal{Q}$, so the Koszul complex yields a resolution of the diagonal by vector bundles as before, namely

$$\begin{aligned} 0 \to \Lambda^r \mathcal{O}(-1) \boxtimes \mathcal{Q}^\vee \to \cdots \\ \cdots \to \Lambda^2 \mathcal{O}(-1) \boxtimes \mathcal{Q}^\vee \to \mathcal{O}(-1) \boxtimes \mathcal{Q}^\vee \to \mathcal{O}_{\mathbb{P}(\mathcal{E}) \times_Y \mathbb{P}(\mathcal{E})} \to \mathcal{O}_\Delta \to 0. \end{aligned}$$

Call $\mathcal{G}^\bullet \to \mathcal{O}_\Delta$ this resolution, so that $\mathcal{G}^{-i} = \mathcal{O}(-i) \boxtimes \Lambda^i \mathcal{Q}^\vee$ as in the case of $\mathbb{P}^n$.

We want to consider again Fourier-Mukai transforms with the vector bundles $\mathcal{G}^{-i}$ as kernels. For $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E}))$ we obtain an expression

$$\Phi_{\mathcal{G}^{-i}}(\mathcal{F}^\bullet) = \mathbb{R}p_*(q^*(\mathcal{F}^\bullet) \otimes p^*\mathcal{O}(-i) \otimes q^*(\Lambda^i \mathcal{Q}^\vee))$$

where q^* need not be derived because it is the pullback of a flat morphism (hence flat) and $\otimes$ need not be derived because the factor on the right hand side is a vector bundle. With the projection formula we obtain

$$\Phi_{\mathcal{G}^{-i}}(\mathcal{F}^\bullet) \cong \mathcal{O}(-i) \otimes \mathbb{R}p_* q^*(\mathcal{F}^\bullet \otimes \Lambda^i \mathcal{Q}^\vee).$$

And finally using flat base change we get

$$\Phi_{\mathcal{G}^{-i}}(\mathcal{F}^\bullet) \cong \mathcal{O}(-i) \otimes f^*(f_*(\mathcal{F}^\bullet \otimes \Lambda^i \mathcal{Q}^\vee)).$$

This complex is in $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{O}(-i)$ by Serre's theorem (see [EGA III_1, Theorem 2.2.1]) combined with a spectral sequence argument involving the usual particular case of the Grothendieck spectral sequence.

Now split the resolution $\mathcal{G}^\bullet \to \mathcal{O}_\Delta$ into short exact sequences to obtain distinguished triangles involving $\Phi_{\mathcal{G}^{-i}}(\mathcal{F}^\bullet)$ and cones thereof. Inductively we see that $\mathcal{F}^\bullet$ is in the smallest triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathcal{E}))$ containing the subcategories $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{O}(-i)$.

As before, we can shift the relative exceptional collection of line bundles back and forth, so this finishes the proof. □

Combined with example 1.60, the previous proposition shows that if $\mathbf{D}^{\mathrm{b}}(Y)$ has a full exceptional collection of length l, then

$$\mathrm{HH}_\bullet(\mathbb{P}(\mathcal{E})) = \mathbb{C}^{l(r+1)}.$$

Ruled surfaces

Orlov's projective bundle formula has immediate applications in the case of ruled surfaces, because every ruled surface $f\colon S \to C$ can be written as $S \cong \mathbb{P}(\mathcal{E})$ for a rank 2 vector bundle on C (see [Har77, Proposition V.2.2]).

Example 2.33 (Hirzebruch surfaces)**.** Consider the vector bundle $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$. The resulting ruled surface $f\colon \mathbb{F}_n \to \mathbb{P}^1$ is a *Hirzebruch surface*. Combining proposition 2.15 and proposition 2.32 we obtain a semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathbb{F}_n) = \langle\langle \mathcal{O}, \mathcal{O}(1)\rangle, \langle \mathcal{O}, \mathcal{O}(1)\rangle \boxtimes \mathcal{O}(1)\rangle.$$

By example 1.60 we have $\mathrm{HH}_\bullet(\mathbb{F}_n) = \mathbb{C}^4$ and hence we can already compute its Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Example 2.34 (Surfaces ruled over curves of any genus)**.** Let $S = \mathbb{P}(\mathcal{E})$ be a ruled surface over a curve C of genus g. Then we have $\mathbf{D}^{\mathrm{b}}(S) = \langle \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(C) \boxtimes \mathcal{O}(1)\rangle$, so we have an isomorphism $\mathrm{HH}_\bullet(S) = \mathrm{HH}_\bullet(C) \oplus \mathrm{HH}_\bullet(C)$. Hodge diamonds of curves are easy to compute: we have $h^{0,0}(C) = 1$ as always and $h^{0,1}(C) = g$ by definition. Therefore $\mathrm{HH}_0(C) = \mathbb{C}^2$ and $\mathrm{HH}_1(C) = \mathbb{C}^g$, from which we obtain $\mathrm{HH}_0(S) \cong \mathbb{C}^4$ and $\mathrm{HH}_1(S) \cong \mathbb{C}^{2g}$. Together with the symmetries of the Hodge diamond, this already allows us to compute it:

$$\begin{matrix} & & 1 & & \\ & g & & g & \\ 0 & & 2 & & 0 \\ & g & & g & \\ & & 1 & & \end{matrix}$$

Grassmann and quadric fibrations

As in the absolute case, the generalisation of proposition 2.32 to Grassmannians was easier than to quadrics. It can already be found in the same paper in which Orlov introduces the projective bundle and blow-up formulas.

Proposition 2.35 ([Orl93, Section 3])**.** *Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective variety X. Let $\mathbb{G}(\mathcal{E}) = G(k, \mathcal{E})$ be the Grassmann bundle of k-planes in $\mathcal{E}$ with its natural projection onto X. From the tautological rank k-vector bundle on $\mathbb{G}(\mathcal{E})$ we can construct a full relative exceptional collection on $\mathbf{D}^{\mathrm{b}}(\mathbb{G}(\mathcal{E}))$ indexed by Young diagrams with at most k rows and at most $r-k$ columns.*

The case of quadric fibrations was later studied by Kuznetsov. We refer to his paper for the precise statement and its proof (see [Kuz08, Theorem 4.2]).

Calabi-Yau fibrations

In light of proposition 1.57, one could ask whether a similar result holds for Calabi-Yau fibrations, which are defined in the analogous way to Fano fibrations. Being a Calabi-Yau fibration is a relative notion and having an indecomposable derived category is an absolute notion, so there is no hope to find such a strong statement in the relative setting, as the following example shows:

Example 2.36. Let $X = \mathbb{P}^1 \times E$ be the product of $\mathbb{P}^1$ and a smooth elliptic curve. The projection $X \to \mathbb{P}^1$ is a Calabi-Yau fibration. But on the other hand $X = \mathbb{P}(\mathcal{O}_E^{\oplus 2}) \to E$ corresponds to the projectivisation of the trivial rank 2 vector bundle on E, hence

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{D}^{\mathrm{b}}(E), \mathbf{D}^{\mathrm{b}}(E) \boxtimes \mathcal{O}(1) \rangle$$

So $\mathbf{D}^{\mathrm{b}}(X)$ is not indecomposable.

Nevertheless, one could still ask whether the arguments in the proof of theorem 2.28 still work for Calabi-Yau fibrations. It turns out that this is not the case:

Proposition 2.37. *Let $f\colon X \to Y$ be a Calabi-Yau fibration. Then there are no f-exceptional objects.*

Proof. Let us first check that being a relative exceptional object is compatible with restriction to open subsets. Let $f\colon X \to Y$ be a morphism between smooth projective varieties and let $\mathcal{E}$ be an f-exceptional object. Let $j\colon V \subseteq Y$ be the inclusion of a dense open subset and let $i\colon U = f^{-1}(V) \to X$ be the inclusion of its preimage. The claim is that $i^*\mathcal{E}$ is an $(f|_U)$-exceptional object. Since our varieties are regular, all complexes are perfect and the canonical morphism in [Stacks, Tag 08I3] is an isomorphism. Combined with flat base change, this yields

$$\mathbb{R}(f|_U)_*\mathbb{R}\mathcal{H}om(i^*\mathcal{E}, i^*\mathcal{E}) \cong \mathbb{R}(f|_U)_* i^*\mathbb{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}) \cong j^*\mathbb{R}f_*\mathbb{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}) \cong j^*\mathcal{O}_Y = \mathcal{O}_V.$$

This computation allows us to reduce to the smooth case via generic smoothness (see [Har77, Corollary III.10.7]), so let now $f\colon X \to Y$ be a smooth and proper morphism between smooth quasi-projective varieties and let $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(X)$ be an f-exceptional object. Let $y \in Y$ be a closed point and let $i\colon X_y \to X$ denote now the inclusion of the fibre over y, which is a smooth Calabi-Yau of dimension at least 1. We will see that $i^*\mathcal{E}$ is an exceptional object in $\mathbf{D}^{\mathrm{b}}(X_y)$, which implies by indecomposability that $\mathbf{D}^{\mathrm{b}}(X_y) = \langle i^*\mathcal{E} \rangle \cong \mathbf{D}^{\mathrm{b}}(\mathbb{C})$, contradicting therefore the assumption on the dimension of X_y.

In [BO95, Lemma 1.3], Bondal and Orlov prove a variant of the base-change theorem for smooth and proper morphisms along arbitrary base change morphisms. So combining this again with the canonical isomorphism from [Stacks, Tag 08I3] we have

$$\mathrm{Ext}^\bullet(i^*\mathcal{E}, i^*\mathcal{E}) \cong \mathbb{C}.$$

Hence $i^*\mathcal{E}$ is indeed an exceptional object in $\mathbf{D}^{\mathrm{b}}(X_y)$, yielding the desired contradiction. □

2.3 Singular Fano varieties and fibrations

Since we are interested in the derived categories appearing in the Minimal Model Program, we need to allow certain kinds of singularities. In the first part of this section we will define and briefly discuss these kind of singularities. Later on we will generalise the results obtained for Fano manifolds and smooth Fano fibrations to this singular setting.

Normal varieties

Recall that an integral scheme is called *normal* if all its local rings are integrally closed in their fraction fields. Regular local rings are unique factorisation domains (see [Stacks, Tag 0AG0]), and unique factorisation domains are integrally closed. Hence every smooth scheme over $\mathbb{C}$ is normal.

Example 2.38. A curve is normal if and only if it is smooth, because a one dimensional noetherian local domain is integrally closed if and only if its maximal ideal is principal.

On the other hand, normal surfaces can already be singular:

Example 2.39 ([Sha94, Section 5.1])**.** Let $X = \{x^2 + y^2 = z^2\} \subseteq \mathbb{A}^3$ be the cone over a smooth conic. The origin is a singular point, because all partial derivatives vanish. Being an integrally closed domain is stable under localisation, so it suffices to check that the coordinate ring $\mathbb{C}[X] = \mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$ is integrally closed. This follows from the following fact, which is nice to produce examples. Let R be a unique factorisation domain with $2 \in R^\times$ a unit and let $x, p \in R$ such that $x^2 - p$ is irreducible in $R[x]$. Then $R[x]/(x^2 - p)$ is integrally closed if and only if p has no repeated prime factors (see [Vak17, Exercise 5.4.H]).

In our case all the assumptions hold, because $x^2 + y^2 = (x + iy)(x - iy)$ has no repeated prime factors in $R = \mathbb{C}[x, y]$.

A key property on normal schemes is that the group of Cartier divisors is a subgroup of the group of Weil divisors in a way which is compatible with linear equivalence (cf. proof of [Har77, Proposition II.6.11]). But some Weil divisors are not Cartier divisors (see [Har77, Examples II.6.5.2 and II.6.11.3]).

If X is a normal variety, its singular locus is a closed subscheme of codimension at least 2 (see [Sha94, Theorem 5.2]). In particular, the canonical divisor on the smooth locus extends to a Weil divisor K_X which is unique up to linear equivalence (see [Har77, Proposition II.6.5]).

Definition 2.40. Let X be a normal variety and let $U \subseteq X$ be its smooth locus. Then the restriction to U defines an isomorphism between the divisor class groups. Let K_U be the usual canonical divisor on the smooth variety U. Any divisor whose restriction to U is K_U is called the *canonical divisor* of X, denoted K_X.

Terminal and canonical singularities

We have already defined a canonical divisor K_X associated to any normal projective variety X, but now we want to be able to do some geometry with it. Two main sources of geometry on a variety are vector bundles and intersection products.

Remark 2.41 (Cartier divisors are nice).

i) Every Cartier divisor D on X corresponds bijectively to a line bundle $\mathcal{O}(D)$ on X (see [Har77, Proposition II.6.15 and Remark II.6.14.1]), but there is no obvious way to attach a line bundle to a Weil divisor in general. In particular, the notion of ample divisor is only well-defined for Cartier divisors.

ii) When we intersect two effective Cartier divisors, the dimension drops at most by 1, because locally we are intersecting the zero loci of two regular elements. This may fail for Weil divisors. For example, let $X = \{xy = zw\} \subseteq \mathbb{A}^4$ and consider the Weil divisors $W = \{x = z = 0\}$ and $W' = \{y = w = 0\}$. Then $W \cap W' = \{0\}$.

So in order to compute intersection products and to pullback divisors via their corresponding line bundles, we would like to require K_X to be a Cartier divisor. But it turns out that this is too much to ask, so we will need the following notion:

Definition 2.42. A Weil divisor D on a normal variety X is called $\mathbb{Q}$-Cartier if mD is a Cartier divisor for some $m \in \mathbb{N}$.

If K_X is $\mathbb{Q}$-Cartier, we can associate a line bundle $\mathcal{O}(mK_X)$ to some multiple of it. With this line bundle we can pull it back along a morphism $f\colon X' \to X$ to the $\mathbb{Q}$-divisor $\frac{1}{m}D'$, where D' is the Cartier divisor on X' corresponding to the line bundle $f^*\mathcal{O}(mK_X)$.

Similarly, if $C \subseteq X$ is a curve then we have a well-defined and well-behaved intersection product

$$K_X.C = \frac{1}{m}(mK_X.C).$$

The example in part *ii)* of remark 2.41 shows that not every Weil divisor is $\mathbb{Q}$-Cartier. A normal variety X on which every Weil divisor is a $\mathbb{Q}$-Cartier divisor is called a $\mathbb{Q}$-*factorial* variety.

Let now X be a normal variety such that mK_X is Cartier. By Hironaka's work, we may find a resolution of singularities $f\colon Y \to X$ such that the exceptional locus of f is a divisor with simple normal crossing support (see [Laz04, Theorem 4.1.3]). Away from this exceptional locus, f is an isomorphism between two smooth quasi-projective varieties, so the restriction of mK_X pulls back to the corresponding restriction of mK_Y. It follows that we can write

$$mK_Y \sim f^*(mK_X) + \sum_i ma(E_i, X)E_i$$

for some rational numbers $a(E_i, X)$, where E_i is the collection of prime exceptional divisors of f.

Definition 2.43. In the previous setting, the rational number $a_i = a(E_i, X)$ is called the *discrepancy* of E_i with respect to X. We say that X has...

i) *Terminal singularities*, if all $a_i > 0$.

ii) *Canonical singularities*, if all $a_i \geqslant 0$.

Terminal singularities are the smallest class of singularities needed to run the Minimal Model Program starting from a smooth projective variety, and canonical singularities are the smallest class of singularities that appear on canonical models of varieties of general type (cf. [Kol13, Definition 2.8]).

Remark 2.44. The discrepancy of a divisor over X does not depend on the particular choice of resolution (cf. [KM98, Remark 2.23]).

Example 2.45. Let $X = \{x^2 + y^2 + z^2 = 0\} \subseteq \mathbf{A}^3$ be the cone over a conic $C \subseteq \mathbb{P}^2$.

Blow-up the cone at the origin to obtain a resolution $f\colon \tilde{X} \to X$ with exceptional locus $E \cong C$ and write

$$K_{\tilde{X}} \sim_{\mathbb{Q}} f^*K_X + aE$$

By construction, $E \subseteq \mathbb{P}^2$ lies in the projective plane with which we replaced the origin of $\mathbb{A}^3$. The normal bundle to this projective plane as a subvariety of the blow-up of the affine space is by construction the tautological line bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence the normal bundle to E inside $\tilde{X}$ is the restriction of $\mathcal{O}_{\mathbb{P}^2}(-1)$, which on $E \cong C \subseteq \mathbb{P}^2$ corresponds to $-L$ for L a line in the plane. Applying [Har77, Example V.1.4.1] we have $E^2 = -L.C = -2$, because a conic and a line meet in 2 points counted with multiplicity. Hence $K_{\tilde{X}}.E = aE^2 = -2a$.

On the other hand, by the adjunction formula we have $\deg K_E = (K_{\tilde{X}} + E).E = -2a - 2$. By Riemann-Roch, this degree must be $2g(C) - 2 = -2$, so we conclude that $a = 0$. Hence X has canonical but not terminal singularities.

For an example of a singular variety with terminal singularities we need to search in higher dimensions, because surfaces with terminal singularities are smooth.

Example 2.46. Let V be the *Veronese surface* in $\mathbb{P}^5$, which is the image of the complete linear system $|\mathcal{O}(2)|$ on $\mathbb{P}^2$, explicitly given by

$$[x, y, z] \mapsto [x^2, y^2, z^2, xy, xz, yz].$$

Let $X \subseteq \mathbb{A}^6$ be the affine cone over V as in the previous examples.

Let $f\colon \tilde{X} \to X$ be the blow-up at the origin. Write $K_{\tilde{X}} \sim_{\mathbb{Q}} f^*K_X + aE$, with $E \cong V \subseteq \mathbb{P}^5$ the exceptional divisor. Let $L \subseteq E \subseteq \mathbb{P}^5$ be a line. Then $L.E = -2$ by similar arguments as in the previous example. By the adjunction formula, $\deg K_L = (K_E+L).L = K_E.L+1$, and since $L \cong \mathbb{P}^1$ we get $K_E.L = -3$. But also by the adjunction formula we have

$$K_E \sim_{\mathbb{Q}} (K_{\tilde{X}} + E)|_E \sim_{\mathbb{Q}} (f^*K_X + (a+1)E)|_E \sim_{\mathbb{Q}} (a+1)E|_E.$$

Therefore $-3 = K_E.L = (a+1)E.L = -2(a+1)$, which implies that $a = \frac{1}{2} > 0$ and thus X has terminal singularities.

Singular Fano varieties

Our goal is to generalise the semiorthogonal decompositions in corollary 2.10 and theorem 2.28 to varieties with terminal singularities. From now on we will refer to such varieties simply as singular varieties.

Definition 2.47. The *index* of a singular Fano variety is the largest rational $r > 0$ such that $-K_X \equiv rH$ for some ample divisor H.

Proposition 2.48. *Let X be a singular Fano variety. Then every line bundle $\mathcal{L}$ on X is an exceptional object on $\mathbf{D}^{\mathrm{b}}(X)$.*

Proof. Line bundles still induce autoequivalences of the bounded derived category. Indeed, the functor $(-) \otimes \mathcal{L}$ is exact, so it need not be derived and thus it does not leave the bounded derived category. And a quasi-inverse is still given by $(-) \otimes \mathcal{L}^{-1}$.

Hence it suffices to show that $\mathrm{Ext}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \cong \mathbb{C}$.

As usual we have $\mathrm{Ext}^\bullet(\mathcal{O}_X, \mathcal{O}_X) \cong H^\bullet(X, \mathcal{O}_X)$. Since X is proper over $\mathbb{C}$, we have $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. For $i > 0$ we can apply the singular Kawamata-Viehweg vanishing (see [Mat02, Theorem 5-2-7]) to deduce $H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{O}_X(K_X - K_X)) = 0$. □

The $\mathbb{C}$-linear triangulated category $\mathbf{D}^{\mathrm{b}}(X)$ is not proper anymore, so we need to be careful when discussing admissible subcategories induced by exceptional objects. More precisely, the problem may be that for an exceptional object $E \in \mathbf{D}^{\mathrm{b}}(X)$ the image of $\oplus_{i\in\mathbb{Z}}(\mathrm{Ext}^i(E, -) \otimes E[-i])$ is not contained in $\mathbf{D}^{\mathrm{b}}(X)$ anymore. In our case, the exceptional objects we are considering are line bundles, so there is no problem with that.

Corollary 2.49. *Let X be an d-dimensional singular Fano variety and let $\mathcal{L}$ be a line bundle on X. Then its thick envelope $\langle\mathcal{L}\rangle \subseteq \mathbf{D}^{\mathrm{b}}(X)$ is a right admissible subcategory.*

Proof. Since $\mathcal{L}$ is a line bundle, it is an exceptional object by the previous proposition. Moreover, the functor $\oplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(\mathcal{L}, -) \cong \oplus_{i\in\mathbb{Z}} H^i(X, (-) \otimes \mathcal{L}^{-1})$ takes values in $\mathbf{D}^{\mathrm{b}}(\mathbb{C})$ on every bounded complex $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$. The second page of the usual spectral sequence $E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet \otimes \mathcal{L}^{-1}))$ can only be non-zero on a rectangle of width $d+1$ and height given by the number of non-zero cohomology sheaves of the bounded complex $\mathcal{F}^\bullet \otimes \mathcal{L}$. Moreover, each $E_2^{p,q}$ is a finite dimensional vector space by Serre's theorem. Hence its limit $H^n(X, \mathcal{F}^\bullet \otimes \mathcal{L}^{-1})$ can only be non-zero for finitely many $n \in \mathbb{Z}$, and for those n it is a finite dimensional $\mathbb{C}$-vector space.

This implies that $\oplus_{i\in\mathbb{Z}}(\mathrm{Ext}^i(\mathcal{L}, \mathcal{F}^\bullet) \otimes \mathcal{L}[-i]) \in \mathbf{D}^{\mathrm{b}}(X)$ for all $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$. Now we can finish the proof as in the smooth case. For all $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$ we consider the evaluation morphism $\oplus_{i\in\mathbb{Z}}(\mathrm{Ext}^i(\mathcal{L}, \mathcal{F}^\bullet) \otimes \mathcal{L}[-i]) \to \mathcal{F}^\bullet$. Take the cone $\mathcal{M}^\bullet$ to complete into a distinguished triangle

$$\oplus_{i\in\mathbb{Z}}(\mathrm{Ext}^i(\mathcal{L}, \mathcal{F}^\bullet) \otimes \mathcal{L}[-1]) \to \mathcal{F}^\bullet \to \mathcal{M}^\bullet \to \oplus_{i\in\mathbb{Z}}(\mathrm{Ext}^i(\mathcal{L}, \mathcal{F}^\bullet) \otimes \mathcal{L}[-i+1]).$$

By lemma A.26 it suffices to show that $\mathcal{M}^\bullet \in \mathcal{L}^\perp$. To see this, apply the cohomological functor $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{L}, -)$ to the previous distinguished triangle. Since $\mathrm{Ext}^i(\mathcal{L}, \mathcal{F}^\bullet) \otimes$

$\mathcal{L}[-i]$ is a direct sum of copies of $\mathcal{L}[-i]$ and $\mathcal{L}$ is an exceptional object, the hom-grop from $\mathcal{L}$ to the first term of the triangle is $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{L}, \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{L}, \mathcal{F}^\bullet) \otimes \mathcal{L})$. The composition with the evaluation yields a canonical isomorphism from this hom-group to $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{L}, \mathcal{F}^\bullet)$. Using embedding theorems to pretend we are in $\mathbf{D}^{\mathrm{b}}(\mathbf{Ab})$, if we are given a morphism in the first hom-vector $e \mapsto f_e \otimes e$, then the composition with the evaluation yields another morphism $e \mapsto f_e(e)$. Given a morphism f in the second hom-vector space we may use it to define $e \mapsto f \otimes e$ in the first hom-vector space, and this canonical correspondence being an isomorphism follows now from $\mathcal{L}$ being exceptional and thus both hom-vector spaces having the same dimension.

By the same argument, the shifted evaluation becomes an isomorphism after applying $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{L}, -)$. Therefore we deduce from the long exact sequence of this cohomological functor that $\mathrm{Ext}^i(\mathcal{L}, \mathcal{M}^\bullet) = 0$ for all $i \in \mathbb{Z}$, hence $\mathcal{M}^\bullet \in \mathcal{L}^\perp$. □

Hence, we can deduce the analogous of corollary 2.10 in the singular setting:

Proposition 2.50 (cf. [BBF16, Proposition 6.8])**.** *Let X be a singular Fano variety of index r and let H be an ample divisor such that $-K_X \equiv rH$. Then we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}, \mathcal{O}_X, \mathcal{O}_X(H), \ldots, \mathcal{O}_X(\lceil r-1 \rceil H) \rangle.$$

Proof. We have seen already that every line bundle is exceptional. Next we need to show that $\mathrm{Ext}^\bullet(\mathcal{O}_X(iH), \mathcal{O}_X(jH)) = 0$ for all $0 \leqslant j < i \leqslant \lceil r-1 \rceil$. We apply again the singular Kawamata-Viehweg vanishing theorem (see [Mat02, Theorem 5-2-7]). As in the smooth case we have $\mathcal{O}_X((j-i)H) = \mathcal{O}_X(K_X + (r-(i-j))H)$, and $(r-(i-j))H$ is still ample for the given range (cf. figure 2.1).

Since each orthogonal complement is a full subcategory of $\mathbf{D}^{\mathrm{b}}(X)$, we can iterate the argument we used to show right admissibility. In particular we obtain the claimed semiorthogonal decomposition by lemma A.26 and lemma 1.43 *a*). □

Singular Fano fibrations

Let us now do the case of singular Fano fibrations.

Definition 2.51. A *singular Fano fibration* is a morphism $f\colon X \to Y$ of singular varieties such that

i) the canonical morphism $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism,

ii) $-K_X$ is f-ample, and

iii) $\dim X > \dim Y$.

The *relative index* of X over Y is defined as the largest rational $r > 0$ such taht $-K_X \equiv_f rH$ for some f-ample Cartier divisor H.

As in the case of singular Fano varieties, we cannot directly apply proposition 1.55, because in the proof we have used the smoothness assumption. But in our situation most of the functors are not derived, hence remain in the bounded derived categories. Let us then directly state the theorem and go through the proof carefully.

Theorem 2.52. *Let $f\colon X \to Y$ be a flat singular Fano fibration of relative index r and let H be an f-ample divisor such that $-K_X \equiv_f H$. Let $\mathcal{L}$ denote $\mathcal{O}_X(H)$. Then we have a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_f, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes 2}, \dots, \mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^{\otimes \lceil r-1 \rceil} \rangle$$

where $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}$ denotes the essential image of $\mathcal{F}^\bullet \mapsto f^*(\mathcal{F}^\bullet) \otimes \mathcal{L}$.

Proof. Since $\mathcal{L}$ is a line bundle, the derived tensor product $\mathcal{L} \otimes^{\mathbb{L}} (-)$ is just the usual one. The same goes for the internal hom $\mathbb{R}\mathcal{H}om(\mathcal{L}, -)$, so we have the usual adjunction $\mathcal{L} \otimes (-) \dashv \mathcal{H}om(\mathcal{L}, -)$.

On the other hand, since f is flat we do not have to derive f^*. Moreover, by properness of f the derived functor $\mathbb{R}f_*$ stays within $\mathbf{D}^{\mathrm{b}}(X)$ (Serre's theorem plus the usual spectral sequence argument). Hence we have an adjunction $f^* \dashv \mathbb{R}f_*$ between the bounded derived categories $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$.

We check then that the functor $\mathcal{F}^\bullet \mapsto f^*(\mathcal{F}^\bullet) \otimes \mathcal{L}$ is right admissible. By composing the adjoint functors we obtain an adjunction

$$\mathcal{L} \otimes f^*(-) \dashv \mathbb{R}f_*\mathcal{H}om(\mathcal{L}, -)$$

between the bounded derived categories. We need to check that the unit of this adjunction is a natural isomorphism. Since $\mathcal{L}$ is a line bundle we have functorial isomorphisms

$$\mathbb{R}f_*(\mathcal{H}om(\mathcal{L}, \mathcal{L} \otimes f^*(\mathcal{F}^\bullet))) \cong \mathbb{R}f_*(\mathcal{H}om(\mathcal{L}, \mathcal{L}) \otimes f^*(\mathcal{F}^\bullet)) \cong \mathbb{R}f_*(\mathcal{O}_X \otimes f^*(\mathcal{F}^\bullet)).$$

By the projection formula in [Stacks, Tag 08EU], the previous expression is functorially isomorphic to $\mathbb{R}f_*\mathcal{O}_X \otimes^{\mathbb{L}} \mathcal{F}^\bullet$, but since f has connected fibres this is just $\mathcal{O}_Y \otimes \mathcal{F}^\bullet \cong \mathcal{F}^\bullet$.

This shows that each $\mathbf{D}^{\mathrm{b}}(Y) \boxtimes \mathcal{L}^i$ is a right admissible subcategory. It remains to check the semiorthogonalities. So let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(Y)$ and let $0 \leqslant j < i \leqslant \lceil r-1 \rceil$. We need to show that

$$\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(f^*\mathcal{F}^\bullet, f^*\mathcal{G}^\bullet \otimes \mathcal{L}^{\otimes j-i}[p]) = 0$$

for all $p \in \mathbb{Z}$. By the adjunction $f^* \dashv \mathbb{R}f_*$ and the projection formula in [Stacks, Tag 08EU], the previous expression is isomorphic to $\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes^{\mathbb{L}} \mathbb{R}f_*(\mathcal{L}^{\otimes j-i})[p])$ So it suffices to show that $\mathbb{R}f_*(\mathcal{L}^{\otimes j-i}) = 0$. That is, we need to show $\mathrm{R}^p f_*(\mathcal{L}^{\otimes j-i}) = 0$ for all $p \in \mathbb{Z}$.

The case $p = 0$ is again a consequence of the theorem on formal functions and flatness of f. By Grauert's direct image theorem $f_*\mathcal{L}^{\otimes j-i}$ is a vector bundle on Y with fibres $H^0(X_y, \mathcal{L}_y^{\otimes j-i}) = 0$, hence $f_*\mathcal{L}^{\otimes j-i} = 0$.

For $p > 0$ we use a relative version of the singular Kawamata-Viehweg vanishing (see [Mat02, Theorem 5-2-8]). We need to check that we can apply the theorem in the desired range of values of $j-i$. Writing $\mathcal{O}_X(\mathcal{O}_X((j-i)H))$ as $\mathcal{O}_X(-rH + (r-(i-j))H)$ shows that this is indeed the case by definition of the relative index r. $\square$

Appendix A

Triangulated categories

A.1 Basic definitions and results

Definition A.1. A *triangulated category* is an additive category $\mathbf{T}$ with an additive automorphism $\Sigma\colon \mathbf{T} \to \mathbf{T}$ called the *suspension functor* and a collection of triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

called *distinguished triangles* such that the follwoing axioms hold:

(TR1) $(\mathrm{id}_X, 0, 0)$ is a distinguished triangle on $(X, X, 0)$ for all $X \in \mathbf{T}$, every morphism $X \xrightarrow{f} Y$ fits into a distinguished triangle (f, g, h) on (X, Y, C) and every triangle isomorphic to a distinguished triangle is itself distinguished. We call C the cone of f.

(TR2) (f, g, h) is a distinguished triangle on (X, Y, Z) if and only if $(g, h, -\Sigma f)$ is a distinguished triangle on $(Y, Z, \Sigma X)$[1].

(TR3) If the diagram below has distinguished rows, then we can find such a dashed morphism[2]:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "\exists h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

(TR4) *Octahedral axiom:* Given a diagram whose rows and column are distinguished triangles as below, we can find dashed arrows making the whole diagram commute and making the new vertical arrow a distinguished triangle as well.

[1]See remark A.4 to see why this minus sign appears.

[2]This morphism is not unique. This problem is often referred to as non functoriality of the cone. One solution is to consider dg enhancements.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & W \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "g\circ f"] & Z \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma X \\
& V \arrow[r, equal] \arrow[d] & V \arrow[d] & \\
& \Sigma Y \arrow[r, "\Sigma g"] & \Sigma W &
\end{tikzcd}$$

Moreover, the composition $C \to V \to \Sigma Y$ is the same as the composition $C \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y$. A good mnemonic to remember this last condition, which also explains the name of the axiom, consists of collapsing the identities and shifting some morphisms to obtain the equivalent diagram

$$\begin{tikzcd}
& & \Sigma^{-1}V \arrow[dl] \arrow[dr] & & \\
& Y \arrow[rr] \arrow[dd] & & W \arrow[dd, dashed] \arrow[dr] & \\
X \arrow[ur] \arrow[dr] & & & & \Sigma X \\
& Z \arrow[rr] \arrow[dr] & & C \arrow[ur] \arrow[dl, dashed] & \\
& & V & &
\end{tikzcd}$$

Now we can glue X with ΣX and $\Sigma^{-1}V$ with V respectively to obtain a 3 dimensional octahedron diagram, whose commutativity already encodes all commutativities stated in the axiom.

A functor $F\colon \mathbf{T} \to \mathbf{S}$ between triangulated categories is called *triangulated* if it commutes with the shift functors and it preserves distinguished triangles. This means more precisely that there is a natural isomorphism $\alpha^F\colon F \circ \Sigma_{\mathbf{T}} \to \Sigma_{\mathbf{S}} \circ F$ and that $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\alpha^F_X \circ Fh} \Sigma_{\mathbf{S}} FX$ is distinguished whenever $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_{\mathbf{T}} X$ is.

A natural transformation $\eta\colon F \to G$ between triangulated functors is called *triangulated* if it is compatible with the natural isomorphisms α^F and α^G, meaning that $\Sigma_{\mathbf{S}}(\eta_X) \circ \alpha^F_X = \alpha^G_X \circ \eta_{\Sigma_{\mathbf{T}} X}$.

Remark A.2. The axiom (TR3) and the converse direction of (TR2) are redundant. See [May01, Section 2].

Being an abelian category is a property, whereas being triangulated is an extra structure that has to be specified on the category. But not every category admits a triangulated structure. To begin with, a triangulated category is additive by definition, and being additive is already an intrinsic property that a category may or may not have[3]. For example, an additive category must have a zero object, which already rules out many trivial (e.g. $\mathbf{T} = \varnothing$) and non trivial (e.g. $\mathbf{T} = \mathbf{Ring}$) examples.

But more interestingly, on a triangulated category every monomorphism is a section and every epimorphism is a retraction (see corollary A.15). In particular, if an abelian category admits a triangulated structure, then every short exact sequence has to split.

Proposition A.3 ([GM03, Theorem IV.1.9])**.** *The homotopy category* $\mathbf{K}(\mathbf{A})$ *is a triangulated category with suspension functor given by the shift* $[1]\colon \mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{A})$ *and with distinguished triangles those isomorphic to a triangle of the form*

$$A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$$

Proof. Using [Fre64, Theorem 7.34] we may work in $\mathbf{A}$ by picking elements[4] as if we were in $\mathbf{Ab}$.

The homotopy category $\mathbf{K}(\mathbf{A})$ is still an additive category, because we have just replaced every hom-group by the quotient group of cochain morphisms modulo null homotopic morphisms.

Let $A^\bullet \in \mathbf{K}(\mathbf{A})$. Then the cone of id_A is contractible, with a null homotopy of the identity given by $(a_1, a_2) \mapsto (a_2, 0)$. The rest of (TR1) follows by choice of the class of distinguished triangles.

To show the only if direction of (TR2) we may assume that the triangle is strict, i.e. of the form $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$. Consider the following diagram:

$$\begin{tikzcd}
B^\bullet \arrow[r, "v"] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, equal] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, dashed, "\alpha"] & B[1]^\bullet \arrow[d, equal] \\
B^\bullet \arrow[r, "v"] & C(f)^\bullet \arrow[r] & C(v)^\bullet \arrow[r] & B[1]^\bullet
\end{tikzcd}$$

The map α is given by $a \mapsto (-fa, a, 0)$. This is a morhpism of complexes, because the differential of $C(v)^\bullet$ sends $(-fa, a, 0) \mapsto (dfa, -da, -fa+fa) = (fda, -da, 0) = \alpha(-da)$. This is a chain homotopy equivalence with inverse β given by $(b_1, a, b_2) \mapsto a$. Indeed, one of the two compositions is the identity and the other one is $(b_1, a, b_2) \mapsto (-fa, a, 0)$, and $(b_1, a, b_2) \mapsto (b_1+fa, 0, b_2)$ is null homotopic via $(b_1, a, b_2) \mapsto (b_2, 0, 0)$. It remains to show that $(\mathrm{id}, \mathrm{id}, \alpha)$ is a morphism of triangles. The right square commutes already in $\mathbf{Ch}(\mathbf{A})$. The middle square commutes up to homotopy, because $(a, b) \mapsto (0, a, b) - (-fa, a, 0) = (fa, 0, b)$ is null homotopic via $(a, b) \mapsto (b, 0, 0)$. So the only if direction of (TR2) holds.

[3]Indeed, a category is called *semiadditive* if it has all finite products and coproducts and the canonical morphisms from coproducts to products are isomorphisms. This is either true or false for any given category, so it is a property and not an extra structure. Every semiadditive category admits a natural enrichment over the category of commutative monoids, and the category is additive precisely when this enrichment is in fact over the category of abelian groups.

[4]One could also avoid these embedding theorems using matrices to write down morphisms or using generalised elements as in [Mac78, Theorem VIII.4.3].

Axiom (TR3) is redundant, but it is easy to verify it in $\mathbf{K}(\mathbf{A})$ using the explicit definition of the mapping cone.

Let us finally show that (TR4) holds. We may assume again that all triangles involved are strictly distinguished. We have then a commutative diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r] \arrow[d, "g"] & C(f)^\bullet \arrow[r] \arrow[d, dashed, "u"] & A[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r, "gf"] & C^\bullet \arrow[r] \arrow[d] & C(gf)^\bullet \arrow[r] \arrow[d, dashed, "v"] & A[1]^\bullet \\
 & C(g)^\bullet \arrow[r, equal] \arrow[d] & C(g)^\bullet \arrow[d] & \\
 & B[1]^\bullet \arrow[r] & C(f)[1]^\bullet &
\end{tikzcd}$$

Define u by $(a, b) \mapsto (a, gb)$ and v by $(a, c) \mapsto (fa, c)$. These are chain morphisms because $(-da, gfa + dgb) = (-da, gfa + db)$ and because $(-fda, gfa + dc) = (-dfa, gfa + dc)$. They make the necessary diagram commute in $\mathbf{Ch}(\mathbf{A})$. So we need to check that u and v form a distinguished triangle with the morphism $C(g)^\bullet \to C(f)[1]^\bullet$, which is given by $(b, c) \mapsto b \mapsto (0, b)$. For this consider the diagram

$$\begin{tikzcd}
C(f)^\bullet \arrow[r, "u"] \arrow[d, equal] & C(gf)^\bullet \arrow[r, "v"] \arrow[d, equal] & C(g)^\bullet \arrow[r] \arrow[d, dashed, "\gamma"] & C(f)[1]^\bullet \arrow[d, equal] \\
C(f)^\bullet \arrow[r, "u"] & C(gf)^\bullet \arrow[r] & C(u)^\bullet \arrow[r] & C(f)[1]^\bullet
\end{tikzcd}$$

The morphism γ is given by the natural inclusion $(b, c) \mapsto (0, b, 0, c)$. The middle square commutes in $\mathbf{K}(\mathbf{A})$, because $(a, c) \mapsto (0, fa, 0, c) - (0, 0, a, c) = (0, fa, -a, 0)$ is null homotopic via $(a, c) \mapsto (-a, 0, 0, 0)$. The right square commutes in $\mathbf{Ch}(\mathbf{A})$. A chain homotopy inverse δ to γ is given by $(a_1, b, a_2, c) \mapsto (b + fa_2, c)$. This is a chain morphism, because $-fa_1 - db + f(a_1 - da_2) = -db - dfa_2$. The composition $\delta\gamma$ is the identity on $C(g)^\bullet$. The morphism $\mathrm{id}_{C(u)} - \gamma\delta$ sends $(a_1, b, a_2, c) \mapsto (a_1, -fa_2, a_2, 0)$ and is null homotopic via $(a_1, b, a_2, c) \mapsto (a_2, 0, 0, 0)$. Hence γ is an isomorphism in $\mathbf{K}(\mathbf{A})$ and the upper row in the previous diagram is distinguished.

Finally, the compositions $C(gf)^\bullet \to C(g)^\bullet \to B[1]^\bullet$ and $C(gf)^\bullet \to A[1]^\bullet \to B[1]^\bullet$ agree by definition of the morphisms involved. □

Remark A.4. The commutative diagram

$$\begin{tikzcd}
A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, "\alpha"] & B[1]^\bullet \arrow[d, equal] \\
C(g)^\bullet \arrow[r] & B[1]^\bullet
\end{tikzcd}$$

in the previous proof shows why we need the minus sign in the axiom (TR2).

Cohomological functors

The notion of distinguished triangle replaces the notion of short exact sequence of complexes. A key property of the latter is the long exact cohomology sequence. This feature is now encoded in the notion of cohomological functor.

Definition A.5. Let $\mathbf{T}$ be a triangulated category and let $\mathbf{A}$ be an abelian category. An additive functor $H\colon \mathbf{T} \to \mathbf{A}$ is called *cohomological* if for every distinguished triangle $X \to Y \to Z \to \Sigma X$ the sequence $H(X) \to H(Y) \to H(Z)$ is exact in $\mathbf{A}$.

The *long exact sequence* associated to the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is then obtained by applying (TR2) and the cohomological functor repeatedly (and modifying some signs):

$$\cdots \to H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}h)} H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \to \cdots$$

Lemma A.6 ([Nee01, Remark 1.1.3])**.** *The composition of two consecutive morphisms on a distinguished triangle is zero.*

Proof. Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle. By (TR2) it suffices to show that the composition $X \to Y \to Z$ is zero. To do this, consider the following diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, equal] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

By (TR3) we can find such a dashed arrow making the diagram commute. But then the composition $X \to Y \to Z$ factors through zero, hence must be zero itself. □

Proposition A.7 ([Nee01, Lemma 1.1.10])**.** *Let* $\mathbf{T}$ *be a triangulated category and let* $W \in \mathbf{T}$*. Then the functor* $\mathrm{Hom}_{\mathbf{T}}(W, -)\colon \mathbf{T} \to \mathbf{Ab}$ *is cohomological. Similarly, the contravariant functor* $\mathrm{Hom}_{\mathbf{T}}(-, W)$ *from* $\mathbf{T}$ *to* $\mathbf{Ab}$ *is cohomological.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle in $\mathbf{T}$. We want to show that the sequence

$$\mathrm{Hom}_{\mathbf{T}}(W, X) \to \mathrm{Hom}_{\mathbf{T}}(W, Y) \to \mathrm{Hom}_{\mathbf{T}}(W, Z)$$

is exact in $\mathbf{Ab}$. The composition is zero because $\mathrm{Hom}_{\mathbf{T}}(W, -)$ is additive by definition of direct sums and $gf = 0$ by the previous lemma. Let $\alpha\colon W \to Y$ such that $g\alpha = 0$. By (TR2) we have a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \Sigma W \arrow[r, "-\mathrm{id}"] \arrow[d, dashed, "\Sigma\beta"] & \Sigma W \arrow[d, "\Sigma\alpha"] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

By (TR3) and fulness of Σ, we get a morphism $\beta\colon W \to X$ such that $\Sigma f \circ \Sigma\beta = \Sigma\alpha$. Hence $f \circ \beta = \alpha$ by faithfulness of Σ and we are done. □

Proposition A.8 ([GM03, Section IV.1.6]). *The zero cohomology functor $H^0 \colon \mathbf{K}(\mathbf{A}) \to \mathbf{A}$ is a cohomological functor.*

Proof. Note first that H^0 is well defined on $\mathbf{K}(\mathbf{A})$, because chain homotopic morphisms induce the same morphisms in cohomology. Since finite direct sums are exact, they commute with cohomology. So H^0 is an additive functor.

To see that this functor is cohomological, it suffices to show that we get an exact sequence for every strictly distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet$. Consider the mapping cone short exact sequence

$$0 \to B^\bullet \to C(f)^\bullet \to A[1]^\bullet \to 0$$

in $\mathbf{Ch}(\mathbf{A})$ and take its long exact sequence in cohomology

$$\cdots \to H^{-1}(A[1]^\bullet) \to H^0(B^\bullet) \to H^0(C(f)^\bullet) \to \cdots$$

Since $H^{-1}(A[1]^\bullet) = H^0(A^\bullet)$ and the coboundary morphism of the mapping cone sequence is the same as the morphism induced in cohomology by f, we get the desired claim. □

Elementary results on distinguished triangles

We begin with the triangulated 5 lemma, from which we will deduce most of the rest of elementary results:

Proposition A.9 ([Nee01, Proposition 1.1.20]). *In a morphism between distinguished triangles, if two out of the first three morphisms are isomorphisms, so is the third one.*

Proof. This follows directly from the Yoneda lemma using the cohomological functors $\mathrm{Hom}_{\mathbf{T}}(X, -)$ and the usual 5 lemma for abelian groups. □

Remark A.10. In particular, the cone of a morphism as defined in (TR1) is well defined up to isomorphism. But as remarked in (TR3), this isomorphism is usually not unique.

Proposition A.11 ([Nee01, Corollary 1.2.6]). *A morphism $f \colon X \to Y$ is an isomorphism if and only if its cone is isomorphic to zero.*

Proof. If f is an isomorphism, using the trivial distinguished triangle on X and the 5 lemma we get $C \cong 0$. Conversely, if $X \xrightarrow{f} Y \to C \to \Sigma X$ is distinguished with $0 \cong C$, then we can apply the 5 lemma to conclude that f is an isomorphism too. Here we use that the any morphism $0 \to C$ is necessarily an isomorphism, since there is only one such morphism and one such isomorphism. □

Proposition A.12 ([Nee01, Proposition 1.2.1] and [Nee01, Proposition 1.2.3]). *The direct sum of two triangles is distinguished if and only if each summand is a distinguished triangle.*

Proof. Let $X_1 \to Y_1 \to Z_1 \to \Sigma X_1$ and $X_2 \to Y_2 \to Z_2 \to \Sigma X_2$ be two distinguished triangles. Since Σ is an additive functor, $\Sigma X_1 \oplus \Sigma X_2 \cong \Sigma(X_1 \oplus X_2)$. Complete the map $X_1 \oplus X_2 \to Y_1 \oplus Y_2$ to an distinguished triangle $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to C \to \Sigma(X_1 \oplus X_2)$. By the universal product of coproducts and by (TR3) we get morphisms

$$\begin{tikzcd}
X_1 \oplus X_2 \arrow[r] \arrow[d] & Y_1 \oplus Y_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\exists h_i"] & \Sigma(X_1 \oplus X_2) \arrow[d] \\
X_i \arrow[r] & Y_i \arrow[r] & Z_i \arrow[r] & \Sigma X_i
\end{tikzcd}$$

By the universal property of products we get morphisms

$$\begin{tikzcd}
X_1 \oplus X_2 \arrow[r] \arrow[d, equal] & Y_1 \oplus Y_2 \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "h_1 \times h_2"] & \Sigma(X_1 \oplus X_2) \arrow[d, equal] \\
X_1 \oplus X_2 \arrow[r] & Y_1 \oplus Y_2 \arrow[r] & Z_1 \oplus Z_2 \arrow[r] & \Sigma(X_1 \oplus X_2)
\end{tikzcd}$$

By the 5 lemma the two triangles are isomorphic and thus by (TR1) the bottom row is an distinguished triangle.

Conversely, suppose that the bottom row is an distinguished triangle. By symmetry it suffices to show that $X_1 \to Y_1 \to Z_1 \to \Sigma X_1$ is distinguished. Complete $X_1 \to Y_1$ to an distinguished triangle $X_1 \to Y_1 \to C \to \Sigma X_1$. Use the inclusions and projections and use (TR3) to get a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] \arrow[dd, bend right, equal] & Y_1 \arrow[r] \arrow[d] \arrow[dd, bend right, equal] & C \arrow[r] \arrow[d, "\exists h"] & \Sigma X_1 \arrow[d] \arrow[dd, bend right, equal] \\
X_1 \oplus X_2 \arrow[r] \arrow[d] & Y_1 \oplus Y_2 \arrow[r] \arrow[d] & Z_1 \oplus Z_2 \arrow[r] \arrow[d] & \Sigma(X_1 \oplus X_2) \arrow[d] \\
X_1 \arrow[r] & Y_1 \arrow[r] & Z_1 \arrow[r] & \Sigma X_1
\end{tikzcd}$$

The vertical compositions are the identities by definition of biproducts. We cannot directly apply the 5 lemma, because the bottom row is not yet known to be distinguished. But in any case, the bottom row induces a long exact sequence when we apply $\mathrm{Hom}_{\mathbf{T}}(T, -)$, because this functor is additive and the direct sum of two sequences is exact if and only if the summands are. Hence we can repeat the proof of the 5 lemma and conclude by (TR1) that the bottom triangle is distinguished. $\square$

We come now to another triangulated version of a popular diagram lemma, namely the 3×3 lemma:

Proposition A.13 ([Wei94, Exercise 10.2.6]). *Consider a comuutative diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & \Sigma X \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] & \Sigma X' \\
X'' \arrow[d] & Y'' \arrow[d] & & \\
\Sigma X \arrow[r] & \Sigma Y & &
\end{tikzcd}$$

in which all rows and columns are distinguished triangles. Then we can find an object Z'' and dashed arrows

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, dashed] & \Sigma X' \arrow[d] \\
X'' \arrow[r, dashed] \arrow[d] & Y'' \arrow[r, dashed] \arrow[d] & Z'' \arrow[r, dashed] \arrow[d, dashed] & \Sigma X'' \arrow[d, "-1"] \\
\Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r, "-1"] & \Sigma^2 X
\end{tikzcd}$$

such that:

i) The diagram commutes except for the bottom right square, which commutes up to a sign.

ii) All rows and columns are distinguished triangles.

Proof. Note that the bottom and rightmost rows are exact, because $(1, -1, 1)$ yields an isomorphism to the respective fully rotated distinguished triangles.

We use (TR3) and (TR1) to obtain the horizontal dashed distinguished triangle. By applying (TR4) to a composition $g \circ f$ we mean applying it to a diagram whose upper part is

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "g\circ f"] & Y' \arrow[r] & C(g \circ f) \arrow[r] & \Sigma X
\end{tikzcd}$$

Now apply (TR1) and (TR4) to the composition $X \to Y \to Y'$ to obtain an object $C \in \mathbf{T}$ together with morphisms $X'' \to C$ and $C \to Y''$. Apply finally (TR4) to this composition to obtain a distinguished triangle $Z' \to Z'' \to \Sigma Z \to \Sigma Z'$ as we wanted.

For the claimed commutativity, note first that the commutativity of all rectangles larger than 1×1 is trivial by lemma A.6. So we only need to check each 1×1 square. Let us enumerate all such squares in the diagram as the entries of a matrix. Commutativity of $(1,1)$ holds by assumption. Commutativity of $(2,1)$ and $(3,1)$ follows from (TR3). For $(1,2)$, by the (TR4) diagram corresponding to the composition $X \to X' \to Y'$, the morphism $Y' \to Z'$ is the same as the composition $Y' \to C \to Z'$. But by the (TR4) diagram corresponding to $X \to Y \to Y'$, the composition $Y \to Y' \to C$ is the same as $Y \to Z \to C$. Hence $(1,2)$ commutes. The commutativity of the square $(2,2)$ follows from a similar argument. The composition $Y'' \to \Sigma Y \to \Sigma Z$ is by construction the morphism $Y'' \to \Sigma Z$, and this is the same by one of the (TR4) diagrams as $Y'' \to Z'' \to \Sigma Z$. Hence $(3,2)$ commutes. The commutativity of $(1,3)$ is a bit trickier, because we need the extra commutativity statement in (TR4) applied to $X \to X' \to Y'$ which is not explicit in the diagram. This tells us that $C \to \Sigma X \to \Sigma X'$ is the same as $C \to \mathbb{Z}' \to \Sigma X'$. And $Z \to \Sigma X$ is the same as $Z \to C \to \Sigma X$, so $(1,3)$ commutes as well. The morphism $Z' \to \Sigma X''$, which by construction is the composition $Z' \to \Sigma X' \to \Sigma X''$, is the same as the composition $Z' \to Z'' \to \Sigma X''$ by the (TR4) diagram of the composition $X'' \to C \to Y''$. Hence $(2,3)$ also commutes. Finally, let us check that $Z'' \to \Sigma X'' \to \Sigma^2 X$ is the opposite of $Z'' \to \Sigma Z \to \Sigma^2 X$. The morphism $\Sigma Z \to \Sigma^2 X$ without the -1 sign is the same as the composition $\Sigma Z \to \Sigma C \to \Sigma^2 X$ by the (TR4) diagram of $X \to Y \to Y'$. And the composition $Z'' \to \Sigma Z \to \Sigma C$ is the same as the composition $\Sigma Z'' \to \Sigma X'' \to \Sigma C$ by the non explicit commutativity in (TR4) applied to $X'' \to C \to Y''$. This finishes the proof. □

We say that a triangle $X \to Y \to Z \to \Sigma X$ *splits* if it is isomorphic to the triangle $X \to X \oplus Z \to Z \xrightarrow{0} \Sigma X$.

Proposition A.14 ([Nee01, Corollary 1.2.7])**.** *If $X \to Y \to Z \xrightarrow{0} \Sigma X$ is distinguished, then it splits.*

Proof. The triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is distinguished, so by the previous proposition the triangle $X \to X \oplus Z \to Z \xrightarrow{0} \Sigma X$ is distinguished as well. By (TR3) and the 5 lemma we find the desired splitting. □

From the last proposition we can deduce the announced property of triangulated categories:

Corollary A.15 (Verdier)**.** *Every monomorphism in a triangulated category is a section, and every epimorphism a retraction.*

Proof. Since the triangulated structure on $\mathbf{T}$ induces naturally a triangulated structure on $\mathbf{T}^{opp}$, it suffices to show the statement for monomorphisms. So let $f\colon X \to Y$ be a monomorphism. Complete it to a distinguished triangle $C[-1] \to X \xrightarrow{f} Y \to C$ with (TR1) and (TR2). The composition $C[-1] \to X \to Y$ is zero, and since f is a monomorphism, the morphism $C[-1] \to X$ is zero. Hence the triangle splits, we get an isomorphism $\alpha\colon Y \to X \oplus C$ and we can write $\mathrm{id}_X = p_X \alpha f$, where $p_X\colon X \oplus C \to X$ is the projection from the biproduct. Hence f is a section. □

A.2 Triangulated functors

In this section we will study some important particular cases of triangulated functors. As mentioned at the beginning of the appendix, these are functors that preserve the triangulated structure. We can think of triangulated categories as objects in a category **Trg** with triangulated functors as morphisms. As in other familiar categories such as **Grp**, we will define special kinds of subobjects and quotients.

An important consequence of this section is that adjoints of triangulated functors are always triangulated. In particular, triangulated equivalences always have at least one triangulated quasi-inverse.

Triangulated subcategories

Definition A.16. Let $\mathbf{T}$ be a triangulated category. A (necessarily non-empty) strictly full subcategory $\mathbf{S} \subseteq \mathbf{T}$ is called a *triangulated subcategory* if it admits a triangulated structure such that the inclusion functor is triangulated. This is equivalent to being closed under taking suspensions and cones. We denote this by $\mathbf{S} \leqslant \mathbf{T}$.

If $\mathbf{S} \leqslant \mathbf{T}$ is also closed under taking direct summands, we call it a *thick* triangulated subcategory, denoted $\mathbf{S} \trianglelefteq \mathbf{T}$.

Definition A.17. Let $\mathbf{T}$ be a triangulated category.

i) Let $H\colon \mathbf{T} \to \mathbf{A}$ be a cohomological functor. Define its *kernel* to be the full subcategory given by

$$\operatorname{Ker} H = \{X \in \mathbf{T} \mid F(\Sigma^m X) = 0 \text{ for all } m \in \mathbb{Z}\}.$$

ii) Let $F\colon \mathbf{T} \to \mathbf{S}$ be a triangulated functor. Define its *kernel* to be the full subcategory given by

$$\operatorname{Ker} F = \{X \in \mathbf{T} \mid FX = 0\}.$$

Lemma A.18 ([Nee01, Remark 2.1.7])**.** *Let* $\mathbf{T}$ *be a triangulated category.*

i) Let $H\colon \mathbf{T} \to \mathbf{A}$ *be a cohomological functor. Then* $\operatorname{Ker} H \trianglelefteq \mathbf{T}$.

ii) Let $F\colon \mathbf{T} \to \mathbf{S}$ *be a triangulated functor. Then* $\operatorname{Ker} F \trianglelefteq \mathbf{T}$.

Proof. Both cases are similar, so let us see for example that $\operatorname{Ker} F \trianglelefteq \mathbf{T}$.

By definition it is a full subcategory. Suspension invariance follows from F being triangulated. Closedness under cones also follows from F being triangulated, because if two out of the first three objects in a distinguished triangle are zero, so is the third one.

It remains to show closedness under direct summands. If $F(X \oplus Y) = 0$, then $F(X) \oplus F(Y) \cong 0$ by additivity of F. So the respective identities of $F(X)$ and $F(Y)$ factor through 0, and thus $F(X) = F(Y) = 0$. □

If X, Y are objects in a triangulated category $\mathbf{T}$, we will denote $\operatorname{Ext}^m(X, Y) = \operatorname{Hom}_{\mathbf{T}}(X, \Sigma^m Y)$ (cf. example 1.9).

Definition A.19. Let $\mathbf{T}$ be a triangulated category and let $\mathbf{E} \subseteq \mathrm{Ob}(\mathbf{T})$ be a collection of objects in $\mathbf{T}$.

i) The *right orthogonal complement* of $\mathbf{E}$, or simply the *orthogonal complement* of $\mathbf{E}$, is the full subcategory of $\mathbf{T}$ given by

$$\mathbf{E}^{\perp} = \{X \in \mathbf{T} \mid \mathrm{Ext}^{\bullet}(E, X) = 0 \text{ for all } E \in \mathbf{E}\}.$$

ii) The *left orthogonal complement* of $\mathbf{E}$ is the full subcategory of $\mathbf{T}$ given by

$$^{\perp}\mathbf{E} = \{X \in \mathbf{T} \mid \mathrm{Ext}^{\bullet}(X, E) = 0 \text{ for all } E \in \mathbf{E}\}.$$

Lemma A.20 ([Kuz11, Section 2.2])**.** *Let $\mathbf{T}$ be a triangulated category and $\mathbf{E} \subseteq \mathrm{Ob}(\mathbf{T})$ be a collection of objects in $\mathbf{T}$. Then $\mathbf{E}^{\perp} \trianglelefteq \mathbf{T}$ and $^{\perp}\mathbf{E} \trianglelefteq \mathbf{T}$.*

Proof. Again both cases are similar, so let us see for example that $\mathbf{E}^{\perp} \trianglelefteq \mathbf{T}$.

If $\mathbf{E} = \{E\}$ is a single object, then $\mathbf{E}^{\perp} = \mathrm{Ker}(\mathrm{Hom}_{\mathbf{T}}(E, -)) \trianglelefteq \mathbf{T}$.

Let us do the general case. By definition $\mathbf{E}^{\perp}$ is a full subcategory. It is suspension invariant also by definition. Let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle with $X, Y \in \mathbf{E}^{\perp}$. Let $E \in \mathbf{E}$. Then $\mathrm{Hom}_{\mathbf{T}}(E, Z[m]) = 0$ for all $m \in \mathbb{Z}$ by the long exact sequence of the cohomological functor $\mathrm{Hom}_{\mathbf{T}}(E, -)$. So $Z \in \mathbf{E}^{\perp}$.

Finally, closedness under direct summands follows from $\mathrm{Hom}_{\mathbf{T}}(E, -)$ being additive and thus preserving direct sums for all $E \in \mathbf{E}$. □

Triangulated adjunctions and equivalences

Proposition A.21 ([BK90, Proposition 1.4])**.** *Let $F\colon \mathbf{T} \to \mathbf{S}$ is a triangulated functor and $F \dashv H$, then $H\colon \mathbf{S} \to \mathbf{T}$ is triangulated as well. And similarly, if $G \dashv F$, then G is also triangulated.*

Proof. Let us show it for H for example. We will follow the proof given in [Huy06, Proposition 1.41].

By Yoneda, H commutes with the shift functor. Let $X \xrightarrow{f} Y \to Z \to \Sigma X$ be an exact triangle in $\mathbf{S}$. Fit $H(f)$ into an exact triangle $H(X) \to H(Y) \to C \to \Sigma H(X)$ by (TR1). Apply then the exact functor F and apply (TR3) to get a map $\chi\colon F(C) \to Z$. Apply H to the resulting morphism of triangles and use the unit of the adjunction η to get a diagram

$$\begin{tikzcd}
H(X) \arrow[r] \arrow[d] \arrow[dd, bend right, equal] & H(Y) \arrow[r] \arrow[d] \arrow[dd, bend right, equal] & C \arrow[r] \arrow[d, "\eta_C"] & \Sigma H(X) \arrow[d] \arrow[dd, bend right, equal] \\
HFH(X) \arrow[r] \arrow[d] & HFH(Y) \arrow[r] \arrow[d] & HF(C) \arrow[r] \arrow[d, "H(\chi)"] & \Sigma HFH(X) \arrow[d] \\
H(X) \arrow[r] & H(Y) \arrow[r] & H(Z) \arrow[r] & \Sigma H(X)
\end{tikzcd}$$

The two vertical composition of the first two columns are the respective identities because of the triangle identities. As in previous proofs, we do not know yet whether the last row is exact or not, so we cannot apply the 5 lemma directly. But we can apply $\mathrm{Hom}(T, -)$ for any T and use adjunction to show that the resulting sequence is exact. So we can again argue with the usual 5 lemma and the Yoneda lemma to see that the remaining vertical arrow is an isomorphism. By (TR1) the bottom row is distinguished and we are done. $\square$

This proposition shows in particular that any triangulated equivalence between triangulated categories has a triangulated quasi-inverse, because we can always improve an equivalence to an adjoint equivalence (see [Mac78, Theorem IV.4.1]).

Admissible subcategories

Admissible subcategories can be thought of as semiorthogonal components of our triangulated category (cf. lemma 1.42). We will first introduce the slightly more general notion of an admissible functor, but immediately after we will see that we can always reduce to the case of subcategories.

Definition A.22. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor between any two categories. We say that F is *right admissible* (resp. *left admissible*) if it admits a right (resp. left) adjoint such that the unit (resp. counit) of the adjunction is a natural isomorphism.

Lemma A.23 ([Kuz16, Lemma 2.3])**.** *Every right (resp. left) admissible functor is fully faithful.*

Proof. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a right admissible functor and let $F \dashv G$. For $X, Y \in \mathbf{A}$ we need to show that $F\colon \mathrm{Hom}_{\mathbf{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{B}}(FX, FY)$ is an isomorphism. This follows from the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(X,Y) \arrow[r, "\eta_Y \circ (-)"] \arrow[dr, "F"'] & \mathrm{Hom}_{\mathbf{B}}(X, GFY) \arrow[d, "\text{adj.}"] \\ & \mathrm{Hom}_{\mathbf{B}}(FX, FY) \end{tikzcd}$$

So let us show that the diagram commutes following [Huy06, Lemma 1.21].

Let $f \in \mathrm{Hom}_{\mathbf{A}}(X, Y)$. By definition of adjunction we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(Y, GFY) \arrow[d, "(-)\circ f"'] & \mathrm{Hom}_{\mathbf{B}}(FY, FY) \arrow[l, "\text{adj.}"] \arrow[d, "(-)\circ F(f)"] \\ \mathrm{Hom}_{\mathbf{A}}(X, GFY) & \mathrm{Hom}_{\mathbf{B}}(FX, FY) \arrow[l, "\text{adj.}"] \end{tikzcd}$$

By definition of the unit of the adjunction, the upper morphism sends $\mathrm{id}_{FY} \mapsto \eta_Y$. Hence $\eta_Y \circ f = \mathrm{id}_{FY} \circ F(f) = F(f)$ and we are done. $\square$

So every admissible functor defines an equivalence with its essential image. Thus we reduce to the following definition:

Definition A.24. Let $\mathbf{T}$ be a triangulated category. We say that $\mathbf{S} \leqslant \mathbf{T}$ is *right admissible* (resp. *left admissible*) if the inclusion admits a right (resp. left) adjoint, and simply *admissible* if it admits both right and left adjoints.

The unit (resp. counit) of this adjunction is automatically an isomorphism by Yoneda.

Remark A.25. By proposition A.21 these adjoints are also triangulated.

Lemma A.26 ([Bon90, Lemma 3.1])**.** *Let* $\mathbf{S} \leqslant \mathbf{T}$*. The following are equivalent:*

i) $\mathbf{S}$ *is right admissible (resp. left).*

ii) For every $X \in \mathbf{T}$ *there is an exact triangle* $Y \to X \to Z \to \Sigma Y$ *with* $Y \in \mathbf{S}$ *and* $Z \in \mathbf{S}^{\perp}$ *(resp.* $Y \in^{\perp} \mathbf{S}$ *and* $Z \in \mathbf{S}$*).*

iii) $\mathbf{S}^{\perp}$ *is left admissible (resp.* ${}^{\perp}\mathbf{S}$ *is right admissible).*

Proof. Both cases are similar, so let us show for example the case in which $\mathbf{S}$ is right admissible. The equivalences *i)* $\Leftrightarrow$ *ii)* and *i)* $\Leftrightarrow$ *iii)* are similar, so let us show the equivalence of the first two statements.

Let $F\colon \mathbf{T} \to \mathbf{S}$ be a right adjoint of the inclusion and let $X \in \mathbf{T}$. Taking the cone of the counit $\varepsilon_X\colon FX \to X$ we obtain a distinguished triangle

$$FX \to X \to C \to \Sigma FX$$

with $FX \in \mathbf{S}$. To see that $C \in \mathbf{S}^{\perp}$, let $S \in \mathbf{S}$ and apply the cohomological functor $\mathrm{Hom}_{\mathbf{T}}(S, -)$. By adjunction we have a functorial isomorphism $\mathrm{Hom}_{\mathbf{T}}(S, FX) \cong \mathrm{Hom}_{\mathbf{T}}(S, X)$. Since this isomorphism is precisely the map in the long exact sequence, we conclude $\mathrm{Hom}_{\mathbf{T}}(S, C) = 0$, hence $C \in \mathbf{S}^{\perp}$.

Suppose now that *ii)* holds. By the axiom of choice we can fix a distinguished triangle $Y \xrightarrow{g} X \to Z \to \Sigma Y$ for each $X \in \mathbf{T}$. So for an object $X \in \mathbf{T}$ we can set $FX = Y \in \mathbf{S}$. Let $f\colon X \to X'$ be a morphism in $\mathbf{T}$. Then we have a diagram of the form

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, dashed, "\exists! h"] & X \arrow[r] \arrow[d, "f"] & Z \\ Y' \arrow[r, "g'"] & X' \arrow[r] & Z' \end{tikzcd}$$

Since $\mathrm{Ext}^{\bullet}_{\mathbf{T}}(Y, Z') = 0$, the composition $f \circ g$ comes from a unique morphism $h\colon Y \to Y'$ in the long exact sequence of the cohomological functor $\mathrm{Hom}_{\mathbf{T}}(Y, -)$. This means precisely that there is a unique dashed arrow making the diagram above commute. We set then $Ff = h$. If f is the identity on X, then the identity on Y makes the diagram commute, so F preserves identities. And similarly F preserves compositions, because if we paste two commutative squares we get a commutative rectangle. So F is indeed a functor. We need to check that it is right adjoint to the identity, so let $X \in \mathbf{T}$ and $S \in \mathbf{S}$. Then the argument we used to define Ff yields an isomorphism $\mathrm{Hom}_{\mathbf{T}}(S, FX) \cong \mathrm{Hom}_{\mathbf{T}}(S, X)$ sending $t \mapsto g \circ t$, with g as in the diagram above. If $r\colon S \to S'$ is any morphism in $\mathbf{S}$ and $t'\colon S' \to FX$ is any morphism in $\mathbf{T}$, then we have $g \circ (t' \circ r) = (g \circ t') \circ r$, so these isomorphisms are functorial in the first variable. If $f\colon X \to X'$ is any morphism in $\mathbf{T}$ and $t\colon S \to FX$ is any morphism in $\mathbf{T}$, then $f \circ g \circ t = g' \circ h \circ t = g' \circ Ff \circ t$, with g, g' and h as in the diagram above. This shows functoriality in the other variable and finishes the proof. □

Corollary A.27 ([BK90, Lemma 1.7])**.** *If $\mathbf{A} \leqslant \mathbf{T}$ is right (resp. left) admissible, then $\mathbf{A} = {}^{\perp}(\mathbf{A}^{\perp})$ (resp. $\mathbf{A} = ({}^{\perp}\mathbf{A})^{\perp}$).*

Proof. Let us do the right admissible case for example. We always have $\mathbf{A} \subseteq {}^{\perp}(\mathbf{A}^{\perp})$, because if $X \in \mathbf{A}$ and $Y \in \mathbf{A}^{\perp}$, then $\mathrm{Ext}^{\bullet}(X, Y) = 0$ by definition of orthogonal complement.

So we have to show that this inclussion is essentially surjective. Let then $X \in {}^{\perp}(\mathbf{A}^{\perp})$. By lemma A.26 we can find a distinguished triangle $Y \to X \to Z \to \Sigma Y$ such that $Y \in \mathbf{A}$ and $Z \in \mathbf{A}^{\perp}$. Since $X \in {}^{\perp}(\mathbf{A}^{\perp})$, the morphism $X \to Z$ is trivial and the triangle splits (cf. proposition A.14). We obtain an isomorphism $Y \cong X \oplus \Sigma^{-1}Z$. The projection $Y \to \Sigma^{-1}Z \in \mathbf{A}^{\perp}$ has to be trivial, so the identity on $\Sigma^{-1}Z$ factors through zero and is therefore zero itself. This means that $Z = 0$, which in turn implies $Y \cong X$ by proposition A.11. Hence $X \in \mathbf{A}$. □

A situation that we will often face is the following: suppose $\mathbf{A} \leqslant \mathbf{B}$ is right admissible and $\mathbf{B} \leqslant \mathbf{C}$ is right admissible. Is $\mathbf{A}$ right admissible in $\mathbf{C}$?

Lemma A.28. *Let $F_1\colon \mathbf{A} \to \mathbf{B}$ and $F_2\colon \mathbf{B} \to \mathbf{C}$ be two right (resp. left) admissible functors. Then $F_2 \circ F_1$ is also right (resp. left) admissible.*

Proof. If $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $F_2 \circ F_1 \dashv G_1 \circ G_2$. Moreover, the unit of this adjunction is obtained from the other units as follows:

$$\mathrm{id}_{\mathbf{A}} \Rightarrow G_1 \circ \mathrm{id}_{\mathbf{B}} \circ F_1 \Rightarrow G_1 \circ G_2 \circ F_2 \circ F_1$$

Hence it is also a natural isomorphism and we are done. □

So the composition of right (resp. left) admissible functors is right (resp. left) admissible. This applies in particular to the case of subcategories.

Lemma A.29. *Let $F_1\colon \mathbf{A} \to \mathbf{B}$ and $F_2\colon \mathbf{B} \to \mathbf{C}$ be two functors and suppose that $F_2 \circ F_1$ and F_2 are right (resp. left) admissible. Then F_1 is also right (resp. left) admissible.*

Proof. Let us show the right admissible case, the other one being similar.

Note first that by lemma A.23 and the two out of three rule for fully faithfulness, all three functors are fully faithful. Moreover, by assumption we have right adjoints $G^!$ and $(GF)^!$. We claim that $F \dashv (GF)^! \circ G$ is an adjunction whose unit is a natural isomorphism.

To check the adjunction, note that

$$\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(GFA, GB) \cong \mathrm{Hom}(A, (GF)^! GB)$$

by fully faithfulness and by adjunction respectively. Both isomorphisms are functorial in the two entries, hence they show the required adjunction. The unit of this adjunction $\eta\colon \mathrm{id}_{\mathbf{A}} \to (GF)^! \circ G \circ F = (GF)^! \circ (GF)$ is the unit of the adjunction $GF \dashv (GF)^!$, which is a natural isomorphism by hypothesis. $\square$

Localisation of triangulated categories

In this subsection we will refer to the literature for the details, e.g. [Ver96, Section II.2].

Definition A.30. Let $\mathbf{C}$ be a category and let S be a collection of morphisms in $\mathbf{C}$. We call S a *multiplicative system* if it satisfies the following axioms:

i) S is closed under finite compositions. In particular, S contains all identities.

ii) *Ore condition*[5]: suppose f, f' are any morphisms in $\mathbf{C}$ and $s, s' \in S$. Then we can find dashed arrows g, g' and $t, t' \in S$ as below making the diagrams commute.

$$\begin{tikzcd}
 & W \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "g"] & \\
X \arrow[dr, "f"'] & & Z \arrow[dl, "s"] \\
 & Y &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & W & \\
X \arrow[ur, dashed, "t'"] & & Z \arrow[ul, dashed, "g'"'] \\
 & Y \arrow[ul, "f'"] \arrow[ur, "s'"'] &
\end{tikzcd}$$

iii) For all parallel morphisms $f, g\colon X \to Y$ we have $sf = sg$ for some $s \in S$ if and only if $ft = gt$ for some $t \in S$.

It is a standard fact from category theory that under such conditions we can localise $\mathbf{C}$ at S via roofs as explained in section 1.1. What we are interested in the behaviour of triangulated categories under this localisation. It turns out that under certain assumptions on the multiplicative system we can guarantee that the resulting localisation is triangulated, both the category and the functor.

Definition A.31. Let $\mathbf{T}$ be a triangulated category and S a multiplicative system on $\mathbf{T}$. We say that S is *compatible with the triangulation* if the following two axioms are satisfied:

v) S is invariant under suspension, i.e. $s \in S$ if and only if $\Sigma s \in S$.

vi) If the two morphisms in (TR3) are in S, we can also find a dashed arrow $h \in S$.

[5] Called like this by similarity with the usual Ore condition.

Let $\mathbf{T}$ be a triangulated category and S a multiplicative system compatible with the triangulation. Then the localisation $\mathbf{T}[S^{-1}]$ admits a unique triangulated structure making the localisation functor $\mathbf{T} \to \mathbf{T}[S^{-1}]$ triangulated.

Let us just mention that the suspension is well defined on the localisation by condition *iv*), and that a triangle is distinguished on $\mathbf{T}[S^{-1}]$ if and only if it is isomorphic to the image of a distinguished triangle under the localisation functor. Moreover, the factorisation obtained from the universal property from a cohomological (resp. triangulated) functor is again cohomological (resp. triangulated).

Example A.32 ([Ver96, Proposition II.2.1.8])**.** Let $\mathbf{S} \leqslant \mathbf{T}$ be a triangulated subcategory. The collection of morphisms $S_{\mathbf{S}}$ whose cone lies in $\mathbf{S}$ is a multiplicative system compatible with the triangulated structure.

Moreover, if $\mathbf{S} \trianglelefteq \mathbf{T}$ is a thick subcategory, then $S_{\mathbf{S}}$ is *saturated*, meaning that $g \in S_{\mathbf{S}}$ whenever f, g and h are composable morphisms in $\mathbf{T}$ such that $fg, gh \in S_{\mathbf{S}}$.

The resulting localisation is called the *Verdier quotient*, and denoted $\mathbf{T}/\mathbf{S}$. If $\mathbf{S} \trianglelefteq \mathbf{T}$ is thick, then the kernel of the localisation functor $\mathbf{T} \to \mathbf{T}/\mathbf{S}$ is precisely $\mathbf{S}$ (see [Ver96, Corollary II.2.2.11]).

A.3 Generators of triangulated categories

We will again skip most of the details and refer to the literature, e.g. [BB03].

Definition A.33. Let $\mathbf{T}$ be a triangulated category and $\mathbf{E} \subseteq \mathbf{T}$ be a collection of objects.

i) We say that $\mathbf{E}$ *generates* $\mathbf{T}$ *as a triangulated category* if the smallest triangulated subcategory of $\mathbf{T}$ containing $\mathbf{E}$ is $\mathbf{T}$ itself.

ii) We say that $\mathbf{E}$ *classically generates* $\mathbf{T}$ if the smallest thick triangulated subcategory of $\mathbf{T}$ containing $\mathbf{E}$, denoted $\langle \mathbf{E} \rangle$, is $\mathbf{T}$ itself.

iii) We say that $\mathbf{E}$ *generates*[6] $\mathbf{T}$ if $\mathbf{E}^{\perp} = \{0\}$.

Remark A.34. The notation $\langle \mathbf{E} \rangle$ is also used for the smallest triangulated subcategory containing $\mathbf{E}$ (see [BK90, Definition 4.6]). Note that even in the case of a single object $\mathbf{E} = \{E\}$ the two definitions may not be equivalent. But the two become equivalent for semiorthogonal collections of admissible subcategories, which is the setting that we will have most of the time (cf. lemma 1.46).

We can rephrase the definition of admissible subcategory in terms of generation:

Proposition A.35 ([Bon90, Lemma 3.1])**.** *A triangulated subcategory* $\mathbf{A} \leqslant \mathbf{T}$ *is right admissible if and only if* $\mathbf{A}$ *and* $\mathbf{A}^{\perp}$ *generate* $\mathbf{T}$ *as a triangulated subcategory.*

[6] *Separates* would perhaps be a better name (cf. [Mac78, Section V.7]).

Proof. Recall that $\mathbf{A}$ is right admissible if and only if every object $X \in \mathbf{T}$ fits into a distinguished triangle

$$Y \to X \to Z \to \Sigma Y$$

with $Y \in \mathbf{A}$ and $Z \in \mathbf{A}^{\perp}$. In particular, $\mathbf{A}$ and $\mathbf{A}^{\perp}$ generate $\mathbf{T}$ as a triangulated category.

Conversely, we need to show that the collection of all objects $X \in \mathbf{T}$ which fit into some distinguished triangle as before is the whole $\mathbf{T}$. The collection of such objects is suspension invariant, so if we show that it is also closed under taking cones, then it must be the smallest triangulated subcategory containing $\mathbf{A}$ and $\mathbf{A}^{\perp}$, which by assumption is the whole $\mathbf{T}$.

This follows from the 3×3 lemma. Suppose $X \to Y$ is a morphism with X and Y such that we can find distinguished triangles $X' \to X \to X'' \to \Sigma X'$ and $Y' \to Y \to Y'' \to \Sigma Y$ such that $X', Y' \in \mathbf{A}$ and $X'', Y'' \in \mathbf{A}^{\perp}$. Applying the cohomological functor $\mathrm{Hom}(X', -)$ to the Y triangle we obtain a dashed arrow making the following square commute:

$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

Consider now the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] & \Sigma X' \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & \Sigma X \arrow[d] \\
X'' \arrow[d] & Y'' \arrow[d] & & \Sigma X'' \arrow[d] \\
\Sigma X' \arrow[r] & \Sigma Y' \arrow[r] & \Sigma Z' \arrow[r] & \Sigma^2 X'
\end{tikzcd}$$

in which all rows and columns are distinguished triangles. Apply the 3×3 lemma to obtain an almost commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, dashed] & \Sigma X' \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d] \\
X'' \arrow[r, dashed] \arrow[d] & Y'' \arrow[r, dashed] \arrow[d] & Z'' \arrow[r, dashed] \arrow[d, dashed] & \Sigma X'' \arrow[d] \\
\Sigma X' \arrow[r] & \Sigma Y' \arrow[r] & \Sigma Z' \arrow[r] & \Sigma^2 X'
\end{tikzcd}$$

in which again all rows and columns are distinguished triangles. Since X'' and Y'' are in $\mathbf{A}^\perp$, so is Z'' by the usual cohomological functor argument. And since $\mathbf{A} \leqslant \mathbf{T}$, the cone of $X' \to Y'$ is also in $\mathbf{A}$. Hence Z fits into a distinguished triangle

$$Z' \to Z \to Z'' \to \Sigma Z'$$

with $Z' \in \mathbf{A}$ and $Z'' \in \mathbf{A}^\perp$, as we wanted to show. □

Lemma A.36. *We have* $\langle \mathbf{E} \rangle^\perp = \mathbf{E}^\perp$.

Proof. The bigger the collection, the smaller its orthogonal complement, hence $\langle \mathbf{E} \rangle^\perp \subseteq \mathbf{E}^\perp$. Conversely, suppose that $\operatorname{Ext}^\bullet(E, X) = 0$ for $X \in \mathbf{T}$ and all $E \in \mathbf{E}$. Then $\operatorname{Ext}^\bullet(\Sigma^i E, X) = \operatorname{Ext}^{\bullet+i}(E, X) = 0$. If $E \cong E_1 \oplus E_2$, then $\operatorname{Ext}^\bullet(E_1, X) = 0$ as well by the universal property of direct sums. Finally, if $E \to F$ is a morphism between objects in $\mathbf{E}$ and $\operatorname{Ext}^\bullet(E, X) = \operatorname{Ext}^\bullet(F, X) = 0$, then its cone C also verifies $\operatorname{Ext}^\bullet(C, X) = 0$ because of the long exact sequence of the cohomological functor $\operatorname{Hom}(-, X)$. □

Proposition A.37. *Let* $\mathbf{E} \subseteq \mathbf{T}$ *be a collection of objects in a triangulated category* $\mathbf{T}$. *In definition A.33, we have implications*

$$i) \Rightarrow ii) \Rightarrow iii)$$

Proof. If $\mathbf{E}$ generates $\mathbf{T}$ as a triangulated category, then it also classically generates $\mathbf{T}$ because the smallest thick triangulated subcategory containing $\mathbf{E}$ has to be in particular a triangulated subcategory containing $\mathbf{E}$, hence is the whole $\mathbf{T}$ already. If $\langle \mathbf{E} \rangle = \mathbf{T}$, then $\langle \mathbf{E} \rangle^\perp = \mathbf{T}^\perp = \{0\}$. So by the previous lemma we have $\mathbf{E}^\perp = 0$. □

Example A.38. Any abelian category $\mathbf{A}$ generates its bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathbf{A})$ as a triangulated category.

Proof. Let $\mathbf{T}$ denote the smallest (full) triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ containing $\mathbf{A}$. Then complexes concentrated on a single degree are trivially in $\mathbf{T}$. To show that every bounded complex is also in $\mathbf{T}$ we proceed by induction on their length $l \in \mathbb{N}$. Consider the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{l-2} \arrow[r, "-d^{l-2}"] \arrow[d] & A^{l-1} \arrow[r, "-d^{l-1}"] \arrow[d] & A^l \arrow[r] \arrow[d, "d^l"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, dashed] & 0 \arrow[r] \arrow[d, dashed] & A^{l+1} \arrow[r] \arrow[d, dashed] & 0 \arrow[r] \arrow[d, dashed] & \cdots \\
\cdots \arrow[r] & A^{l-1} \arrow[r, "d^{l-1}"] & A^l \arrow[r, "d^l"] & A^{l+1} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The last row is the mapping cone of the morphism between the first two rows. The first two rows are in $\mathbf{T}$ by induction, so the cone of this morphism must also be in $\mathbf{T}$. Hence every bounded complex is in $\mathbf{T}$ and $\mathbf{T} = \mathbf{D}^{\mathrm{b}}(X)$. □

The notion of classical generator is often reserved for single objects[7] instead of larger collections. In that case we can construct the smallest thick subcategory $\langle E \rangle \trianglelefteq \mathbf{T}$ containing $E \in \mathbf{T}$ inductively as follows. We let $\langle E \rangle_1$ denote the strictly full subcategory of $\mathbf{T}$ whose objects are isomorphic to direct summands of finite direct sums of suspensions of E, i.e. direct summands of

$$\bigoplus_{i=1}^{m} \Sigma^{a_i} E$$

For $n > 1$, let $\langle E \rangle_n$ denote the full subcategory of $\mathbf{T}$ of all objects X which fit into a distinguished triangle

$$Y \to X \to Z \to \Sigma Y$$

with $Y \in \langle E \rangle_1$ and $Z \in \langle E \rangle_{n-1}$. Finally, let

$$\langle E \rangle = \bigcup_{n \in \mathbb{N}} \langle E \rangle_n$$

Lemma A.39 ([Stacks, Tag 0ATG])**.** *With the previous notation, suppose that*

$$Y \to X \to Z \to \Sigma Y$$

is a distinguished triangle with $Y \in \langle E \rangle_a$ and with $Z \in \langle E \rangle_b$. Then $X \in \langle E \rangle_{a+b}$.

Proof. We proceed by induction on $a \in \mathbb{N}$. The case $a = 1$ holds by construction. Suppose it is true for some natural number $a \geqslant 1$ and let

$$Y \to X \to Z \to \Sigma Y$$

be a distinguished triangle with $Y \in \langle E \rangle_{a+1}$ and $Z \in \langle E \rangle_b$. We apply (TR1) and (TR4) to obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dashed] & \Sigma Z \arrow[d, equal] \\
Z \arrow[r] & E_a \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & \Sigma Z \\
 & \Sigma E_1 \arrow[r, equal] \arrow[d] & \Sigma E_1 & \\
 & \Sigma Y \arrow[r] & \Sigma X &
\end{tikzcd}$$

where $E_1 \in \langle E \rangle_1$, $E_a \in \langle E \rangle_a$ and all vertical and horizontal arrows are distinguished triangles. From the induction hypothesis and the lower horizontal triangle we obtain $C \in \langle E \rangle_{a+b}$. From the rightmost vertical triangle we obtain then $X \in \langle E \rangle_{a+b+1}$ by construction of $\langle E \rangle_{a+b+1}$. □

[7]Or equivalently finitely many objects. Taking their direct sum shows that the two are equivalent.

Proposition A.40 ([Stacks, Tag 0ATG])**.** *Keeping the notation as above, $\langle E\rangle$ is the smallest thick triangulated subcategory of $\mathbf{T}$ containing E.*

Proof. Being closed under supsensions and direct summands follows directly from the construction. Being triangulated follows directly from the previous lemma. And it is indeed the smallest such subcategory, because each $\langle E\rangle_n$ must be contained in the smallest thick trianguluated subcategory containing E by definition. □

Hence E classically generates $\mathbf{T}$ if and only if $\langle E\rangle = \mathbf{T}$.

Compact generation

Definition A.41. An object K in a triangulated category $\mathbf{T}$ with small coproducts is called *compact* if the functor $\mathrm{Hom}_{\mathbf{T}}(K, -)$ preserves all small coproducts.

In such a triangulated category we may define the full subcategory $\mathbf{T}^c \trianglelefteq \mathbf{T}$ of compact objects, which is a thick triangulated subcategory.

Lemma A.42 ([Stacks, Tag 09SR])**.** *Let $\mathbf{T}$ be a triangulated category with small coproducts and let $E \in \mathbf{T}$ be a compact object. Then the following are equivalent:*

i) E classically generates $\mathbf{T}^c$ and $\mathbf{T}$ is compactly generated.

ii) E generates $\mathbf{T}$.

The relevance of compact objects and generation on triangulated categories is the following result due to Neeman (see [Nee96, Theorem 3.1]).

Theorem A.43 (Brown representability)**.** *Let $\mathbf{T}$ be a compactly generated triangulated category. Then every contravariant cohomological functor which transforms coproducts into products is representable.*

Corollary A.44. *Let $\mathbf{T}$ be a compactly generated triangulated category. Let $F\colon \mathbf{T} \to \mathbf{T}'$ be a triangulated functor which preserves coproducts. Then F has a triangulated right adjoint.*

Proof. Fix $Y \in \mathbf{T}'$. By Brown representability we can represent $\mathrm{Hom}_{\mathbf{T}}(F(-), Y)$. This means precisely that F has a right adjoint, which assigns to each $Y \in \mathbf{T}'$ the corresponding representing object. The fact that this right adjoint is triangulated is automatic in the context of triangulated categories, as we have already seen. □

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