A Unified Framework for Monetary Theory and Policy Analysis

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Introduction

- Reduced-form monetary macro models: not explicit about the role of money in overcoming spatial, temporal or informational frictions.
- Search models have explicit micro-foundations.
- Previous search models: ill-suited for the analysis of monetary policy due to the extreme restrictions on money holding.
- This model: no extreme restrictions on money holding.
Main feature of the model

• Previous models w/o restrictions on money holding are complicated by the endogenous distribution of money holding, $F(m)$.

• Assumption of quasi-linear preference makes $F(m)$ degenerate: No wealth effects in the demand for money.

• This framework is as easy to use as standard reduced-form models (e.g. study the cost of inflation)
Model: market structure and preferences

Market structure \( \begin{cases} \text{Day – DM (search): special goods} \\ \text{Night – CM (Walrasian): general goods} \end{cases} \)

- Preferences: \( U(x, h, X, H) = u(x) - c(h) + U(X) - H. \)
  - \( x, X: \) consumption. \( h, H: \) labor supply.
  - \( \exists q^* \in (0, \infty) \text{ s.t. } u'(q^*) = c'(q^*). \)
  - \( \exists X^* \in (0, \infty) \text{ s.t. } U'(X^*) = 1 \text{ with } U(X^*) > X^* \)
Model: DM and CM

- **DM**: decentralized and anonymous $\rightarrow$ no credit.
  $\alpha$: prob of meeting.
- **special goods**:
  prob(double coincidence of wants) $= \delta$.
  prob(single coincidence of wants) $= \sigma$.
  prob(both agents have opposite wants) $= 1 - 2\sigma - \delta$.
- **CM**: All agents produce and consume a general good.
- Special goods and general goods are divisible and non-storable
  $\rightarrow$ no commodity money.
- money: perfectly divisible and storable in any quantity $m \geq 0$.
  $M$: total money stock
Model: distribution of money holdings

- $F_t(\tilde{m}) (G_t(\tilde{m}))$: measure of agents starting the DM (CM) holding $m \leq \tilde{m}$, $F_0, G_0$ exogenously given.
- $\int m \, dF_t(m) = \int m \, dG_t(m) = M, \forall t.$
- $\phi_t$: value of money in terms of general goods in CM.
- No uncertainty in the basic model except for random matching.
- Aggregate variables such as $F_t, G_t$ and prices are taken as given, an agent’s decisions depend only on his money holdings, $m$. 

Value function: DM

An agent with \( m \) entering DM:

\[
V_t(m) = \alpha \sigma \int \{ u[q_t(m, \bar{m})] + W_t[m - d_t(m, \bar{m})] \} dF_t(\bar{m}) \\
+ \alpha \sigma \int \{-c[q_t(\bar{m}, m)] + W_t[m + d_t(\bar{m}, m)] \} dF_t(\bar{m}) \\
+ \alpha \delta \int B_t(m, \bar{m}) \ dF_t(\bar{m}) \\
+ (1 - 2\alpha \sigma - \alpha \delta) W_t(m).
\] (1)
An agent with $m$ entering CM:

$$W_t(m) = \max_{X, H, m'} \{ U(X) - H + \beta V_{t+1}(m') \}$$

(2)

s.t. $X = H + \phi_t m - \phi_t m'$

$X \geq 0, 0 \leq H \leq \bar{H}, m' \geq 0$.

$m'$: money taken out of the market.

• Assume interior solution for $X, H$, characterize equilibrium and then check $0 < H < \bar{H}$ is satisfied.
Agents with $m$ meets someone with $\tilde{m}$

- In a double-coincidence-of-wants meeting: symmetric Nash bargaining with threat point the continuation value $W_t$:

$$B_t(m, \tilde{m}) = u(q^*) - c(q^*) + W_t(m).$$

- In a single-coincidence-of-wants meeting: Nash bargaining power $\theta$:

$$\max_{q,d} \left[ u(q_t) + W_t(m - d_t) - W_t(m) \right]^\theta$$
$$\left[ -c(q_t) + W_t(\tilde{m} + d_t) - W_t(\tilde{m}) \right]^{1-\theta}$$

s.t. $d \leq m$, $q \geq 0$.

- definition of equilibrium (p.468).
How to find an equilibrium?

- Derive some properties of the sol. to the CM problem.
- Solve the bargaining problem.
- Simplify $V_t$ and solve for individual’s problem of choosing $m'_t(m)$: $m'_t = M$ for all agents regardless of $m_t$, $\Rightarrow F_{t+1}$ degenerate
- combine the sol. to CM and DM problems to reduce the model to a single difference equation.
Linearity of $W(m)$

(2) \Rightarrow 

\[ W_t(m) = \phi_t m + \max_{X,m'} \{ U(X) - X - \phi_t m' + \beta V_{t+1}(m') \} \]

where

\[ \max_{X,m'} \{ U(X) - X - \phi_t m' + \beta V_{t+1}(m') \} \equiv W(0). \]

Notes:

- $X(m) = X^*$ where $U'(X^*) = 1$.
- $m_t'(m)$ does not depend on $m$.
  (quasi-linear utility rules out the wealth effect)
- $W_t$ is linear in $m$ with slope $\phi_t$. \( \rightarrow \) \( W(m) = W(0) + \phi m \)
Bargaining solution

• Given that $W(m) = W(0) + \phi m$, the bargaining problem becomes:

$$\max_{q,d} [u(q) - \phi_t d]^\theta [-c(q) + \phi_t d]^{1-\theta}.$$  

s.t. $d \leq m$, $q \geq 0$.

• Bargaining solution:

$$q_t(m, \tilde{m}) = \begin{cases} 
\hat{q}_t(m) & \text{if } m < m^*_t \\
q^* & \text{if } m \geq m^*_t 
\end{cases}$$

$$d_t(m, \tilde{m}) = \begin{cases} 
m & \text{if } m < m^*_t \\
m^* & \text{if } m \geq m^*_t 
\end{cases}$$
Bargaining solution (con’t)

• $\hat{q}_t(m)$ solves $\phi_t m = z(q_t)$

\[
z(q) \equiv \frac{\theta c(q)u'(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)}.
\]

\[
m^*_t = z(q^*)/\phi_t
\]

• constraint not binding: $q_t = q^*$,
  \[
d_t = m^*_t = [\theta c(q^*) + (1 - \theta)u(q^*)]/\phi_t.
\]
  (spend $m^*_t$ dollars to get $q^*$)

• constraint binding: $q_t$ is given by $\hat{q}_t(m)$ with
  \[
d_t = m \Rightarrow \phi_t m = z(q_t).
\]
  (spend all his money to get $\hat{q}_t(m)$)

• Solutions do not depend on sellers’ money holdings $\tilde{m}$!
Equilibrium property

- $q_t(m) = \hat{q}_t(m)$ is strictly increasing for $m < m_t^*$, is continuous at $m_t^*$, and is constant at $q_t(m) = q^*$ for all $m \geq m_t^*$.
- Any equilibrium must satisfy $\phi_t \geq \beta \phi_{t+1}$ ⇒ The minimal inflation consistent with equilibrium is $\phi_t/\phi_{t+1} = \beta \sim$ Friedman rule.
Equilibrium path: $\phi_t$

- $\theta \approx 1$ or $u'$ is log concave ($u'u''' \leq (u'')^2$) $\Rightarrow v''_{t+1} < 0$
  $\Rightarrow$ a unique choice of $m_{t+1}$ in any equilibrium;
  i.e. $F_{t+1}$ degenerate at $m_{t+1} = M$.
  (Thus, $d_{t+1} = M$, the buyer exchanges all his money, and $q_{t+1} = q_{t+1}(M)$.)

- In any monetary equilibrium, FOC evaluated at $m_{t+1} = M$ is

$$\phi_t = \beta[v'_{t+1}(M) + \phi_{t+1}].$$

$\Rightarrow$

$$\phi_t = \{\alpha\sigma u'[q_{t+1}(M)]q'_{t+1}(M) + (1 - \alpha\sigma)\phi_{t+1}\} \quad (3)$$
Equilibrium path: $q_t$

- Inserting $\phi_t = z(q)/M$ and $q'(M) = \phi_t/z'(q_t)$ from the bargaining sol, (3) ⇒

$$z(q_t) = \beta z(q_{t+1})[\alpha \sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha \sigma]. \quad (4)$$

a difference equation in $q_t$.

- A monetary equilibrium is characterized by any path $\{q_t\}$ satisfying (4) that stays in $(0, q^*)$, since $q_t < q^*$ follows from $m_t < m^*_t$.

- Note: the argument that $F_{t+1}$ is degenerate did not use stationarity.
Steady State

(4) \Rightarrow
\frac{u'(q)}{z'(q)} = 1 + \frac{1 - \beta}{\alpha \sigma \beta}.

(5)

- \theta = 1 : z(q) = c(q) \ \exists! \ q \text{ solves (5) if, e.g. } u'(0) = \infty
- \theta < 1 : \frac{u'(q)}{z'(q)} \text{ may not be monotone.}
  If, e.g. \ \theta \approx 1 \text{ or } c \text{ is linear and } u' \text{ log concave,}
  \Rightarrow \ \exists! \ q \text{ solves (19) and}
  \frac{\partial q}{\partial \theta} > 0, \frac{\partial q}{\partial \sigma} > 0, \frac{\partial q}{\partial \alpha} > 0 \text{ and } \frac{\partial q}{\partial \beta} > 0.
- \theta = 1 : q \rightarrow q^* \text{ as } \beta \rightarrow 1
  \theta < 1 : q < q^* \text{ as } \beta \rightarrow 1
- \text{Steady state is efficient iff } q = q^*, \text{ which requires } \beta = 1 \text{ and } \theta = 1.
Changes in the money Supply

• New money is injected in CM: $M_{t+1} = (1 + \tau)M_t$.

• Consider S-S where $q$ and real balances $\phi M = z(q)$ are constant; i.e., $\phi_t/\phi_{t+1} = 1 + \tau$.

• S-S condition:

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1 + \tau - \beta}{\alpha \sigma \beta}.$$  \hspace{1cm} (6)

• $1 + i = (1 + r)(1 + \pi)$; $\pi = \tau$: equilibrium inflation rate $r = \frac{1-\beta}{\beta}$ equilibrium real interest rate.

$$(6) \Rightarrow \quad \frac{u'(q)}{z'(q)} = 1 + \frac{i}{\alpha \sigma}$$

• Assume a unique monetary S.S: $\frac{\partial q}{\partial \tau} < 0$; $\frac{\partial q}{\partial i} < 0$. 

Results

• $\theta = 1$: $z(q) = c(q)$, get $q^*$ iff $\tau = \tau^F$ ($i = 0$)
• $\theta < 1$: $q < q^*$ at $\tau^F$ since a necessary condition for monetary equilibrium is $\tau \geq \tau^F$ ($i \geq 0$).
  The Friedman rule is optimal here but does not achieve the efficient outcomes $q^*$.
• Why?
Two types of inefficiencies

• due to $\beta < 1 : q < q^*$
• due to $\theta < 1$: holdup problem.
• Hosios (1990) condition for efficiency:
  The bargaining solution should split the surplus so that each party is compensated for his contribution to the surplus in a match.
• The surplus in a single-match is all due to the buyer, since the outcome depends on $m$ but not on $\tilde{m}$. Hence, efficiency requires $\theta = 1$ here.
• The wedge due to $\theta < 1$ is important for issue such as the welfare cost of inflation (see Fig. 1).