Suggested Answers for Introduction to Quantitative Methods Final Exam.
September 10, 2001

1a The eigenvalues of an idempotent matrix are either 1 or 0. Since \( A \) is of full rank, all the eigenvalues are 1. Therefore, \( \text{trace}(A) = \sum_{i=1}^{K} \lambda_i = K \).

1b \( |A| = \prod_{i=1}^{K} \lambda_i = 1. \)

1c Since \( \Lambda = \text{diag}(1, 1, \cdots, 1) = I \), then \( A = C \Lambda C' = CC' = I \). \( A - I = 0. \)

2 This is a homogeneous system, to have a nonzero solution, we have
\[
\begin{vmatrix}
3 & 1 & -\lambda \\
4 & -2 & -3 \\
2\lambda & 4 & \lambda
\end{vmatrix}
= 0
\]
This gives \( \lambda^2 + 8\lambda - 9 = 0 \), or \( \lambda = 1 \) or \( \lambda = -9 \). For \( \lambda = 1 \), the solution is \((x_1, x_2, x_3) = \pm \frac{1}{\sqrt{6}} (1, 1, 2) \). For \( \lambda = -9 \), the solution is \((x_1, x_2, x_3) = \pm \frac{1}{\sqrt{94}} (3, 9, -2) \).

3a The first order condition is \( 2Ax + a = 0 \), and \( x = -\frac{1}{2} A^{-1}a \).

3b \( x = -\frac{1}{2} \begin{bmatrix} 25 & 7 \\ 7 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{552} \begin{bmatrix} 13 & -7 \\ -7 & 25 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{552} \\ \frac{61}{552} \end{bmatrix} \).

3c The Hessian matrix is \( 2A \). We need to show that \( A \) is a positive definite matrix. One way of checking this is to find the eigenvalues of \( A \).
\[
\begin{vmatrix}
25 - \lambda & 7 \\
7 & 13 - \lambda
\end{vmatrix}
= \lambda^2 - 38\lambda + 276 = 0
\]
\( \lambda_1 + \lambda_2 = 38, \lambda_1\lambda_2 = 276 \)
Therefore, \( \lambda_1, \lambda_2 > 0 \). \( A \) is positive definite.

4a All of these estimators are unbiased.

4b \( \text{Var}[\hat{\mu}_1] = \frac{\sigma^2}{n}, \text{Var}[\hat{\mu}_2] = \sigma^2, \text{Var}[\hat{\mu}_3] = \frac{n\sigma^2}{4(n-1)} \). Therefore, \( \hat{\mu}_1 \) is the most efficient.

4c Only \( \text{Var}[\hat{\mu}_1] \) converges to 0, it converges in mean square which implies plim \( \hat{\mu}_1 = \mu \). Therefore, \( \hat{\mu}_1 \) is consistent.

5a \( \hat{\beta} = (X'X)^{-1}X'(X\beta + \epsilon) = \beta + (X'X)^{-1}X'\epsilon, E[\hat{\beta}] = \beta + (X'X)^{-1}X'E[\epsilon] = \beta. \hat{\beta} \) is unbiased.
5b

\[ M' = I - (X'X)^{-1}X' = I - X'X^{-1}X' = M, M \text{ is symmetric.} \]

\[ MM = (I - X'X^{-1}X')(I - X'X^{-1}X') \]
\[ = I - X'X^{-1}X' - X'X^{-1}X' + X'X^{-1}X'X'X^{-1}X' \]
\[ = I - X'X^{-1}X' = M, M \text{ is idempotent.} \]

5c

\[ \hat{\epsilon}'X = (My)'X = y'MX = y'(I - X'X^{-1}X')X \]
\[ = y'(X - X'X^{-1}X') = y'(X - X) = 0 \]

5d We have E[\hat{\beta}] = \beta, and

\[ \text{Var}[\hat{\beta}] = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \]
\[ = E[((X'X)^{-1}X')\epsilon((X'X)^{-1}X')\epsilon'] \]
\[ = (X'X)^{-1}X'E[\epsilon\epsilon']X(X^{-1}X)^{-1} \]
\[ = (X'X)^{-1}X'\Sigma X(X^{-1}X)^{-1} \]

Therefore, the distribution of \( \hat{\beta} \) is

\[ \hat{\beta} \sim N(\beta, (X'X)^{-1}X'\Sigma X(X^{-1}X)^{-1}) \]

6a 1,8,27 and c1, c2 and c3.

6b Since \( A \) is symmetric, then \( A' = A \). \( (A^{-1})' = (A')^{-1} = A^{-1} \), so \( A^{-1} \) is also symmetric.

6c Let \( A : n \times K, B : K \times n, C = AB \) and \( D = BA \). Since \( c_{ii} = a_i^T b_i = \sum_{k=1}^{K} a_{ik}b_{ki} \), and \( d_{kk} = b_k^T a_k = \sum_{i=1}^{n} b_{ki}a_{ik} \),

\[ \text{tr}(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{K} a_{ik}b_{ki} = \sum_{k=1}^{K} \sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{K} d_{kk} = \text{tr}(BA) \]