Options on Stock Indices, Currencies, and Futures

Chapter 14
European Options on Stocks Providing a Dividend Yield

We get the same probability distribution for the stock price at time $T$ in each of the following cases:

1. The stock starts at price $S_0$ and provides a dividend yield $= q$
2. The stock starts at price $S_0 e^{-qT}$ and provides no income
European Options on Stocks Providing Dividend Yield continued

We can value European options by reducing the stock price to $S_0e^{-qT}$ and then behaving as though there is no dividend.
Extension of Chapter 9 Results
(Equations 14.1 to 14.3)

Lower Bound for calls:

\[ c \geq S_0 e^{-qT} - Ke^{-rT} \]

Lower Bound for puts

\[ p \geq Ke^{-rT} - S_0 e^{-qT} \]

Put Call Parity

\[ c + Ke^{-rT} = p + S_0 e^{-qT} \]
Extension of Chapter 13 Results
(Equations 14.4 and 14.5)

\[ c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \]

\[ p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \]

where

\[ d_1 = \frac{\ln(\frac{S_0}{K}) + (r - q + \sigma^2 / 2)T}{\sigma \sqrt{T}} \]

\[ d_2 = \frac{\ln(\frac{S_0}{K}) + (r - q - \sigma^2 / 2)T}{\sigma \sqrt{T}} \]
The Binomial Model

\[ S_0 \xrightarrow{p} S_0u \quad S_0 \xrightarrow{(1-p)} S_0d \]

\[ f = e^{-rT}[pf_u + (1-p)f_d] \]
The Binomial Model continued

- In a risk-neutral world the stock price grows at $r-q$ rather than at $r$ when there is a dividend yield at rate $q$
- The probability, $p$, of an up movement must therefore satisfy
  \[ pS_0u + (1-p)S_0d = S_0e^{(r-q)T} \]
  so that
  \[ p = \frac{e^{(r-q)T} - d}{u - d} \]
The most popular underlying indices in the U.S. are:
- The Dow Jones Index times 0.01 (DJX)
- The Nasdaq 100 Index (NDX)
- The Russell 2000 Index (RUT)
- The S&P 100 Index (OEX)
- The S&P 500 Index (SPX)

Contracts are on 100 times index; they are settled in cash; OEX is American and the rest are European.
LEAPS

- Long-term Equity AnticiPation Securities
- Leaps are options on stock indices that last up to 3 years
- They have December expiration dates
- The index is divided by five for the purposes of quoting the strike price and the option price
- Leaps also trade on some individual stocks
Index Option Example

- Consider a call option on an index with a strike price of 560.
- Suppose 1 contract is exercised when the index level is 580.
- What is the payoff?
Using Index Options for Portfolio Insurance

- Suppose the value of the index is $S_0$ and the strike price is $K$
- If a portfolio has a $\beta$ of 1.0, the portfolio insurance is obtained by buying 1 put option contract on the index for each $100S_0$ dollars held
- If the $\beta$ is not 1.0, the portfolio manager buys $\beta$ put options for each $100S_0$ dollars held
- In both cases, $K$ is chosen to give the appropriate insurance level
Example 1

- Portfolio has a beta of 1.0
- It is currently worth $500,000
- The index currently stands at 1000
- What trade is necessary to provide insurance against the portfolio value falling below $450,000?
Example 2

- Portfolio has a beta of 2.0
- It is currently worth $500,000 and index stands at 1000
- The risk-free rate is 12% per annum
- The dividend yield on both the portfolio and the index is 4%
- How many put option contracts should be purchased for portfolio insurance?
Calculating Relation Between Index Level and Portfolio Value in 3 months

- If index rises to 1040, it provides a 40/1000 or 4% return in 3 months
- Total return (incl dividends)=5%
- Excess return over risk-free rate=2%
- Excess return for portfolio=4%
- Increase in Portfolio Value=4+3-1=6%
- Portfolio value=$530,000
Determining the Strike Price (Table 14.2, page 320)

<table>
<thead>
<tr>
<th>Value of Index in 3 months</th>
<th>Expected Portfolio Value in 3 months ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,080</td>
<td>570,000</td>
</tr>
<tr>
<td>1,040</td>
<td>530,000</td>
</tr>
<tr>
<td>1,000</td>
<td>490,000</td>
</tr>
<tr>
<td>960</td>
<td>450,000</td>
</tr>
<tr>
<td>920</td>
<td>410,000</td>
</tr>
</tbody>
</table>

An option with a strike price of 960 will provide protection against a 10% decline in the portfolio value
Valuing European Index Options

We can use the formula for an option on a stock paying a dividend yield

Set $S_0 = \text{current index level}$
Set $q = \text{average dividend yield expected during the life of the option}$
Currency Options

- Currency options trade on the Philadelphia Exchange (PHLX)
- There also exists an active over-the-counter (OTC) market
- Currency options are used by corporations to buy insurance when they have an FX exposure
The Foreign Interest Rate

- We denote the foreign interest rate by $r_f$
- When a U.S. company buys one unit of the foreign currency it has an investment of $S_0$ dollars
- The return from investing at the foreign rate is $r_fS_0$ dollars
- This shows that the foreign currency provides a “dividend yield” at rate $r_f$
Valuing European Currency Options

- A foreign currency is an asset that provides a “dividend yield” equal to $r_f$
- We can use the formula for an option on a stock paying a dividend yield:
  
  Set $S_0 = \text{current exchange rate}$
  
  Set $q = r_f$
Formulas for European Currency Options  
(Equations 14.7 and 14.8, page 322)  

\[ c = S_0 e^{-r_f T} N(d_1) - K e^{-r T} N(d_2) \]
\[ p = K e^{-r T} N(-d_2) - S_0 e^{-r_f T} N(-d_1) \]

where \[ d_1 = \frac{\ln(S_0 / K) + (r - r_f + \sigma^2 / 2)T}{\sigma \sqrt{T}} \]
\[ d_2 = \frac{\ln(S_0 / K) + (r - r_f - \sigma^2 / 2)T}{\sigma \sqrt{T}} \]
Alternative Formulas
(Equations 14.9 and 14.10, page 322)

Using \[ F_0 = S_0 e^{(r - r_f)T} \]

\[ c = e^{-rT} [F_0 N(d_1) - KN(d_2)] \]

\[ p = e^{-rT} [KN(-d_2) - F_0 N(-d_1)] \]

\[ d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]
For Binomial Tree

\[ p = \frac{a - d}{u - d} \]

\[ a = e^{r \Delta t} \text{ for a nondividend paying stock} \]

\[ a = e^{(r - q) \Delta t} \text{ for a stock index where } q \text{ is the dividend yield on the index} \]

\[ a = e^{(r - r_f) \Delta t} \text{ for a currency where } r_f \text{ is the foreign risk-free rate} \]
Mechanics of Call Futures Options

When a call futures option is exercised the holder acquires

1. A long position in the futures
2. A cash amount equal to the excess of the futures price over the strike price
Mechanics of Put Futures Option

When a put futures option is exercised the holder acquires
1. A short position in the futures
2. A cash amount equal to the excess of the strike price over the futures price
The Payoffs

If the futures position is closed out immediately:

Payoff from call = $F_0 - K$

Payoff from put = $K - F_0$

where $F_0$ is futures price at time of exercise
Consider the following two portfolios:

1. European call plus $Ke^{-rT}$ of cash
2. European put plus long futures plus cash equal to $F_0e^{-rT}$

They must be worth the same at time $T$ so that

$$c + Ke^{-rT} = p + F_0e^{-rT}$$
Binomial Tree Example

A 1-month call option on futures has a strike price of 29.

Futures price = $30
Option Price = ?

Futures Price = $33
Option Price = $4

Futures Price = $28
Option Price = $0
Setting Up a Riskless Portfolio

- Consider the Portfolio: long $\Delta$ futures, short 1 call option

- Portfolio is riskless when $3\Delta - 4 = -2\Delta$ or $\Delta = 0.8$
Valuing the Portfolio
(Risk-Free Rate is 6%)

- The riskless portfolio is:
  - long 0.8 futures
  - short 1 call option
- The value of the portfolio in 1 month is -1.6
- The value of the portfolio today is $-1.6e^{-0.06/12} = -1.592$
Valuing the Option

- The portfolio that is long 0.8 futures short 1 option is worth -1.592.
- The value of the futures is zero.
- The value of the option must therefore be 1.592.
Generalization of Binomial Tree Example (Figure 14.2, page 330)

- A derivative lasts for time $T$ and is dependent on a futures price $F_0$.
Generalization (continued)

- Consider the portfolio that is long $\Delta$ futures and short 1 derivative

\[ F_0u \Delta - F_0 \Delta - f_u \]
\[ F_0d \Delta - F_0 \Delta - f_d \]

- The portfolio is riskless when

\[ \Delta = \frac{f_u - f_d}{F_0u - F_0d} \]
Generalization
(continued)

- Value of the portfolio at time $T$ is $F_0 u \Delta - F_0 \Delta - f_u$
- Value of portfolio today is $-f$
- Hence

$$f = - [F_0 u \Delta - F_0 \Delta - f_u]e^{-rT}$$
Generalization
(continued)

- Substituting for $\Delta$ we obtain

$$f = \left[ p f_u + (1 - p) f_d \right] e^{-rT}$$

where

$$p = \frac{1 - d}{u - d}$$
Valuing European Futures Options

- We can use the formula for an option on a stock paying a dividend yield.
  Set $S_0 = \text{current futures price} \ (F_0)$
  Set $q = \text{domestic risk-free rate} \ (r)$

- Setting $q = r$ ensures that the expected growth of $F$ in a risk-neutral world is zero.
Growth Rates For Futures Prices

- A futures contract requires no initial investment
- In the risk-neutral world, the expected return should be zero
- The expected growth rate of the futures price is therefore zero
- The futures price can therefore be treated like a stock paying a dividend yield of $r$ in the risk-neutral world
Black’s Formula
(Equations 14.16 and 14.17, page 333)

The formulas for European options on futures are known as Black’s formulas

\[ c = e^{-rT} \left[ F_0 \ N(d_1) - K \ N(d_2) \right] \]
\[ p = e^{-rT} \left[ K \ N(-d_2) - F_0 \ N(-d_1) \right] \]

where

\[ d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma \sqrt{T}} \]
\[ d_2 = \frac{\ln(F_0 / K) - \sigma^2 T / 2}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \]
Futures Option Prices vs Spot Option Prices

● If futures prices are higher than spot prices (normal market), an American call on futures is worth more than a similar American call on spot. An American put on futures is worth less than a similar American put on spot.

● When futures prices are lower than spot prices (inverted market) the reverse is true.
Summary of Key Results

- We can treat stock indices, currencies, and futures like a stock paying a dividend yield of $q$
  - For stock indices, $q = \text{average dividend yield on the index over the option life}$
  - For currencies, $q = r_f$
  - For futures, $q = r$