Types of Rates

- Treasury rates
- LIBOR and LIBID rates
  - London Interbank Offer Rate
  - London Interbank Bid Rate
- Repo rates (repurchase agreement rate)
Measuring Interest Rates

- There are many kinds of compounding frequencies used for an interest rate, for example, quarterly or annual compounding.
Discretely Compounding

\[ 1 \times \left( 1 + \frac{R}{m} \right)^{mn} \]

\( m = \text{number of compounding intervals per year} \)
\( n = \text{number of years} \)
\( R = \text{annual interest rate} \)
Continuously Compounding

Let $m \rightarrow \infty$

$$\lim_{m \rightarrow \infty} \left(1 + \frac{R}{m}\right)^{mn} = e^{Rn}$$

where $e = 2.718281828$
<table>
<thead>
<tr>
<th>$m$</th>
<th>Final Sum (n=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.1200$</td>
</tr>
<tr>
<td>2</td>
<td>$1.1236$</td>
</tr>
<tr>
<td>12</td>
<td>$1.126825$</td>
</tr>
<tr>
<td>365</td>
<td>$1.1274746$</td>
</tr>
<tr>
<td>∞</td>
<td>$1.1274969$</td>
</tr>
</tbody>
</table>
Continuous Compounding
(Page 79)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- $100$ grows to $100e^{RT}$ when invested at a continuously compounded rate $R$ for time $T$
- $100$ received at time $T$ discounts to $100e^{-RT}$ at time zero when the continuously compounded discount rate is $R$
Define

\( R_c \): continuously compounded rate
\( R_m \): same rate with compounding \( m \) times per year

\[
P \cdot e^{R_c n} = P \left( 1 + \frac{R_m}{m} \right)^{mn}
\]

\[
e^{R_c} = \left( 1 + \frac{R_m}{m} \right)^m
\]

\[
R_c = m \cdot \ln \left( 1 + \frac{R_m}{m} \right)
\]

\[
R_m = m \cdot (e^{R_c / m} - 1)
\]
Zero Rates

A zero rate (or spot rate), for maturity $T$ is the rate of interest earned on an investment that provides a payoff only at time $T$. 
Example (Table 4.2, page 81)

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Zero Rate (% cont comp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>
Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate.
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

\[
3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} \\
+ 103e^{-0.068 \times 2.0} = 98.39
\]
Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond.

- Suppose that the market price of the bond in our example equals its theoretical price of 98.39.

- The bond yield (continuously compounded) is given by solving
  \[ 3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39 \]
  to get \( y = 0.0676 \) or 6.76%. 

4.12
Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.

- In our example we solve

\[
\frac{c}{2} e^{-0.05 \times 0.5} + \frac{c}{2} e^{-0.058 \times 1.0} + \frac{c}{2} e^{-0.064 \times 1.5} \\
+ \left(100 + \frac{c}{2}\right) e^{-0.068 \times 2.0} = 100
\]

to get \( c = 6.87 \) (with s.a. compounding)
In general if $m$ is the number of coupon payments per year, $d$ is the present value of $1$ received at maturity and $A$ is the present value of an annuity of $1$ on each coupon date

\[
100 = A \frac{c}{m} + 100d
\]

\[
c = \frac{(100 - 100d)m}{A}
\]
### Sample Data (Table 4.3, page 82)

<table>
<thead>
<tr>
<th>Bond Principal (dollars)</th>
<th>Time to Maturity (years)</th>
<th>Annual Coupon (dollars)</th>
<th>Bond Cash Price (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>
The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- The 3-month rate is 4 times $2.5/97.5$ or 10.256% with quarterly compounding
- This is 10.127% with continuous compounding
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding
The Bootstrap Method continued

- To calculate the 1.5 year rate we solve

\[ 4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96 \]

to get \( R = 0.10681 \) or 10.681%

- Similarly the two-year rate is 10.808%
Zero Curve Calculated from the Data (Figure 4.1, page 84)
Forward Rates

The forward rate is the future zero rate implied by today’s term structure of interest rates.
### Calculation of Forward Rates

**Table 4.5, page 85**

<table>
<thead>
<tr>
<th>Year ((n)</th>
<th>Zero Rate for an (n) -year Investment (% per annum)</th>
<th>Forward Rate for (n) th Year (% per annum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>4.6</td>
<td>5.8</td>
</tr>
<tr>
<td>4</td>
<td>5.0</td>
<td>6.2</td>
</tr>
<tr>
<td>5</td>
<td>5.3</td>
<td>6.5</td>
</tr>
</tbody>
</table>
Suppose that the zero rates for time periods $T_1$ and $T_2$ are $R_1$ and $R_2$ with both rates continuously compounded.

The forward rate for the period between times $T_1$ and $T_2$ is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$
Upward vs Downward Sloping Yield Curve

- For an upward sloping yield curve: \( \text{Fwd Rate} > \text{Zero Rate} \)

- For a downward sloping yield curve: \( \text{Zero Rate} > \text{Fwd Rate} \)
A forward rate agreement (FRA) is an agreement at $t = 0$ that a certain rate $R_k$ will apply to a certain principal during a certain future time period.

FRA are usually settled at $T_1$, assuming $R_M$ is the actual LIBOR rate at $T_1$, the payoff of this FRA is

$$\frac{L (R_M - R_K) (T_2 - T_1)}{1 + R_M (T_2 - T_1)}$$
Forward Rate Agreement continued

- An FRA at $t = 0$ can be valued by assuming that the forward interest rate is certain to be realized:

$$V_{FRA} = L(R_F - R_k)(T_2 - T_1)e^{-R_2T_2}$$

- Usually $R_k$ is set to be $R_F$ such that the value of FRA at $t = 0$ is zero.
Duration (page 89)

- Duration of a bond that provides cash flow $c_i$ at time $t_i$ is

$$B = \sum_{t_i} c_{t_i} e^{-yt_i}$$

$$D = \sum_{t_i} t_i \left[ \frac{c_{t_i} e^{-yt_i}}{B} \right]$$

where $B$ is its price and $y$ is its yield (continuously compounded)

- This leads to

$$- \frac{dB}{dy} / B = D$$
Duration Continued

\[ B = \sum_{t_i} c_{t_i} e^{-y t_i}, \quad D = \sum_{t_i} t_i \frac{c_{t_i} e^{-y t_i}}{B} \]

\[- \frac{dB}{dy} = \frac{1}{B} \left( - \sum_{t_i} c_{t_i} e^{-y t_i} (-t_i) \right) \]

\[ = \frac{1}{B} \left( \sum_{t_i} t_i c_{t_i} e^{-y t_i} \right) \]

\[ = \sum_{t_i} t_i \frac{c_{t_i} e^{-y t_i}}{B} \]

\[ = D \]
Duration Continued

- When the yield $y$ is expressed with compounding $m$ times per year

$$- \frac{dB/B}{dy} = \frac{1}{1 + y/m} D$$

- The expression

$$D^* = \frac{1}{1 + y/m} D$$

is referred to as the “modified duration”
Duration Continued

\[ B = \sum_{t_i} \frac{c_{t_i}}{(1 + y / m)^{m \cdot t_i}}, \quad D = \sum_{t_i} t_i \frac{c_{t_i}/(1 + y / m)^{m \cdot t_i}}{B} \]

\[- \frac{dB}{dy} / B = \frac{1}{B} \left( - \sum_{t_i} \frac{c_{t_i}}{(1 + y / m)^{m \cdot t_i + 1}} \cdot (-m \cdot t_i) \cdot \left( \frac{1}{m} \right) \right)\]

\[= \frac{1}{B} \left( \sum_{t_i} t_i \frac{c_{t_i}}{(1 + y / m)^{m \cdot t_i + 1}} \right)\]

\[= \frac{1}{1 + y/m} \left( \sum_{t_i} t_i \frac{c_{t_i}/(1 + y / m)^{m \cdot t_i}}{B} \right)\]

\[= \frac{1}{1 + y/m} D\]

\[D^* = - \frac{dB}{dy} / B = \frac{1}{1 + y/m} D\]
Duration Matching

- This involves hedging against interest rate risk by matching the durations of assets and liabilities
- It provides protection against small parallel shifts in the zero curve
Convexity

The convexity of a bond is defined as

\[ C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}}{B} \]

so that

\[ \frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2 \]
Theories of the Term Structure
Page 93

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates
  - Investors prefer to preserve their liquidity and invest funds for short periods of time
  - Borrowers prefer to borrow at fixed rates for long periods of time
  - Banks raise long-term rate relative to expected future short-term rate to reduce the demand for long-term fixed-rate borrowing and encourages investors to deposit their fund for long terms