數值分析
Chapter 8
Approximation Theory
8.2 Discrete Least Squares Approximation

“Best Fit”: it needs not agree precisely with the data at any point.

1. \( \min(\max |y_i - (a_1 x_i + a_0)|) \)
2. \( \min \sum |y_i - (a_1 x_i + a_0)| \)
3. \( \min \sum (y_i - (a_1 x_i + a_0))^2 \)

Linear Least Squares

The general problem of fitting the best least squares line to a collection of data \( \{(x_i, y_i)\}_{i=1}^{m} \) involves minimizing the total error \( E_2(a_0, a_1) = \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0))^2 \) with respect to the parameters \( a_0 \) and \( a_1 \). For a minimum to occur, we need

\[
0 = \frac{\partial}{\partial a_0} \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0))^2 = 2 \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0)) (-1)
\]

and

\[
0 = \frac{\partial}{\partial a_1} \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0))^2 = 2 \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0)) (-x_i)
\]

These equations simplify to the normal equations

\[
a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i \quad \text{and} \quad a_0 m + a_1 \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i
\]

The linear least squares solution for a given collection of data \( \{(x_i, y_i)\}_{i=1}^{m} \) has the form \( y = a_1 x + a_0 \), where

\[
a_0 = \frac{\sum_{i=1}^{m} x_i^2 (\sum_{i=1}^{m} y_i) - (\sum_{i=1}^{m} x_i y_i)(\sum_{i=1}^{m} x_i)}{m(\sum_{i=1}^{m} x_i^2) - (\sum_{i=1}^{m} x_i)^2}
\]

and

\[
a_1 = \frac{m(\sum_{i=1}^{m} x_i y_i) - (\sum_{i=1}^{m} x_i)(\sum_{i=1}^{m} y_i)}{m(\sum_{i=1}^{m} x_i^2) - (\sum_{i=1}^{m} x_i)^2}
\]

P.326 Example 1
• Polynomial Least Squares

The problem of approximating a set of data, \( \{(x_i, y_i)|i = 1, 2, ..., m\} \) with

\[ P_n(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \]

of degree \( n < m - 1 \) using least squares is handled in a similar manner. It requires choosing the constants \( a_0, a_1, \ldots, a_n \) to minimize the total least squares error:

\[ E_2 = \sum_{i=1}^{m} (y_i - P_n(x_i))^2 \]

For \( E_2 \) to be minimized, it is necessary that \( \frac{\partial E_2}{\partial a_j} = 0 \) for each \( j = 0, 1, ..., n \). This gives \( n+1 \) normal equations in the \( n+1 \) unknowns, \( a_j \),

\[
\begin{align*}
    a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \ldots + a_n \sum_{i=1}^{m} x_i^n &= \sum_{i=1}^{m} y_i x_i^0 \\
    a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + \ldots + a_n \sum_{i=1}^{m} x_i^{n+1} &= \sum_{i=1}^{m} y_i x_i^1 \\
    \vdots & \quad \vdots \\
    a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + \ldots + a_n \sum_{i=1}^{m} x_i^{2n} &= \sum_{i=1}^{m} y_i x_i^n
\end{align*}
\]

P.328 Example 2.

• Least Square of Arbitrary Function \( \phi_j(x) \)

\[
\min_{a_j} \sum_{i=1}^{m} [y_i - a_n \phi_n(x_i) - a_{n-1} \phi_{n-1}(x_i) - \ldots - a_0 \phi_0(x_i)]^2
\]

\[
\frac{\partial}{\partial a_j} = 2 \sum_{i=1}^{m} [y_i - a_n \phi_n(x_i) - a_{n-1} \phi_{n-1}(x_i) - \ldots - a_0 \phi_0(x_i)] (-\phi_j(x_i)) = 0
\]
\[ \sum_{k=0}^{n} a_k \left[ \sum_{i=1}^{m} \phi_k(x_i) \phi_j(x_i) \right] = \sum_{i=1}^{m} y_i \phi_j(x_i) \]

\[
\begin{bmatrix}
\sum_{i=1}^{m} \phi_0(x_i) \phi_j(x_i) \\
\sum_{i=1}^{m} \phi_1(x_i) \phi_j(x_i) \\
\vdots \\
\sum_{i=1}^{m} \phi_n(x_i) \phi_j(x_i)
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_j \\
\vdots \\
a_n
\end{bmatrix}
= 
\begin{bmatrix}
\sum_{i=1}^{m} y_i \phi_0(x_i) \\
\sum_{i=1}^{m} y_i \phi_1(x_i) \\
\vdots \\
\sum_{i=1}^{m} y_i \phi_j(x_i) \\
\vdots \\
\sum_{i=1}^{m} y_i \phi_n(x_i)
\end{bmatrix}
\]

8.3 Continuous Least Squares Approximation

• 不只是 m 個點逼近，而是整個函數逼近，所以並非已知 \((x_i, y_i)\)，而是已知 \(f(x)\) 與區間 \([a, b] \]

• To minimize the error

\[
E(a_0, a_1, \ldots, a_n) = \int_{a}^{b} (f(x) - P_n(x))^2 \, dx = \int_{a}^{b} (f(x) - \sum_{k=0}^{n} a_k x^k)^2 \, dx
\]

A necessary condition for the numbers \(a_0, a_1, \ldots, a_n\) to minimize the total error \(E\) is that

\[
\frac{\partial E}{\partial a_j} = 0 \text{ for each } j = 0, 1, \ldots, n
\]
We can expand the integrand in this expression to
\[ E = \int_{a}^{b} [f(x)]^2 dx - 2 \sum_{k=0}^{n} a_k \int_{a}^{b} x^k f(x) dx + \int_{a}^{b} \left( \sum_{k=0}^{n} a_k x^k \right)^2 dx \]
so
\[ \frac{\partial E}{\partial a_j} = -2 \int_{a}^{b} x^j f(x) dx + 2 \sum_{k=0}^{n} a_k \int_{a}^{b} x^{j+k} dx \]
\[ \Rightarrow \sum_{k=0}^{n} a_k \int_{a}^{b} x^{j+k} dx = \int_{a}^{b} x^j f(x) dx, \text{ for each } j = 0, 1, ..., n \]

P.333 Example 1

• The set of function \( \{ \phi_0, \phi_1...\phi_n \} \) is said to be linearly independent on \([a, b]\) if,
\[ c_0 \phi_0(x) + c_1 \phi_1(x) + ... + c_n \phi_n(x) = 0 \text{ for all } x \in [a, b] \]
only when \( c_0 = c_1 = ... = c_n = 0 \)

• Weight function (assign varying degrees of importance to approximations on certain portions of the intervals)
\[ w(x) = \frac{1}{\sqrt{1-x^2}} \text{ on } [-1, 1] \]

Suppose \( \{ \phi_0, \phi_1, ..., \phi_n \} \) is a set of linearly indep functions on \([a, b]\), \( w \) is a weight function for \([a, b]\), and, for \( f \in C[a, b] \), a linear combination
\[ P(x) = \sum_{k=0}^{n} a_k \phi_k(x) \]
is sought to minimize the error
\[ E(a_0, a_1, ..., a_n) = \int_{a}^{b} w(x) |f(x) - \sum_{k=0}^{n} a_k \phi_k(x)|^2 dx \]

This problem reduces to the situation considered at the beginning of this section in the special case when \( w(x) \equiv 1 \) and \( \phi_k(x) = x^k \).
FOC $\Rightarrow 0 = \frac{\partial E}{\partial a_j} = 2 \int_a^b w(x)[f(x) - \sum_{k=0}^n a_k \phi_k(x)]\phi_j(x)dx$

$\Rightarrow \int_a^b w(x)f(x)\phi_j(x)dx = \sum_{k=0}^n a_k \int_a^b w(x)\phi_k(x)\phi_j(x)dx$ for each $j = 0, 1, ..., n$

The set of functions $\{\phi_0, \phi_1, ..., \phi_n\}$ is said to be orthogonal for the interval $[a, b]$ with respect to the weight function $w$ if

$$\int_a^b w(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_k > 0, & \text{when } j = k \end{cases}$$

$\Rightarrow \int_a^b w(x)f(x)\phi_j(x)dx = a_j \int_a^b w(x)[\phi_j(x)]^2dx = a_j\alpha_j$

$\Rightarrow a_j = \frac{1}{\alpha_j} \int_a^b w(x)f(x)\phi_j(x)dx$

$$= \int_a^b w(x)f(x)\phi_j(x)dx \left(= \frac{\int_a^b w(x)^2dx}{\alpha_j}\right)$$

- Recursive Generation of Orthogonal Polynomials

The set of polynomials $\{\phi_0, \phi_1, ..., \phi_n\}$ defined in the following way is linearly independent and orthogonal on $[a, b]$ with respect to the weight function $w$

$\phi_0(x) \equiv 1$, $\phi_1(x) = x - B_1$

where

$$B_1 = \frac{\int_a^b xw(x)[\phi_0(x)]^2dx}{\int_a^b w(x)[\phi_0(x)]^2dx}$$

and when $k \geq 2$

$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x)$ (recursive equation) where

$$B_k = \frac{\int_a^b xw(x)[\phi_{k-1}(x)]^2dx}{\int_a^b w(x)[\phi_{k-1}(x)]^2dx}$$ and $C_k = \frac{\int_a^b xw(x)\phi_{k-1}(x)\phi_{k-2}(x)dx}{\int_a^b w(x)[\phi_{k-2}(x)]^2dx}$

Moreover, for any polynomial $\phi_k(x)$ of degree $k < n$

$$\int_a^b w(x)\phi_n(x)\phi_k(x)dx = 0$$
EX 3. \( \{P_n(x)\} \) is orthogonal on \([-1,1]\) with respect to the weight function \( w(x) \equiv 1 \). \( P_n(1) = 1 \) for each \( n \). Using the recursive procedure, \( P_0(x) \equiv 1 \), so

\[
B_1 = \frac{\int_{-1}^{1} x \, dx}{\int_{-1}^{1} dx} = 0 \quad \text{and} \quad P_1(x) = (x - B_1)P_0(x) = x
\]

Also

\[
B_2 = \frac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} x^2 \, dx} = 0 \quad \text{and} \quad C_2 = \frac{\int_{-1}^{1} x^2 \, dx}{\int_{-1}^{1} dx} = \frac{1}{3}
\]

so

\[
P_2(x) = (x - B_2)P_1(x) - C_2P_0(x) = (x - 0)x - \frac{1}{3} \times 1 = x^2 - \frac{1}{3}x
\]

\[
B_3 = 0, \quad C_3 = \frac{4}{15} \Rightarrow P_3(x) = xP_2(x) - \frac{4}{15}P_1(x) = x^3 - \frac{3}{5}x
\]

\[
P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}
\]

\[
P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x
\]

8.4 Chebyshev Polynomials

- \( \{T_n(x)\} \) are orthogonal on \((-1, 1)\) with respect to the weight function \( w(x) = (1 - x^2)^{-1/2} \)

For \( x \in [-1,1] \), define

\[
T_n(x) = \cos(n \cos^{-1} x) \quad \text{for each} \quad n \geq 0
\]

\[
T_0(x) = \cos 0 = 1 \quad \text{and} \quad T_1(x) = \cos(\cos^{-1} x) = x
\]

\( \theta = \cos^{-1} x, \quad T_n(\theta(x)) \equiv T_n(\theta) = \cos(n\theta), \quad \text{where} \quad \theta \in [0, \pi] \)

(因 \( x \in (-1, 1) \)

because \( T_{n+1}(\theta) = \cos(n\theta + \theta) = \cos(n\theta)\cos \theta - \sin(n\theta)\sin \theta \)

and

\[
T_{n-1}(\theta) = \cos(n\theta - \theta) = \cos(n\theta)\cos \theta + \sin(n\theta)\sin \theta
\]

therefore \( T_{n+1}(\theta) = 2\cos(n\theta)\cos \theta - T_{n-1}(\theta) \)

\Rightarrow \quad T_{n+1} = 2\cos(n\cos^{-1} x)x - T_{n-1}(x) = 2T_n(x)x - T_{n-1}(x)

Since \( T_0(x) \) and \( T_1(x) \) are both polynomials in \( x \), \( T_{n+1}(x) \) will be a polynomial in \( x \) for each \( n \)

P.341 Figure 8.8
To show the orthogonal of the Chebyshev polynomials, consider

\[ \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \, dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)}{\sqrt{1-x^2}} \, dx \]

\( \theta = \cos^{-1}x \Rightarrow d\theta = \frac{1}{\sqrt{1-x^2}} \, dx \) (三角形圖示)

and

\[ \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \, dx = -\int_{0}^{\pi} \cos(n\theta)\cos(m\theta) \, d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta) \, d\theta \]

\[ \| \cos(n\theta)\cos(m\theta) \, d\theta = \frac{1}{2} [\cos((n+m)\theta) + \cos((n-m)\theta)] \]

\[ \Rightarrow \int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \, dx = \frac{1}{2} \int_{0}^{\pi} \cos((n+m)\theta) \, d\theta + \frac{1}{2} \int_{0}^{\pi} \cos((n-m)\theta) \, d\theta \]

\[ = \left[ \frac{1}{2(n+m)} \sin(n+m)\theta + \frac{1}{2(n-m)} \sin(n-m)\theta \right]_{0}^{\pi} = 0 \]

Similarly, \( \int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} \) for each \( n \geq 1 \)

Zeros and Extrema of Chebyshev Polynomials

The Chebyshev polynomial \( T_n(x) \), of degree \( n \geq 1 \), has \( n \) simple zeros in \([-1,1]\) at

\[ x_k = \cos(\frac{2k-1}{2n}\pi) \] for each \( k = 1, 2, ..., n \)

Moreover, \( T_n \) assumes its absolute extrema at

\[ x_k' = \cos(\frac{k\pi}{n}) \] with \( T_n(x_k') = (-1)^{-k} \) for each \( k = 0, 1, ..., n \)

P.341 Figure 8.8

\[
\begin{align*}
\text{if } n &= 1 \Rightarrow \, k = 1 \quad \Rightarrow \, x_1 = \cos \frac{\pi}{2} = 0 \\
& \quad k = 0, 1 \quad \Rightarrow \, x_0' = \cos 0, x_1' = \cos \pi = -1
\end{align*}
\]

\[
\begin{align*}
\text{if } n &= 2 \Rightarrow k = 1, 2 \quad \Rightarrow \, x_1 = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, x_2 = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}
\end{align*}
\]

\[
\begin{align*}
& \quad k = 0, 1, 2 \quad \Rightarrow \, x_0' = \cos 0 = 1, x_1' = \cos \frac{\pi}{2} = 0, \\
& \quad x_2' = \cos \pi = -1
\end{align*}
\]
monic Chebyshev polynomial (使 leading coefficient = 1)

\[ T_0 = 1, \ T_n = \frac{1}{2^n} T_n(x) \]

此時 zeros 與 extrem 之 \( x_k \) 與 \( \bar{x}_k \) 與原本之 Chebyshev 都一樣. 但極值會隨 \( n \uparrow \) 而 \( \downarrow \), \( \bar{T}_n(x_k') = \frac{(-1)^k}{2^n} \)

(最大最小值之變動會越來越小)

- The error form for the Lagrange polynomial applied to the interval \([-1, 1]\) states that if \( x_0, x_1, \ldots, x_n \) are distinct numbers in the interval \([-1, 1]\) and if \( f \in C^{n+1}[-1, 1] \), then, each \( x \in [-1, 1] \), a number \( \xi(x) \) exists in \((-1, 1)\) with

\[
    f(x) - P(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)\ldots(x - x_n)
\]

where \( P(x) \) is the Lagrange interpolating polynomial. There is no control over \( \xi(x) \), so to minimize the error by shrewd placement of the nodes \( x_0, x_1, \ldots, x_n \) is equivalent to choosing \( x_0, x_1, \ldots, x_n \) to minimize the quantity

\[
    |(x - x_0)\ldots(x - x_n)|
\]

throughout the interval \([-1, 1]\)

此時若取 \( x_0, x_1, \ldots, x_n \) 爲 zeros of \( \bar{T}_{n+1}(x) \)

\[
    (x - \bar{x}_0)(x - \bar{x}_1)\ldots(x - \bar{x}_n) = \bar{T}_{n+1}(x) \quad \text{(因 \( \bar{T}_{n+1}(x) \) 爲 n 次 polynomial , 且在 \( \bar{x}_0, \bar{x}_1\ldots, \bar{x}_n \) 都為 0)}
\]

且因 \( \max_{x \in [-1, 1]} |\bar{T}_{n+1}| = \frac{1}{2^n} \)

\[
    \Rightarrow \max_{x \in [-1, 1]} |f(x) - P(x)| \leq \frac{1}{2^{n+1}} \cdot \max_{x \in [-1, 1]} |f^{n+1}(x)|
\]

- The technique for choosing points to minimize the interpolating error can be easily extended to a general closed interval \([a, b]\) by using the change if variable

\[
    \tilde{x}_k = \frac{1}{2}[(b - a)\bar{x}_k + a + b]
\]

to transform the numbers \( \bar{x}_k \) in the interval \([-1, 1]\) into the corresponding numbers in the interval \([a, b] \)

P.344 Example 1, P.345 Table 8.4
8.5 Rational Function Approximation

- A rational function $r$ of degree $N$ has the form
  \[ r(x) = \frac{p(x)}{q(x)} \]
  where $p(x)$ and $q(x)$ are polynomials whose degree sum to $N$.

Consider the Padé approximation technique:

\[ f(x) - r(x) = f(x) - \frac{p(x)}{q(x)} = \frac{f(x)q(x) - p(x)}{q(x)} = \frac{f(x) \sum_{i=0}^{m} q_{i} x^{i} - \sum_{i=0}^{n} p_{i} x^{i}}{q(x)} \]

This is the Maclaurin Expansion, and we wish to find $q_{i}, p_{i}, \text{s.t. } f^{(k)}(0) - r^{(k)}(0) = 0$, for $k=0,1,...,N$.

- Ex1. The Maclaurin series expansion for $e^{-x}$ is $\sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!} x^{i}$.

To find the Padé approximation to $e^{-x}$ of degree 5 with $n = 3$ and $m = 2$ requires choosing $p_{0}, p_{1}, p_{2}, p_{3}, q_{1}$, and $q_{2}$ so that the coefficients of $x^{k}$ for $k = 0, 1, ..., 5$ are zero in the expression:

\[ (1 - x + \frac{x^{2}}{2} - \frac{x^{3}}{6} + \ldots) (1 + q_{1} x + q_{2} x^{2}) - (p_{0} + p_{1} x + p_{2} x^{2} + p_{3} x^{3}) \]

Solving the system of equations:

\[ p_{0} = 1, \quad p_{1} = \frac{-3}{5}, \quad p_{2} = \frac{3}{20}, \quad p_{3} = \frac{-1}{60}, \quad q_{1} = \frac{2}{5}, \quad q_{2} = \frac{1}{20} \]

so the Padé approximation is:

\[ r(x) = \frac{1 - \frac{3}{5} x + \frac{3}{20} x^{2} - \frac{1}{60} x^{3}}{1 + \frac{2}{5} x + \frac{1}{20} x^{2}} \]

P.349 Table 8.5 ($r(x)$ vs. $P_{5}(x)$)
Using nested multiplication:

\[ P_5(x) = (((1/120)x + 1/24)x - 1/6)x + 1 \]

5个“×”，5个“+”、“-”

\[ r(x) = \frac{((1/60)x + 3/20)x + 1}{(5/6)x + 1} \]

5个“×”，5个“+”、“-”，1个“/”

(continuous-fraction 可增進計算效率)

用 \( T_k \) 取代 Pade approximation 中的 \( x_k \), 形成 general Cheby-
shev rational function.

8.6 Trigonometric Polynomial Approximation

- 適合用在周期性函數之估計.

- For each positive \( n \), the set \( \tau_n \) of trigonometric polynomials of degree less than or equal to \( n \) is the set of all linear combinations of \( \{\phi_0, \phi_1, ..., \phi_{2n-1}\} \), where
  \[
  \phi_0(x) = \frac{1}{\sqrt{2\pi}} (\text{共1個})
  \]
  \[
  \phi_k(x) = \frac{1}{\sqrt{\pi}} \cos kx \text{ for each } k = 1, 2, ..., n \text{ (共 n 個)}
  \]
  \[
  \phi_{n+k}(x) = \frac{1}{\sqrt{\pi}} \sin kx \text{ for each } k = 1, 2, ..., n - 1 \text{ (共 n-1個)}
  \]

\[
\{\phi_0, \phi_1, ..., \phi_{2n-1}\} \text{ is orthonormal on } [-\pi, \pi]
\]

with respect to the weight function \( w(x) \equiv 1 \)

If \( k \neq j \) and \( j \neq 0 \)

\[
\int_{-\pi}^{\pi} \phi_{n+k}(x)\phi_j(x)dx
\]

\[
= \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin kx \frac{1}{\sqrt{\pi}} \cos jxdx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \cos jxdx
\]

\[
= (\text{by積合合差}) \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sin(k+j)x + \sin(k-j)x]dx
\]

\[
= \frac{1}{2\pi \pi} [\frac{-
\cos(k+j)x}{k+j} + \frac{\cos(k-j)x}{k-j}]_{-\pi}^{\pi} = 0
\]

其他之證明類似
Given \( f \in C[-\pi, \pi] \), the continuous least squares approximation by function in \( \tau_n \) is defined by
\[
S_n(x) = \frac{2n-1}{\pi} \sum_{k=0}^{2n-1} a_k \phi_k(x)
\]
(if \( n \to \infty \), \( S_n(x) \) is called the Fourier series of \( f \))
where
\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \phi_k(x) \, dx \quad \text{for each} \quad k = 0, 1, \ldots, 2n - 1
\]
(P.337)
\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} w(x) \phi_k(x) f(x) \, dx \quad \text{where} \quad w(x) = 1
\]
\[
\alpha_k = \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \phi_j(x) \, dx \right)^2
\]
\[
\alpha_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos 2kx + \cos 0) \, dx
\]
\[
= \frac{1}{2} \left( \int_{-\pi}^{\pi} \frac{\sin 2kx}{2k} + x \right) \bigg|_{-\pi}^{\pi}
\]
\[
\alpha_k = \frac{1}{2}
\]

- **Ex 1.** \( f(x) = |x| \) for \(-\pi < x < \pi\)
  \[
a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left( \frac{1}{\sqrt{2\pi}} \right) dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi} x \, dx = \frac{\sqrt{2\pi}^2}{2\sqrt{\pi}}
\]
  \[
a_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \cos kx \, dx = \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} x \cos kx \, dx
\]
  \[
  = \frac{2}{\sqrt{\pi}k^2} \left( (-1)^k - 1 \right) , \text{for each} \ k = 1, 2, \ldots, n
\]
  \[
b_k = a_{n+k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} |x| \sin kx \, dx = 0
\]
  \[
  \Rightarrow S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k^2} \cos kx
\]

- 之前的 \( S_n(x) \) 是 match 整個 \( f(x) \), 但 Trigonometric Polynomial 也可用在 discrete varsion least square (意即 match 點, 而非 match \( f(x) \))
\[ x_j = -\pi + \frac{j}{m}\pi, \text{ for } j = 0, 1, \ldots, 2m - 1 \]

- 取 \( S_n(x) = \sum_{k=0}^{2n-1} a_k \phi_k(x) \)
  其中 \( \hat{\phi}_0(x) = \frac{1}{2} \)
  \( \hat{\phi}_k(x) = \cos kx, \ k = 1, 2, \ldots, n \)
  \( \hat{\phi}_{n+k}(x) = \sin kx, \ k = 1, 2, \ldots, n - 1 \)

- \( \min_{a_k,b_k} \sum_{j=0}^{2m-1} \left( y_j - \left[ \frac{a_0}{2} + a_n \cos nx_j + \sum_{k=1}^{n-1} (a_k \cos kx_j + b_k \sin kx_j) \right] \right)^2 \)

  因 \( \text{equally space in } [-\pi, \pi] \)
  \( \Rightarrow \sum_{j=0}^{2m-1} \hat{\phi}_k(x_j) \hat{\phi}_j(x_j) = 0 \)
  \( \Rightarrow \text{orthogonal set of functions} \)
  \( \Rightarrow a_k = \frac{\sum_{j=0}^{2m-1} \phi_k(x_j)f(x_j)}{\sum_{j=0}^{2m-1} \phi_k(x_j)^2} \)（其中 \( \sum_{j=0}^{m} \phi_k(x_j)^2 = m \)）

\[ a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j \]
\[ b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j \]
8.7 Fast Fourier Transforms

- 加速上小節用 Trigonometric Polynomial 來做 discrete version least square approximation.

- \( S_n(x) = \frac{a_0}{2} + a_n \cos nx + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) \)

where
\[
a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j \text{ for } k = 0, 1, ..., n
\]

and
\[
b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j \text{ for } k = 1, ..., n - 1
\]

Use the form with \( n = m \) for interpolation if we make a minor modification. Replacing the term \( a_m \) with \( \frac{a_m}{2} \)

\[\Rightarrow S_m(x) = \frac{a_0 + a_m \cos mx}{2} + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx) \]

The nodes are given, for each \( j = 0, 1, ..., 2m - 1 \), by
\[x_j = -\pi + \left(\frac{j}{m}\right)\pi\]

and the coefficients, for each \( k = 0, 1, ..., m \), as
\[
a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos kx_j \text{ and } b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin kx_j
\]

用 Trigonometric Polynomial, 需 2m 個點 ⇒ 計算量太多, 且 round off error 也太大.

Instead of directly evaluating the constants \( a_k \) and \( b_k \), the Fast Fourier Transform (FFT) procedure computes the complex coefficients \( c_k \) in the formula
\[
F(x) = \frac{1}{m} \sum_{k=0}^{2m-1} c_k e^{ikx}
\]

where
\[ c_k = \frac{2^{m-1}}{\sum_{j=0}^{y_j e^{\pi i j k/m}} j} \text{ for each } k = 0...2^m - 1 \]

\[ \| e^{iz} = \cos z + i \sin z \]

\[ \Rightarrow \frac{1}{m} c_k e^{-i\pi k} = \frac{1}{m} \sum_{j=0}^{2^{m-1} - 1} y_j e^{\pi i j k/m} e^{-i\pi k} = \frac{1}{m} \sum_{j=0}^{2^{m-1} - 1} y_j e^{ik(-\pi + (\pi j/m))} = \frac{1}{m} \sum_{j=0}^{2^{m-1} - 1} y_j (\cos kx_j + i \sin kx_j) \]

\[ \Rightarrow \frac{1}{m} c_k e^{-i\pi k} = a_k + ib_k \]

**Ex 1.** \( m = 2 \) and \( x_j = -\pi + (\frac{j}{2})\pi \) for \( j = 0, 1, 2, 3 \)

The polynomial is given by

\[ S_2(x) = a_0 + a_2 \cos 2x + a_1 \cos x + b_1 \sin x \]

where the coefficient are

\[ a_0 = \frac{1}{2} \left[ y_0 \cos 0 + y_1 \cos 0 + y_2 \cos 0 + y_3 \cos 0 \right] \]

\[ a_1 = \frac{1}{2} \left[ y_0 \cos x_0 + y_1 \cos x_1 + y_2 \cos x_2 + y_3 \cos x_3 \right] \]

\[ a_2 = \frac{1}{2} \left[ y_0 \cos 2x_0 + y_1 \cos 2x_1 + y_2 \cos 2x_2 + y_3 \cos 2x_3 \right] \]

\[ b_1 = \frac{1}{2} \left[ y_0 \sin x_0 + y_1 \sin x_1 + y_2 \sin x_2 + y_3 \sin x_3 \right] \]

**若用 FFT 來看:**

\[ c_0 = y_0 e^0 + y_1 e^0 + y_2 e^0 + y_3 e^0 \]

\[ c_1 = y_0 e^0 + y_1 e^{\pi i} + y_2 e^{\pi i} + y_3 e^{3\pi i} \]

\[ c_2 = y_0 e^0 + y_1 e^{\pi i} + y_2 e^{2\pi i} + y_3 e^{3\pi i} \]

\[ c_3 = y_0 e^0 + y_1 e^{3\pi i} + y_2 e^{3\pi i} + y_3 e^{9\pi i} \]

and \( a_k + ib_k = \frac{1}{2} c_k e^{-i\pi k} \)

\[ a_0 = \frac{1}{2} \text{Re}(c_0) \]
\[ a_1 = \frac{1}{2} \text{Re}(c_1 e^{-\pi i}) \]
\[ a_2 = \frac{1}{2} \text{Re}(c_2 e^{-2\pi i}) \]
\[ b_1 = \frac{1}{2} \text{Im}(c_1 e^{-\pi i}) \]

- **Discrete Least Square Approximation**

Given \((x_i, y_i), i = 1, \ldots, m\)

(i) 用 \( Y = a_1 X + a_0 \)

(ii) 用 \( Y = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \)

(iii) 用 \( Y = a_n \phi_n(X) + a_{n-1} \phi_{n-1}(X) + \cdots + a_1 \phi_1(X) + a_0 \phi_0(X) \)

用 \( n + 1 \) 個函數來逼近 \( j = 0, \ldots, n \)

\[
\min_{a_j, j=0, \ldots, n} \sum_{i=1}^{m} [y_i - a_n \phi_n(x_i) - a_{n-1} \phi_{n-1}(x_i) - \cdots - a_j \phi_j(x_i) - \cdots - a_0 \phi_0(x_i)]^2
\]

\[
\frac{\partial}{\partial a_j} = 2 \sum_{i=1}^{m} [y_i - a_n \phi_n(x_i) - a_{n-1} \phi_{n-1}(x_i) - \cdots - a_j \phi_j(x_i) - \cdots - a_0 \phi_0(x_i)][-\phi_j(x_i)] = 0
\]

\[
\Rightarrow \sum_{k=0}^{n} a_k [\sum_{i=1}^{m} \phi_k(x_i) \phi_j(x_i)] = \sum_{i=1}^{m} y_i \phi_j(x_i)
\]

- **Continuous Least Square Approximation**

Given \( f(x) \) on \([a, b]\)
(i) 用 n 次 polynomial 作 Approximation.

（可用，但缺點 1. n=2 之計算，n=3 不行用
2. n 增加，矩陣大，不好解）

解救法: 原本 n+1 equation 中都含有 n+1 變數，
現在希望每個 equation 中只有一個變數。
(ii) 用 orthogonal functions 即可達成
(Linearly independent sets of polynomials)

orthogonal functions \{ \text{Chebyshev Polynomials, Trignometric functions } \tau_n \}
若 \( n \to \infty \) \( \longrightarrow \) Fourier Series

- Rational Function Approximation
（可改善 polynomial 之 oscillation error）
可改善 Taylor series 之 error term (使其更小)。