數值分析
Chapter 2
Solutions of Equations of One Variable
2.2 The Bisection Method

• To begin the Bisection method, set $a_1 = a$ and $b_1 = b$, as shown in Figure 2.1, and let $p_1$ be the midpoint of the interval $[a, b]$:

1. define $p_1 = a_1 + \frac{b_1 - a_1}{2}$
2. If $f(p_1) = 0$, then the root $p$ is given by $p = p_1$;
   if $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
3. If $f(a_1)$ and $f(p_1)$ have the same sign, then $p$ is in the interval $(p_1, b_1)$, and we set $a_2 = p_1$ and $b_2 = b_1$
4. If, on the other hand, $f(p_1)$ and $f(a_1)$ have opposite signs, then $p$ is in the interval $(a_1, p_1)$, and we set $a_2 = a_1$ and $b_2 = p_1$

We rapply the process to the interval $[a_2, b_2]$, and continue forming $[a_3, b_3]$, $[a_4, b_4]$,...

(P.31 Figure 2.1) (P.32 ~ P.34) Example 1.

• Bisection Method

An interval $[a_{i+1}, b_{i+1}]$ containing an approximation to a root of $f(x) = 0$ is constructed from an interval $[a_i, b_i]$ containing the root by first letting

$p_i = a_i + \frac{b_i - a_i}{2}$

Then set

$a_{i+1} = a_i$ and $b_{i+1} = p_i$ if $f(a_i)f(p_i) < 0$
and
$a_{i+1} = p_i$ and $b_{i+1} = b_i$ otherwise
2.3 The Secant Method

- (Figure 2.4) Setting \( p_0 = a \) and \( p_1 = b \). The equation of the secant line through \((p_0, f(p_0))\) and \((p_1, f(p_1))\) is

\[
y = f(p_1) + \frac{f(p_1) - f(p_0)}{p_1 - p_0} (x - p_1)
\]

The x-intercept \((p_2,0)\) of this satisfies

\[
0 = f(p_1) + \frac{f(p_1) - f(p_0)}{p_1 - p_0} (p_2 - p_1)
\]

and solving for \( p_2 \) gives

\[
p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}
\]

- Secant Method

The approximation \( p_{n+1} \), for \( n > 1 \), to a root of \( f(x) = 0 \) is computed from the approximation \( p_n \) and \( p_{n-1} \) using the equation

\[
p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}
\]

P.37 Figure 2.4
P.38 Example 1

- Method of False Position

(結合 Secant Method 與 Bisection Method)

An interval \([a_{n+1}, b_{n+1}]\) for \( n > 1 \), containing an approximation to a root of \( f(x) = 0 \) is found from an interval \([a_n, b_n]\) containing the root by first computing

\[
p_{n+1} = a_n - \frac{f(a_n)(b_n - a_n)}{f(b_n) - f(a_n)}.
\]

Then set

\[
a_{n+1} = a_n \text{ and } b_{n+1} = p_{n+1} \text{ if } f(a_n)f(p_{n+1}) < 0
\]

(⇒ \([a_n, p_{n+1}]\))

and

\[
a_{n+1} = p_{n+1} \text{ and } b_{n+1} = b_n \text{ otherwise (⇒ \([p_{n+1}, b_n]\))}
\]

P.39 Figure 2.5 Secant method vs. False position
2.4 Newton’s Method

Newton’s Method

The approximation \( p_{n+1} \) to a root of \( f(x) = 0 \) is computed from the approximation \( p_n \) using the equation

\[
p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}
\]

Expanding \( f \) in the first Taylor polynomial at \( p_n \) and evaluating at \( x = p \) gives

\[
0 = f(p) = f(p_n) + f'(p_n)(p - p_n) + \frac{f''(\xi)}{2}(p - p_n)^2
\]

where \( \xi \) lies between \( p_n \) and \( p \). Consequently, if \( f'(p_n) \neq 0 \), we have

\[
p - p_n + \frac{f(p_n)}{f'(p_n)} = -\frac{f''(\xi)}{2f'(p_n)}(p - p_n)^2
\]

If \( f'' \) is bounded by \( M \) then

\[
|p - p_{n+1}| \leq -\frac{M}{2f'(p_n)} |p - p_n|^2 \Rightarrow \text{converge quadratically. (因每次的 error 會比之前 error 之平方還小)}
\]

Example 1

(i) If \( f'(p) \neq 0 \), then the method will converge if \( f''(p) \neq 0 \) at \( p \). P.46 Figure 2.7

(ii) 可能不收斂. 例: 反曲點.
2.5 Error Analysis and Accelerating Convergence

- Aitken’s $\Delta^2$ Method (用来加速 linearity convergent)

\[ \hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \]

(because $\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}$, 非鋸齒型逼近才有此性質)

P.51 Example 2 與 Table 2.8

2.6 Müller’s Method

- Müller’s Method: (二次之 Secant method) 有機會找到所有的根 (包括實數, 複數), 之前的方法除非開始就猜複數根, 否則是無法找出複數根的.

Given initial approximations $p_0$, $p_1$ and $p_2$, generate

\[ p_3 = p_2 - \frac{2c}{b + \text{sgn}(b) \sqrt{b^2 - 4ac}} \]

where

\[ c = f(p_2) \]
\[ b = \frac{(p_0 - p_2)^2[f(p_1) - f(p_2)] - (p_1 - p_2)^2[f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)} \]

and

\[ a = \frac{(p_1 - p_2)[f(p_0) - f(p_2)] - (p_0 - p_2)[f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)} \]

\[ \text{sgn}(b) \] 之原因: 原本應該有兩個解為 0, 取與 $b$ 同號使得分母不為 0 有幾乎為 0 之情況, 使得 $p_3$ 接近 $p_2$.

Then continue the iteration, with $p_1$, $p_2$ and $p_3$ replacing $p_0$, $p_1$ and $p_2$

P.55 Example 1
<table>
<thead>
<tr>
<th>Method</th>
<th>Initial interval containing the root</th>
<th>Continuity of $f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bisection</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>Secant</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>False Position</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Newton’s</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>Müller’s</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

**Others**
- Bisection: Robust; need a good initial interval; singulartority may be caught as if were a root; 一定收敛, 但速度很慢, 因只用到 sgn(f), 而没用到 f 之值.

- Secant: 未必收敛, 但如果收敛速度比 Bisection 快, 因为其为 Newton’s 一种变形.

- False Position: 未必比 Bisection 快.

- Newton’s: need a good initial guess; iterative convergence rate is high; 可用於 complex roots, if initial variable is a complex number.

- Müller’s: 適合来 approximation polynomial 之 roots, 因其 roots 通常是 complex number; need three initial points; 且 initial points 是 real number 依然可找到 complex roots; 但 three initial points 之選擇依然是問題; 效率比 Newton’s 差, 但比 Secant 好.

- 高階多根時, 每找到一個根, 就降階, 再繼續找下個根. (如此可以減少計算)

- 最後再將找到的所有根, 當作 Newton’s Method 之 initial guess, 放回原式中, 找到精準的根. (如此可去掉降階之誤差)

- BS formula, Hull, Ch 12.
• Implied Volatility.