數值分析
Chapter 1
Mathematical Preliminaries
and
Error Analysis
• Ex. 証明 $0.\bar{9} = 1$
  
  Proof: Let $x = 0.\bar{9}$
  
  $10x = 9.\bar{9}$
  
  $9x = 9$
  
  $\Rightarrow x = 1$

• This book examines problems that can be solved by methods of approximation, which we call numerical methods. For example, numerical differential.

• Ex. $f(x) = x^5 + 5x^4 + 3x^3 - 2x^2 + x - 1$, 求 $f(\pi)$.
  
  Sol: $(((x + 5)x + 3)x - 2)x + 1 - 1$
  
  (因 truncate 或 round off 之次數少, 所以誤差小)

• Continuity and Differentiability (P.2∼P.3)

1.2 Review of Calculus

• Mean Value Thorem
  
  If $f \in C[a,b]$ and $f$ is differentiable on $(a,b)$, then a number $c$ in $(a,b)$ exists such that (see P.4 Figure 1.3)
  
  $f'(c) = \frac{f(b) - f(a)}{b - a}$

• Extreme Value Thorem
  
  If $f \in C[a,b]$, then $c_1$ and $c_2$ in $[a,b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$ for all $x$ in $[a,b]$. If, in addition, $f$ is differentiable on $(a,b)$, then the number $c_1$ and $c_2$ occur either at endpoints of $[a,b]$ or where $f'$ is zero.
  
  (極值不是在邊界, 就是在 $f' = 0$)
• Mean Value Thorem for Integrals and Intermediate Value Theorem
If $f \in C[a, b]$, $g$ is integrable on $[a, b]$ and $g(x)$ does not change sign on $[a, b]$, then there exists a number $c$ in $(a, b)$ with
\[
\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx
\]
(可找到一個等面積之長方形)

• Taylor’s Thorem
Suppose $f \in C^n[a, b]$ and $f^{n+1}$ exists on $[a, b]$. Let $x_0$ be a number in $[a, b]$. For every $x$ in $[a, b]$ , there exists a number $\xi(x)$ between $x_0$ and $x$ with
\[
f(x) = P_n(x) + R_n(x)
\]
where
\[
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{n}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!}(x - x_0)^k
\]
and
\[
R_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}
\]
(用 $x_0$ 這個點上的各階動差來找出整個函數)
(P.9 Figure 1.7)

• $f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(y - y_0)^2 + \frac{\partial^2 f}{\partial x\partial y}(x - x_0)(y - y_0)$ (二維)
(Itô’s Lemma)
1.3 Round-off Error and Computer Arithmetic

- To save storage and provide a unique representation for each floating-point number, a normalization is imposed. Using this system gives a floating-point number of the form

\((-1)^s \times 2^{c-1023} \times (1 + f)\)

Consider for example, the machine number

\((s \ c \ f) = (\text{sign characteristic mantissa})\)

\[0 \ 10000000011 \ 101100100010000000000000000000000000000\]

The leftmost bit is zero, which indicates that the number is positive. The next 11 bits, 10000000011, giving the characteristic, are equivalent to the decimal number

\[c = 1 \times 2^{10} + 0 \times 2^9 + ... + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1024 + 2 + 1 = 1027\]

The exponential part of the number is, therefore, \(2^{1027-1023} = 2^4\). The final 52 bits specify that the mantissa is

\[f = 1 \times \left(\frac{1}{2}\right)^1 + 1 \times \left(\frac{1}{2}\right)^3 + 1 \times \left(\frac{1}{2}\right)^4 + 1 \times \left(\frac{1}{2}\right)^5 + 1 \times \left(\frac{1}{2}\right)^8 + 1 \times \left(\frac{1}{2}\right)^{12}\]

As a consequence, this machine number precisely represents the decimal number

\[(-1)^s \times 2^{c-1023} \times (1 + f) = (-1)^0 \times 2^{1027-1023}(1 + \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{256} + \frac{1}{4096}\right)) = 27.56640625\]

However, the next smaller machine number is

\[0 \ 10000000011 \ 101110010001000000000000000000000000001\]

and the next larger machine number is

\[0 \ 10000000011 \ 101110010001000000000000000000000000001\]

This means that our original machine number represents not only 27.56640625, but also half of the real numbers that are between 27.56640625 and its two nearest machine-number neighbors.
The smallest normalized positive number that can be represented has all zeros except for the rightmost bit of 1 and is equivalent to
\[ 2^{-1023}(1 + 2^{-52}) \approx 10^{-308} \]
and the largest has a leading 0 followed by all 1s and is equivalent to
\[ 2^{1024}(2 - 2^{-52}) \approx 10^{308} \]
Numbers occurring in calculations that have a magnitude less than \( 2^{-1023}(1 + 2^{-52}) \) result in underflow and are generally set to zero. Numbers greater than \( 2^{1024}(2 - 2^{-52}) \) result in overflow and typically cause the computation to halt.

Rounding and Chopping
(The error that results from replacing with its floating-point form is called round-off error)

1.4 Errors in Scientific Computation

Ex 1. \( x^2 + 62.10x + 1 = 0 \), whose roots are approximately \( x_1 = -0.0160723 \) and \( x_2 = -62.08390 \)
In this equation, \( b^2 \) is much larger than \( 4ac \), so the numerator in the calculation for \( x_1 \) involves the subtraction of nearly equal numbers. Since
\[ \sqrt{b^2 - 4ac} = \sqrt{(62.10)^2 - (4)(1)(1)} = 62.06 \]
we have
\[ fl(x_1) = \frac{-62.10 + 62.06}{2.000} = \frac{-0.0004}{2.000} = -0.0002 \]
(a poor approximation to \( x_1 = -0.01611 \) with the larger relative error
\[ \frac{|-0.01611 + 0.0002|}{|-0.01611|} = 2.4 \times 10^{-1} \]
On the other hand, the calculation for \( x_2 \) involves the addition of nearly equal numbers \(-b\) and \(-\sqrt{b^2 - 4ac}\). The presents no problem since

\[
fl(x_2) = \frac{-62.10 - 62.06}{2.000} = \frac{-124.2}{2.000} = -62.10,
\]

has the small relative error

\[
\frac{|-62.08 + 62.10|}{|62.08|} = 3.2 \times 10^{-4} \quad \text{(誤差很小)}
\]

要獲得更精確的 \( x_1 \) 逼近值，使用反有理化。

\[
x_1 = -\frac{b + \sqrt{b^2 - 4ac}}{2a} = -\frac{-2c}{b + \sqrt{b^2 - 4ac}}
\]

\[
\Rightarrow fl(x_1) = \frac{-2.000}{62.10 + 62.06} = -0.01610
\]

which has the small relative error \( 6.2 \times 10^{-4} \)

(反有理化後，誤差變小)

但同時，\( fl(x_2) \) 若也做反有理化，會從誤差小 —— 誤差大.