5.5 Applications of Inner Product Spaces

- Find the cross product of two vectors in $\mathbb{R}^3$.
- Find the linear or quadratic least squares approximation of a function.
- Find the $n$th-order Fourier approximation of a function.

THE CROSS PRODUCT OF TWO VECTORS IN $\mathbb{R}^3$

Here you will look at a vector product that yields a vector in $\mathbb{R}^3$ orthogonal to two vectors. This vector product is called the **cross product**, and it is most conveniently defined and calculated with vectors written in standard unit vector form.

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$  

**REMARK**

The cross product is defined only for vectors in $\mathbb{R}^3$. The cross product of two vectors in $\mathbb{R}^n$, $n \neq 3$, is not defined here.

**Definition of the Cross Product of Two Vectors**

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in $\mathbb{R}^3$. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$  

A convenient way to remember the formula for the cross product $\mathbf{u} \times \mathbf{v}$ is to use the following determinant form.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

```markdown
\text{Components of } \mathbf{u} \\
\text{Components of } \mathbf{v}
```

Technically this is not a determinant because it represents a vector and not a real number. Nevertheless, it is useful because it can help you remember the cross product formula. Using cofactor expansion in the first row produces

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k},$$

which yields the formula in the definition. Be sure to note that the $\mathbf{j}$-component is preceded by a minus sign.

**LINEAR ALGEBRA APPLIED**

In physics, the cross product can be used to measure **torque**—the moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point $A$, as shown in the figure below. When the point of application of the force is $B$, the moment of $\mathbf{F}$ about $A$ is given by

$$\mathbf{M} = \overrightarrow{AB} \times \mathbf{F}$$

where $\overrightarrow{AB}$ represents the vector whose initial point is $A$ and whose terminal point is $B$. The magnitude of the moment $\mathbf{M}$ measures the tendency of $\overrightarrow{AB}$ to rotate counterclockwise about an axis directed along the vector $\mathbf{M}$.
Finding the Cross Product of Two Vectors

Let \( u = i - 2j + k \) and \( v = 3i + j - 2k \). Find each cross product.

a. \( u \times v \)  
   
   **SOLUTION**
   
   a. \( u \times v = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 \end{vmatrix} - \begin{vmatrix} -2 \end{vmatrix} i - \begin{vmatrix} 1 \end{vmatrix} j + \begin{vmatrix} 1 \end{vmatrix} k = 3i + 5j + 7k \)

   Note that this result is the negative of that in part (a).

b. \( v \times u \)
   
   **SOLUTION**
   
   b. \( v \times u = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 \end{vmatrix} - \begin{vmatrix} 1 \end{vmatrix} i - \begin{vmatrix} 3 \end{vmatrix} j + \begin{vmatrix} 1 \end{vmatrix} k = -3i - 5j - 7k \)

   The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance, 
   
   \[ u \times v = -(v \times u) \quad \text{and} \quad v \times v = 0. \]

   These properties, along with several others, are stated in Theorem 5.17.

**EXAMPLE 1** Finding the Cross Product of Two Vectors

**TECHNOLOGY**

Many graphing utilities and software programs can find a cross product. For instance, if you use a graphing utility to verify the result of Example 1(b), then you may see something similar to the following.

**Simulation**

Explore this concept further with an electronic simulation, and for syntax regarding specific programs involving Example 1, please visit www.cengagebrain.com. Similar exercises and projects are also available on the website.

**THEOREM 5.17 Algebraic Properties of the Cross Product**

If \( u, v, \) and \( w \) are vectors in \( \mathbb{R}^3 \) and \( c \) is a scalar, then the following properties are true.

1. \( u \times v = -(v \times u) \)
2. \( u \times (v + w) = (u \times v) + (u \times w) \)
3. \( c(u \times v) = cu \times v = u \times cv \)
4. \( u \times 0 = 0 \times u = 0 \)
5. \( u \times u = 0 \)
6. \( u \cdot (v \times w) = (u \times v) \cdot w \)

**PROOF**

The proof of the first property is given here. The proofs of the other properties are left to you. (See Exercises 53–57.) Let \( u \) and \( v \) be

\[ u = u_1i + u_2j + u_3k \]

and

\[ v = v_1i + v_2j + v_3k. \]
5.5 Applications of Inner Product Spaces

THEOREM 5.18 Geometric Properties of the Cross Product

If \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors in \( \mathbb{R}^3 \), then the following properties are true.

1. \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).
2. The angle \( \theta \) between \( \mathbf{u} \) and \( \mathbf{v} \) is given by \( \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta \).
3. \( \mathbf{u} \) and \( \mathbf{v} \) are parallel if and only if \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \).
4. The parallelogram having \( \mathbf{u} \) and \( \mathbf{v} \) as adjacent sides has an area of \( \| \mathbf{u} \times \mathbf{v} \| \).

**Proof**

The proof of Property 4 follows. The proofs of the other properties are left to you. (See Exercises 58–60.) Let \( \mathbf{u} \) and \( \mathbf{v} \) represent adjacent sides of a parallelogram, as shown in Figure 5.28. By Property 2, the area of the parallelogram is

\[
\text{Area} = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = \| \mathbf{u} \times \mathbf{v} \|.
\]

Property 1 states that the vector \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \). This implies that \( \mathbf{u} \times \mathbf{v} \) (and \( \mathbf{v} \times \mathbf{u} \)) is orthogonal to the plane determined by \( \mathbf{u} \) and \( \mathbf{v} \). One way to remember the orientation of the vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{u} \times \mathbf{v} \) is to compare them with the unit vectors \( \mathbf{i} \), \( \mathbf{j} \), and \( \mathbf{k} \), as shown in Figure 5.29. The three vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{u} \times \mathbf{v} \) form a **right-handed system**, whereas the three vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{v} \times \mathbf{u} \) form a **left-handed system**.
EXAMPLE 2  Finding a Vector Orthogonal to Two Given Vectors

Find a unit vector orthogonal to both \( \mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \)
and
\( \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} \).

**SOLUTION**

From Property 1 of Theorem 5.18, you know that the cross product

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}
\]

is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \), as shown in Figure 5.30. Then, by dividing by the length of \( \mathbf{u} \times \mathbf{v} \),

\[
\| \mathbf{u} \times \mathbf{v} \| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}
\]

you obtain the unit vector

\[
\frac{\mathbf{u} \times \mathbf{v}}{\| \mathbf{u} \times \mathbf{v} \|} = \left( -\frac{3}{\sqrt{134}} \right) \mathbf{i} + \left( \frac{2}{\sqrt{134}} \right) \mathbf{j} + \left( \frac{11}{\sqrt{134}} \right) \mathbf{k}
\]

which is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \), because

\[
\left( -\frac{3}{\sqrt{134}} \right) \mathbf{i} + \left( \frac{2}{\sqrt{134}} \right) \mathbf{j} + \left( \frac{11}{\sqrt{134}} \right) \mathbf{k} \cdot (1, -4, 1) = 0
\]

and

\[
\left( -\frac{3}{\sqrt{134}} \right) \mathbf{i} + \left( \frac{2}{\sqrt{134}} \right) \mathbf{j} + \left( \frac{11}{\sqrt{134}} \right) \mathbf{k} \cdot (2, 3, 0) = 0.
\]

EXAMPLE 3  Finding the Area of a Parallelogram

Find the area of the parallelogram that has

\( \mathbf{u} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \)
and
\( \mathbf{v} = -2\mathbf{j} + 6\mathbf{k} \)
as adjacent sides, as shown in Figure 5.31.

**SOLUTION**

From Property 4 of Theorem 5.18, you know that the area of this parallelogram is \( \| \mathbf{u} \times \mathbf{v} \| \). Because

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}
\]

the area of the parallelogram is

\[
\| \mathbf{u} \times \mathbf{v} \| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19 \text{ square units.}
\]
LEAST SQUARES APPROXIMATIONS (CALCULUS)

Many problems in the physical sciences and engineering involve an approximation of a function \( f \) by another function \( g \). If \( f \) is in the inner product space of all continuous functions on \([a, b]\), then \( g \) is usually chosen from a subspace \( W \) of \( C[a, b] \).

For instance, to approximate the function

\[
f(x) = e^x, \quad 0 \leq x \leq 1
\]

you could choose one of the following forms of \( g \).

1. \( g(x) = a_0 + a_1x, \quad 0 \leq x \leq 1 \) \hspace{1cm} \text{Linear}
2. \( g(x) = a_0 + a_1x + a_2x^2, \quad 0 \leq x \leq 1 \) \hspace{1cm} \text{Quadratic}
3. \( g(x) = a_0 + a_1 \cos x + a_2 \sin x, \quad 0 \leq x \leq 1 \) \hspace{1cm} \text{Trigonometric}

Before discussing ways of finding the function \( g \), you must define how one function can “best” approximate another function. One natural way would require the area bounded by the graphs of \( f \) and \( g \) on the interval \([a, b]\) to be a minimum with respect to other functions in the subspace \( W \), as shown in Figure 5.32.

![Figure 5.32](image)

Because integrands involving absolute value are often difficult to evaluate, however, it is more common to square the integrand to obtain

\[
\int_{a}^{b} [f(x) - g(x)]^2 \, dx.
\]

With this criterion, the function \( g \) is called the least squares approximation of \( f \) with respect to the inner product space \( W \).

**Definition of Least Squares Approximation**

Let \( f \) be continuous on \([a, b]\), and let \( W \) be a subspace of \( C[a, b] \). A function \( g \) in \( W \) is called a least squares approximation of \( f \) with respect to \( W \) when the value of

\[
I = \int_{a}^{b} [f(x) - g(x)]^2 \, dx
\]

is a minimum with respect to all other functions in \( W \).

Note that if the subspace \( W \) in this definition is the entire space \( C[a, b] \), then \( g(x) = f(x) \), which implies that \( I = 0 \).
EXAMPLE 4  Finding a Least Squares Approximation

Find the least squares approximation $g(x) = a_0 + a_1 x$ of $f(x) = e^x$, $0 \leq x \leq 1$.

**SOLUTION**

For this approximation you need to find the constants $a_0$ and $a_1$ that minimize the value of

$$I = \int_0^1 [f(x) - g(x)]^2 \, dx$$

$$= \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx.$$  

Evaluating this integral, you have

$$I = \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx$$

$$= \int_0^1 (e^{2x} - 2a_0 e^x - 2a_1 x e^x + a_0^2 + 2a_0 a_1 x + a_1^2 x^2) \, dx$$

$$= \left[ \frac{1}{2} e^{2x} - 2a_0 e^x - 2a_1 x e^x + a_0^2 x + a_0 a_1 x^2 + a_1^2 x^3 \right]_0^1$$

$$= \frac{1}{2} (e^2 - 1) - 2a_0 (e - 1) - 2a_1 + a_0^2 + a_0 a_1 + \frac{1}{3} a_1^2.$$  

Now, considering $I$ to be a function of the variables $a_0$ and $a_1$, use calculus to determine the values of $a_0$ and $a_1$ that minimize $I$. Specifically, by setting the partial derivatives

$$\frac{\partial I}{\partial a_0} = 2a_0 - 2e + 2 + a_1$$

$$\frac{\partial I}{\partial a_1} = a_0 + \frac{2}{3} a_1 - 2$$

equal to zero, you obtain the following two linear equations in $a_0$ and $a_1$.

$$2a_0 + a_1 = 2(e - 1)$$

$$3a_0 + 2a_1 = 6$$

The solution of this system is

$$a_0 = 4e - 10 \approx 0.873 \quad \text{and} \quad a_1 = 18 - 6e \approx 1.690.$$  

(Verify this.) So, the best linear approximation of $f(x) = e^x$ on the interval $[0, 1]$ is

$$g(x) = 4e - 10 + (18 - 6e)x \approx 0.873 + 1.690x.$$  

Figure 5.33 shows the graphs of $f$ and $g$ on $[0, 1]$.

Of course, whether the approximation obtained in Example 4 is the best approximation depends on the definition of the best approximation. For instance, if the definition of the best approximation had been the Taylor polynomial of degree 1 centered at 0.5, then the approximating function $g$ would have been

$$g(x) = f(0.5) + f'(0.5)(x - 0.5)$$

$$= e^{0.5} + e^{0.5}(x - 0.5)$$

$$\approx 0.824 + 1.649x.$$  

Moreover, the function $g$ obtained in Example 4 is only the best linear approximation of $f$ (according to the least squares criterion). In Example 5 you will find the best quadratic approximation.
5.5 Applications of Inner Product Spaces

**EXAMPLE 5** Finding a Least Squares Approximation

Find the least squares approximation \( g(x) = a_0 + a_1x + a_2x^2 \) of \( f(x) = e^x, \ 0 \leq x \leq 1 \).

**SOLUTION**

For this approximation you need to find the values of \( a_0, a_1, \) and \( a_2 \) that minimize the value of

\[
I = \int_0^1 [f(x) - g(x)]^2 \, dx
\]

\[
= \int_0^1 (e^x - a_0 - a_1x - a_2x^2)^2 \, dx
\]

\[
= \frac{1}{2}(e^2 - 1) + 2a_0(1 - e) + 2a_2(2 - e)
\]

\[
+ a_0^2 + a_1a_1 + \frac{2}{3}a_0a_2 + \frac{1}{2}a_1^2 + \frac{1}{3}a_2^2 - 2a_1.
\]

Setting the partial derivatives of \( I \) (with respect to \( a_0, a_1, \) and \( a_2 \)) equal to zero produces the following system of linear equations.

\[
6a_0 + 3a_1 + 2a_2 = 6(e - 1)
\]

\[
6a_0 + 4a_1 + 3a_2 = 12
\]

\[
20a_0 + 15a_1 + 12a_2 = 60(e - 2)
\]

The solution of this system is

\[
a_0 = -105 + 39e = 1.013
\]

\[
a_1 = 588 - 216e = 0.851
\]

\[
a_2 = -570 + 210e = 0.839.
\]

(Verify this.) So, the approximating function \( g \) is \( g(x) = 1.013 + 0.851x + 0.839x^2 \). Figure 5.34 shows the graphs of \( f \) and \( g \).

The integral \( I \) (given in the definition of the least squares approximation) can be expressed in vector form. To do this, use the inner product defined in Example 5 in Section 5.2:

\[
\langle f, g \rangle = \int_a^b f(x)g(x) \, dx
\]

With this inner product you have

\[
I = \int_a^b [f(x) - g(x)]^2 \, dx = \langle f - g, f - g \rangle = \| f - g \|^2.
\]

This means that the least squares approximating function \( g \) is the function that minimizes \( \| f - g \|^2 \) or, equivalently, minimizes \( \| f - g \| \). In other words, the least squares approximation of a function \( f \) is the function \( g \) (in the subspace \( W \)) closest to \( f \) in terms of the inner product \( \langle f, g \rangle \). The next theorem gives you a way of determining the function \( g \).

**THEOREM 5.19** Least Squares Approximation

Let \( f \) be continuous on \([a, b]\), and let \( W \) be a finite-dimensional subspace of \( C[a, b] \).

The least squares approximating function of \( f \) with respect to \( W \) is given by

\[
g = \langle f, w_1 \rangle w_1 + \langle f, w_2 \rangle w_2 + \cdots + \langle f, w_n \rangle w_n
\]

where \( B = \{ w_1, w_2, \ldots, w_n \} \) is an orthonormal basis for \( W \).
PROOF
To show that $g$ is the least squares approximating function of $f$, prove that the inequality
\[ \|f - g\| \leq \|f - w\| \]
is true for any vector $w$ in $W$. By writing $f - g$ as
\[ f - g = f - (f, w_1)w_1 - (f, w_2)w_2 - \cdots - (f, w_n)w_n \]
you can see that $f - g$ is orthogonal to each $w_i$, which in turn implies that it is orthogonal to each vector in $W$. In particular, $f - g$ is orthogonal to $g - w$. This allows you to apply the Pythagorean Theorem to the vector sum $f - w = (f - g) + (g - w)$ to conclude that
\[ \|f - w\|^2 = \|f - g\|^2 + \|g - w\|^2. \]
So, it follows that $\|f - g\|^2 \leq \|f - w\|^2$, which then implies that $\|f - g\| \leq \|f - w\|$.

Now observe how Theorem 5.19 can be used to produce the least squares approximation obtained in Example 4. First apply the Gram-Schmidt orthonormalization process to the standard basis \(\{1, x\}\) to obtain the orthonormal basis $B = \{1, \sqrt{3}(2x - 1)\}$. (Verify this.) Then, by Theorem 5.19, the least squares approximation of $e^x$ in the subspace of all linear functions is
\[ g(x) = \langle e^x, 1 \rangle(1) + \langle e^x, \sqrt{3}(2x - 1) \rangle \sqrt{3}(2x - 1) \]
\[ = \int_0^1 e^x dx + \sqrt{3}(2x - 1) \int_0^1 e^x(2x - 1) dx \]
\[ = \int_0^1 e^x dx + 3(2x - 1) \int_0^1 e^x(2x - 1) dx \]
\[ = 4e - 10 + (18 - 6e)x \]
which agrees with the result obtained in Example 4.

**EXAMPLE 6** Finding a Least Squares Approximation

Find the least squares approximation of $f(x) = \sin x$, $0 \leq x \leq \pi$, with respect to the subspace $W$ of polynomial functions of degree 2 or less.

**SOLUTION**
To use Theorem 5.19, apply the Gram-Schmidt orthonormalization process to the standard basis for $W$, \(\{1, x, x^2\}\), to obtain the orthonormal basis
\[ B = \{w_1, w_2, w_3\} = \left\{ \frac{1}{\pi} \sqrt{\frac{3}{\pi}} (2x - \pi), \frac{-\sqrt{5}}{\pi \sqrt{\pi}} (6x^2 - 6\pi x + \pi^2) \right\}. \]
(Verify this.) The least squares approximating function $g$ is
\[ g(x) = \langle f, w_1 \rangle w_1 + \langle f, w_2 \rangle w_2 + \langle f, w_3 \rangle w_3 \]
and you have
\[ \langle f, w_1 \rangle = \frac{1}{\sqrt{\pi}} \int_0^\pi \sin x \ dx = \frac{2}{\sqrt{\pi}} \]
\[ \langle f, w_2 \rangle = \frac{\sqrt{3}}{\pi \sqrt{\pi}} \int_0^\pi \sin(2x - \pi) \ dx = 0 \]
\[ \langle f, w_3 \rangle = \frac{\sqrt{5}}{\pi^2 \sqrt{\pi}} \int_0^\pi \sin(x(6x^2 - 6\pi x + \pi^2)) \ dx = \frac{2\sqrt{5}}{\pi^2 \sqrt{\pi}} (\pi^2 - 12) \]
So, $g$ is
\[ g(x) = \frac{2}{\pi} + \frac{10(\pi^2 - 12)}{\pi^5} (6x^2 - 6\pi x + \pi^2) \approx -0.4177x^2 + 1.3122x - 0.0505. \]
Figure 5.35 shows the graphs of $f$ and $g$. 

![Figure 5.35](image_url)
FOURIER APPROXIMATIONS (CALCULUS)

You will now look at a special type of least squares approximation called a Fourier approximation. For this approximation, consider functions of the form

\[ g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx \]

in the subspace \( W \) of

\[ C[0, 2\pi] \]

spanned by the basis

\[ S = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}. \]

These \( 2n + 1 \) vectors are orthogonal in the inner product space \( C[0, 2\pi] \) because

\[ \langle f, g \rangle = \int_0^{2\pi} f(x)g(x) \, dx = 0, \quad f \neq g \]

as demonstrated in Example 3 in Section 5.3. Moreover, by normalizing each function in this basis, you obtain the orthonormal basis

\[ B = \{w_0, w_1, \ldots, w_n, w_{n+1}, \ldots, w_{2n}\} \]

\[ = \left\{ \frac{1}{\sqrt{2\pi}} \cos x, \ldots, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin x, \ldots, \frac{1}{\sqrt{\pi}} \sin nx \right\}. \]

With this orthonormal basis, you can apply Theorem 5.19 to write

\[ g(x) = \langle f, w_0 \rangle w_0 + \langle f, w_1 \rangle w_1 + \cdots + \langle f, w_{2n} \rangle w_{2n}. \]

The coefficients

\[ a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \]

for \( g(x) \) in the equation

\[ g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx \]

are given by the following integrals.

\[ a_0 = \langle f, w_0 \rangle \frac{2}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \left( \frac{1}{\sqrt{2\pi}} \right) dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \]

\[ a_1 = \langle f, w_1 \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \left( \frac{1}{\sqrt{\pi}} \right) \cos x \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx \]

\[ \vdots \]

\[ a_n = \langle f, w_n \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \left( \frac{1}{\sqrt{\pi}} \right) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \]

\[ b_1 = \langle f, w_{n+1} \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \left( \frac{1}{\sqrt{\pi}} \right) \sin x \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x \, dx \]

\[ \vdots \]

\[ b_n = \langle f, w_{2n} \rangle \frac{1}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \left( \frac{1}{\sqrt{\pi}} \right) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \]

The function \( g(x) \) is called the \( n \)-th-order Fourier approximation of \( f \) on the interval \([0, 2\pi]\). Like Fourier coefficients, this function is named after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830). This brings you to Theorem 5.20.
THEOREM 5.20  Fourier Approximation

On the interval $[0, 2\pi]$, the least squares approximation of a continuous function $f$ with respect to the vector space spanned by

\[ \{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\} \]

is

\[ g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx \]

where the Fourier coefficients $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are

\[ a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \]
\[ a_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx \, dx, \quad j = 1, 2, \ldots, n \]
\[ b_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin jx \, dx, \quad j = 1, 2, \ldots, n. \]

EXAMPLE 7  Finding a Fourier Approximation

Find the third-order Fourier approximation of $f(x) = x$, $0 \leq x \leq 2\pi$.

SOLUTION

Using Theorem 5.20, you have

\[ g(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \]

where

\[ a_0 = \frac{1}{\pi} \int_0^{2\pi} x \, dx = \frac{1}{\pi} 2\pi^2 = 2\pi \]
\[ a_j = \frac{1}{\pi} \int_0^{2\pi} x \cos jx \, dx = \left[ \frac{1}{\pi j^2} \cos jx + \frac{x}{\pi j} \sin jx \right]_0^{2\pi} = 0 \]
\[ b_j = \frac{1}{\pi} \int_0^{2\pi} x \sin jx \, dx = \left[ \frac{1}{\pi j^2} \sin jx - \frac{x}{\pi j} \cos jx \right]_0^{2\pi} = -\frac{2}{j}. \]

This implies that $a_0 = 2\pi$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $b_1 = -2$, $b_2 = -\frac{2}{2} = -1$, and $b_3 = -\frac{2}{3}$. So, you have

\[ g(x) = \frac{2\pi}{2} - 2 \sin x - \sin 2x - \frac{2}{3} \sin 3x \]
\[ = \pi - 2 \sin x - \sin 2x - \frac{2}{3} \sin 3x. \]

Figure 5.36 compares the graphs of $f$ and $g$. 

![Third-Order Fourier Approximation](image-url)
In Example 7, the pattern for the Fourier coefficients appears to be \( a_0 = 2\pi, \ a_1 = a_2 = \ldots = a_n = 0, \) and

\[ b_1 = -\frac{2}{1}, \ b_2 = -\frac{2}{2}, \ldots, \ b_n = -\frac{2}{n}. \]

The \( n \)-th order Fourier approximation of \( f(x) \) is

\[ g(x) = \pi - 2\left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots + \frac{1}{n} \sin nx\right). \]

As \( n \) increases, the Fourier approximation improves. For instance, Figure 5.37 shows the fourth- and fifth-order Fourier approximations of \( f(x) = x, \ 0 \leq x \leq 2\pi. \)

In advanced courses it is shown that as \( n \to \infty \), the approximation error \( \|f - g\| \) approaches zero for all \( x \) in the interval \((0, 2\pi)\). The infinite series for \( g(x) \) is called a Fourier series.

**EXAMPLE 8**  
**Finding a Fourier Approximation**

Find the fourth-order Fourier approximation of \( f(x) = |x - \pi|, \ 0 \leq x \leq 2\pi. \)

**SOLUTION**

Using Theorem 5.20, find the Fourier coefficients as follows.

\[ a_0 = \frac{1}{\pi} \int_0^{2\pi} |x - \pi| \, dx = \pi \]

\[ a_j = \frac{1}{\pi} \int_0^{2\pi} |x - \pi| \cos jx \, dx \]

\[ = \frac{2}{\pi} \int_0^\pi (\pi - x) \cos jx \, dx \]

\[ = \frac{2}{\pi j} \left(1 - \cos j\pi\right) \]

\[ b_j = \frac{1}{\pi} \int_0^{2\pi} |x - \pi| \sin jx \, dx = 0 \]

So, \( a_0 = \pi, \ a_1 = 4/\pi, \ a_2 = 0, \ a_3 = 4/9\pi, \ a_4 = 0, \ b_1 = 0, \ b_2 = 0, \ b_3 = 0, \) and \( b_4 = 0, \) which means that the fourth-order Fourier approximation of \( f \) is

\[ g(x) = \frac{\pi}{2} + \frac{4}{\pi} \cos x + \frac{4}{9\pi} \cos 3x. \]

Figure 5.38 compares the graphs of \( f \) and \( g. \)