Chapter 2

Planar Optical Waveguides

Planar optical waveguides are the key devices to construct integrated optical circuits and semiconductor lasers. Generally, rectangular waveguides consist of a square or rectangular core surrounded by a cladding with a lower refractive index than that of the core. Three-dimensional analysis is necessary to investigate the transmission characteristics of rectangular waveguides. However, rigorous three-dimensional analysis usually requires numerical calculations and does not always give a clear insight into the problem. Therefore, this chapter first describes two-dimensional slab waveguides to acquire a fundamental understanding of optical waveguides. Then several analytical approximations are presented to analyze the three-dimensional rectangular waveguides. Although these are approximate methods, the essential lightwave transmission mechanism in rectangular waveguides can be fully investigated. The rigorous treatment of three-dimensional rectangular waveguides by the finite element method will be presented in Chapter 6.

2.1 Slab waveguides

2.1.1 Derivation of basic equations

In this section, the wave analysis is described for the slab waveguide (Fig. 2.1) whose propagation characteristics have been explained [1–3]. Taking into account the fact that we treat dielectric optical waveguides, we set permittivity and permeability as \( \varepsilon = \varepsilon_0 n^2 \) and \( \mu = \mu_0 \) in Maxwell’s equations (1.17) and (1.18) as

\[
\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \quad (2.1a)
\]

\[
\nabla \times \mathbf{H} = \varepsilon_0 n^2 \frac{\partial \mathbf{E}}{\partial t}, \quad (2.1b)
\]

where \( n \) is the refractive index. We are interested in plane-wave propagation in

\[
\mathbf{E} = \mathbf{E}(x, y)e^{j(k_0 n x - \beta y)}, \quad (2.2a)
\]

\[
\mathbf{H} = \mathbf{H}(x, y)e^{j(k_0 n x - \beta y)}, \quad (2.2b)
\]

Substituting Eqs. (2.2a) and (2.2b) into Eqs. (2.1a) and (2.1b), we obtain the following set of equations for the electromagnetic field components:

\[
\frac{\partial E_z}{\partial y} + j\beta E_x = -j\omega \mu_0 H_z, \quad (2.3)
\]

\[
-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega \mu_0 H_y
\]

\[
\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial y} = \omega \varepsilon_0 \mu_0 H_z
\]

\[
\frac{\partial H_z}{\partial y} + j\beta H_x = j\omega \varepsilon_0 n^2 E_x
\]

\[
-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega \varepsilon_0 n^2 E_y \quad (2.4)
\]

In the slab waveguide, as shown in Fig. 2.1, electromagnetic fields \( \mathbf{E} \) and \( \mathbf{H} \) do not have \( y \)-axis dependency. Therefore, we set \( \partial E_z/\partial y = 0 \) and \( \partial H_z/\partial y = 0 \). Putting these relations into Eqs. (2.3) and (2.4), two independent electromagnetic modes are obtained, which are denoted as TE mode and TM mode, respectively.

The TE mode satisfies the following wave equation:

\[
\frac{d^2 E_x}{dx^2} + (k^2 n^2 - \beta^2) E_x = 0, \quad (2.5a)
\]
where

\[ H_z = -\frac{\beta}{\omega \mu_0} E_z, \quad (2.5b) \]
\[ H_z = j \frac{dE_z}{\omega \mu_0 \, dx}, \quad (2.5c) \]
\[ E_z = E_z = H_z = 0. \quad (2.5d) \]

Also the tangential components \( E_z \) and \( H_z \) should be continuous at the boundaries of two different media. As shown in Eq. (2.5d), the electric field component along the z-axis is zero \((E_z = 0)\). Since the electric field lies in the plane that is perpendicular to the z-axis, this electromagnetic field distribution is called transverse electric (TE) mode.

The TM mode satisfies the following wave equation:

\[ \frac{d}{dx} \left( \frac{1}{n^2} \frac{dH_x}{dx} \right) + \left( k^2 - \frac{\beta^2}{n^2} \right) H_y = 0, \quad (2.6a) \]

where

\[ E_x = \frac{\beta}{\omega \mu_0 n^2} H_y, \quad (2.6b) \]
\[ E_y = -\frac{j}{\omega \mu_0 n^2} \frac{dH_y}{dx}, \quad (2.6c) \]
\[ E_z = H_x = H_z = 0. \quad (2.6d) \]

As shown in Eq. (2.6d), the magnetic field component along the z-axis is zero \((H_z = 0)\). Since the magnetic field lies in the plane that is perpendicular to the z-axis, this electromagnetic field distribution is called transverse magnetic (TM) mode.

### 2.1.2 Dispersion equations for TE and TM modes

Propagation constants and electromagnetic fields for TE and TM modes can be obtained by solving Eq. (2.5) or (2.6). Here the derivation method to calculate the dispersion equation (also called the eigenvalue equation) and the electromagnetic field distribution is given. We consider the slab waveguide with uniform refractive-index profile in the core, as shown in Fig. 2.2. Considering the fact that the guided electromagnetic fields are confined in the core and exponentially decay in the cladding, the electric field distribution is expressed as

\[ E_x = \begin{cases} 
A \cos(\kappa a + \phi)e^{\xi(x-a)} & (x > a) \\
A \cos(\kappa x - \phi) & (-a \leq x \leq a) \\
A \cos(\kappa a + \phi)e^{\xi(-x+a)} & (x < -a) 
\end{cases} \quad (2.7) \]

where \( \kappa, \sigma, \) and \( \xi \) are wavenumbers along the x-axis in the core and cladding regions and are given by

\[ \kappa = \sqrt{k^2 n_1^2 - \beta^2}, \quad \sigma = \sqrt{\beta^2 - k^2 n_2^2}, \quad \xi = \sqrt{\beta^2 - k^2 n_1^2}. \quad (2.8) \]

The electric field component \( E_y \) in Eq. (2.7) is continuous at the boundaries of core–cladding interfaces \((x = \pm a)\). There is another boundary condition, that the magnetic field component \( H_z \) should be continuous at the boundaries. \( H_y \) is given by Eq. (2.5c). Neglecting the terms independent of \( x \), the boundary condition for \( H_z \) is treated by the continuity condition of \( dE_x/dx \) as

\[ \frac{dE_x}{dx} = \begin{cases} 
\kappa A \cos(\kappa x - \phi)e^{-\sigma(x-a)} & (x > a) \\
\kappa A \sin(\kappa x - \phi) & (-a \leq x \leq a) \\
\kappa A \cos(\kappa x + \phi)e^{\xi(x+a)} & (x < -a) 
\end{cases} \quad (2.9) \]

From the condition that \( dE_y/dx \) are continuous at \( x = \pm a \), the following equations are obtained:

\[ \kappa A \sin(\kappa a + \phi) = \xi A \cos(\kappa a + \phi) \]
\[ \sigma A \cos(\kappa a - \phi) = \kappa A \sin(\kappa a - \phi). \]

Eliminating the constant \( A \) we have

\[ \tan(u + \phi) = \frac{w}{w'}, \quad (2.10a) \]
\[ \tan(u - \phi) = \frac{w'}{w}, \quad (2.10b) \]
where
\[
\begin{align*}
  u &= \kappa a \\
  w &= \xi a \\
  w' &= \sigma a.
\end{align*}
\] (2.11)

From Eqs. (2.10) we obtain the eigenvalue equations as
\[
\begin{align*}
  u &= \frac{m\pi}{2} + \frac{1}{2} \tan^{-1} \left( \frac{w}{u} \right) + \frac{1}{2} \tan^{-1} \left( \frac{w'}{u} \right) \quad (m = 0, 1, 2, \ldots) \\
  \phi &= \frac{m\pi}{2} + \frac{1}{2} \tan^{-1} \left( \frac{w'}{u} \right) - \frac{1}{2} \tan^{-1} \left( \frac{w'}{u} \right).
\end{align*}
\] (2.12) (2.13)

The normalized transverse wavenumbers \( u, w, \) and \( w' \) are not independent. Using Eqs. (2.8) and (2.11) it is known that they are related by the following equations:
\[
\begin{align*}
  u^2 + w^2 &= k^2 a^2 (n_1^2 - n_0^2) = v^2, \\
  w' &= \sqrt{\gamma v^2 + w^2}, \\
  \gamma &= \frac{n_0^2 - n_{\infty}^2}{n_1^2 - n_0^2},
\end{align*}
\] (2.14) (2.15a) (2.15b)

where \( v \) is the normalized frequency, defined in Eq. (1.15), and \( \gamma \) is a measure of the asymmetry of the cladding refractive indices. Once the wavelength of the light signal and the geometrical parameters of the waveguide are determined, the normalized frequency \( v \) and \( \gamma \) are determined. Therefore \( u, w, w', \) and \( \phi \) are given by solving the eigenvalue equations Eqs. (2.12) and (2.13) under the constraints of Eqs. (2.14)–(2.15). In the asymmetrical waveguide \( (n_c > n_0) \) as shown in Fig. 2.2, the higher refractive index \( n_c \) is used as the cladding refractive index, which is adopted for the definition of the normalized frequency \( v \). It is preferable to use the higher refractive index \( n_c \) because the cutoff conditions are determined when the normalized propagation constant \( \beta/k \) coincides with the higher cladding refractive index. Equations (2.12), (2.14), and (2.15) are the dispersion equations or eigenvalue equations for the TE\(_{lm}\) modes. When the wavelength of the light signal and the geometrical parameters of the waveguide are determined—in other words, when the normalized frequency \( v \) and asymmetrical parameter \( \gamma \) are determined—the propagation constant \( \beta \) can be determined from these equations. As is known from Fig. 2.2 or Eqs. (2.7) and (2.8), the transverse wavenumber \( \kappa \) should be a real number for the main part of the optical field to be confined in the core region. Then the following condition should be satisfied:
\[
  n_c \leq \frac{\beta}{k} \leq n_1,
\] (2.16)

\( \beta/k \) is a dimensionless value and is a refractive index itself for the plane wave.

Therefore it is called the effective index and is usually expressed as
\[
  n_e = \frac{\beta}{k}.
\] (2.17)

When \( n_c < n_0 \), the electromagnetic field in the cladding becomes oscillatory along the transverse direction; that is, the field is dissipated as the radiation mode. Since the condition \( \beta = \kappa n_0 \) represents the critical condition under which the field is cut off and becomes the nonguided mode (radiation mode), it is called as cutoff condition. Here we introduce a new parameter, which is defined by
\[
  b = \frac{n_0^2 - n_0^2}{n_1^2 - n_0^2}.
\] (2.18)

Then the conditions for the guided modes are expressed, from Eqs. (2.16) and (2.17), by
\[
  0 \leq b \leq 1,
\] (2.19)

and the cutoff condition is expressed as
\[
  b = 0.
\] (2.20)

\( b \) is called the normalized propagation constant. Rewriting the dispersion Eq. (2.12) by using the normalized frequency \( v \) and the normalized propagation constant \( b \), we obtain
\[
  2v\sqrt{1 - b} = m\pi + \tan^{-1} \sqrt{1 - b} + \tan^{-1} \left( \frac{b + \gamma}{1 - b} \right).
\] (2.21)

Also, Eq. (2.8) is rewritten as
\[
\begin{align*}
  u &= v\sqrt{1 - b} \\
  w &= v\sqrt{b} \\
  w' &= v\sqrt{b + \gamma}.
\end{align*}
\] (2.22)

For the symmetrical waveguides with \( n_0 = n_\infty \), we have \( \gamma = 0 \) and the dispersion equations (2.12) and (2.13) are reduced to
\[
\begin{align*}
  u &= \frac{m\pi}{2} + \tan^{-1} \left( \frac{w'}{u} \right), \\
  \phi &= \frac{m\pi}{2}.
\end{align*}
\] (2.23a) (2.23b)
Equation (2.23a) is also expressed by
\[ w = u \tan \left( u - \frac{m \pi}{2} \right), \]  
(2.24)

or
\[ \nu \sqrt{1 - b} = \frac{m \pi}{2} + \tan^{-1} \frac{b}{\sqrt{1 - b}}. \]  
(2.25)

If we notice that the transverse wavenumber \( kn_x a \sin \phi \) in Eq. (1.12) can be expressed by using the present parameters as \( u = k a = kn_x a \sin \phi \), then Eq. (1.12) coincides completely with Eq. (2.24).

2.1.3 Computation of propagation constant

First the graphical method to obtain qualitatively the propagation constant of the symmetrical slab waveguide is shown, and then the quantitative numerical method to calculate accurately the propagation constant is described. The relationship between \( u \) and \( w \) for the symmetrical slab waveguide, which is shown in Eq. (2.24), is plotted in Fig. 2.3. Transverse wavenumbers \( u \) and \( w \) should satisfy Eq. (2.14) for a given normalized frequency \( \nu \). This relation is also plotted in Fig. 2.3 for the case of \( \nu = 4 \) as the semicircle with the radius of 4. The solutions of the dispersion equation are then given as the crossing points in Fig. 2.3. For example, the transverse wavenumbers \( u \) and \( w \) for the fundamental mode are given by the crossing point of the curve tangential with \( m = 0 \) and the semicircle. The propagation constant (or eigenvalue) \( \beta \) is then obtained by using Eqs. (2.8) and (2.11). In Fig. 2.3, there is only one crossing point for the case of \( \nu < \pi/2 \). This means that the propagation mode is the only one when the waveguide structure and the wavelength of light satisfy the inequality \( \nu < \pi/2 \). The value of \( \nu = \pi/2 \) then gives the critical point at which the higher-order modes are cut off in the symmetrical slab waveguide. \( \nu \) is called the cutoff normalized frequency, which is obtained from the cutoff condition for the \( m = 1 \) mode,

\[ \begin{cases} b = w = 0 \\ u = \nu = \frac{\pi}{2} \end{cases} \]
(2.26a, 2.26b)

where Eqs. (2.20) and (2.22) have been used. Generally, the cutoff \( \nu \)-value for the TE mode is given by Eq. (2.21) as

\[ \nu_{c, TE} = \frac{m \pi}{2} + \frac{1}{2} \tan^{-1} \frac{1}{\sqrt{\gamma}}, \]
(2.27)

and that for the TM mode is given by Eq. (2.38) (explained in Section 2.1.5) as

\[ \nu_{c, TM} = \frac{m \pi}{2} + \frac{1}{2} \tan^{-1} \left( \frac{n_2^2}{n_1^2} \frac{1}{\gamma} \right). \]
(2.28)

A qualitative value can be obtained by this graphical solution for the dispersion equation. However, in order to obtain an accurate solution of the dispersion equation, we should rely on the numerical method. Here, we show the numerical treatment for the symmetrical slab waveguide so as to compare with the previous graphical method. We first rewrite the dispersion equation (2.25) in the following form:

\[ f(\nu, m, b) = \nu \sqrt{1 - b} - \frac{m \pi}{2} - \tan^{-1} \frac{b}{\sqrt{1 - b}} = 0. \]
(2.29)

Figure 2.4 shows the plot of \( f(\nu, m, b) \) for \( \nu = 4 \). The \( b \)-value at which \( f = 0 \) gives the normalized propagation constant \( b \) for the given \( \nu \)-value. The solution of Eq. (2.29) is obtained by the Newton–Raphson method or the bisection method or the like. Here, the subroutine program of the most simple bisection method is shown in Fig. 2.5.

The normalized propagation constant \( b \) is calculated for each normalized frequency \( \nu \). Figure 2.6 shows the \( \nu-b \) relationship, which is called the dispersion curve, for the TE mode. The mode number is expressed by the subscript \( m \), such as \( \text{TE}_m \) or \( \text{TM}_m \) mode. The parameter in Fig. 2.6 is the measure of asymmetry \( \gamma \).
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Figure 2.4: Plot of $f_r(m, b)$ for the calculation of the eigenvalue.

Figure 2.5: Subroutine program of the bisection method to calculate the eigenvalue.

Figure 2.6: Dispersion curves for the TE modes in the slab waveguide.

It is known that there is no cutoff for the lowest $TE_0$ mode in the symmetrical waveguide ($\gamma = 0$). On the other hand, cutoff exists for the $TE_0$ mode in the asymmetrical waveguide ($\gamma \neq 0$).

2.1.4 Electric field distribution

Once the eigenvalue of the waveguide is obtained, the electric field distribution given by Eq. (2.7) is determined except for the arbitrary constant $A$. Constant $A$ is determined when we specify the optical power $P$ carried by the waveguide. Power $P$ is expressed, by using Eq. (1.45), as

$$ P = \int_{-\infty}^{\infty} \left( E \times H^* \right) \cdot u_z \, dx = \int_{-\infty}^{\infty} \frac{1}{2} (E_z H_x^* - E_x H_z^*) \, dx. \quad (2.30) $$

For the TE mode we can rewrite Eq. (2.30), by using Eqs. (2.5), as

$$ P = \frac{\beta}{2\omega \varepsilon_0} \left( \int_{-\infty}^{\infty} |E_z|^2 \, dx. \quad (2.31) \right. $$

Substituting Eq. (2.7) into Eq. (2.31) we obtain the power fractions in the core,
substrate, and cladding regions, respectively, as

\[
P_{\text{core}} = \frac{\beta a A^2}{2 \omega \mu_0} \left\{ 1 + \frac{\sin^2(u + \phi)}{2w} + \frac{\sin^2(u - \phi)}{2w'} \right\} \quad (-a \leq x \leq a) \quad (2.32a)
\]

\[
P_{\text{sub}} = \frac{\beta a A^2 \cos^2(u + \phi)}{2 \omega \mu_0} \quad (x \leq -a) \quad (2.32b)
\]

\[
P_{\text{clad}} = \frac{\beta a A^2 \cos^2(u - \phi)}{2 \omega \mu_0} \quad (x > a). \quad (2.32c)
\]

For the calculation of Eq. (2.32a) we use Eqs. (2.10). The total power \( P \) is then given by

\[
P = P_{\text{core}} + P_{\text{sub}} + P_{\text{clad}} = \frac{\beta a A^2}{2 \omega \mu_0} \left\{ 1 + \frac{1}{2w} + \frac{1}{2w'} \right\}. \quad (2.33)
\]

Here the constant \( A \) is determined by

\[
A = \sqrt{\frac{2 \omega \mu_0 P}{\beta a (1/2w + 1/2w')}}. \quad (2.34)
\]

Figure 2.7 shows the electric field distributions of the TE mode for \( v = 4 \) in the waveguide with \( n_1 = 3.38, n = 3.17, n_0 = 1.0 \) (\( \gamma = 6.6 \)). The power confinement factor in the core is important to calculate the threshold current density \( j_0 \) of semiconductor lasers [4]. The confinement factor is calculated, by using Eqs. (2.32) and (2.33), as

\[
\Gamma = \frac{P_{\text{core}}}{P} = \frac{1 + \frac{\sin^2(u + \phi)}{2w} + \frac{\sin^2(u - \phi)}{2w'}}{1 + \frac{1}{2w} + \frac{1}{2w'}}. \quad (2.35)
\]

The power confinement factor \( \Gamma \) and the ratio to the core width \( 2a/\Gamma \) for the fundamental mode are shown in Fig. 2.8. The vertical lines in the figure express the single mode core width.

2.1.5 Dispersion equation for TM mode

Based on Eqs. (2.6), the dispersion equation for the TM mode is obtained in a similar manner to that for the TE mode. We first express the magnetic field distribution \( H_y \) as

\[
H_y = \begin{cases} 
A \cos(kx - \phi) e^{-\alpha(x-a)} & (x > a) \\
A \cos(kx - \phi) & (-a \leq x \leq a) \\
A \cos(kx + \phi) e^{\alpha(x+a)} & (x < -a).
\end{cases} \quad (2.36)
\]

Applying the boundary condition that \( H_y \) and \( E_z \) should be continuous at \( x = \pm a \), the following dispersion equation is obtained:

\[
u = \frac{m \pi}{2} + \tan^{-1} \left( \frac{n_1^2 w}{n_0^2 u} \right) + \frac{1}{2} \tan^{-1} \left( \frac{n_1^2 w'}{n_0^2 u} \right). \quad (2.37)
\]

Rewriting this equation by using the normalized frequency \( \nu \) and the normalized propagation constant \( b \), we reduce it to

\[
2\nu \sqrt{1 - b} = m \pi + \tan^{-1} \left( \frac{n_1^2 \sqrt{b}}{n_0^2 \sqrt{1-b}} \right) + \tan^{-1} \left( \frac{n_1^2 \sqrt{b + \gamma}}{n_0^2 \sqrt{1-b}} \right). \quad (2.38)
\]

The dispersion curves for the TM modes in the waveguide with \( n_1 = 3.38, \).
$n_s = n_0 = 3.17 \ (\gamma = 0)$ are shown in Fig. 2.9 and compared with those for the TE modes. It is known that the normalized propagation constant $b$ for the TM mode is smaller than that for the TE mode with respect to the same $v$. That means the TE mode is slightly better confined in the core than is the TM mode. The power carried by the TM mode is obtained, from Eqs. (2.6) and (2.30), by

$$P = \frac{\beta}{2\omega\varepsilon_0} \int_{-\infty}^{\infty} \frac{1}{H_y^2} \ dx.$$  \hspace{1cm} (2.39)

### 2.2 Rectangular waveguides

#### 2.2.1 Basic equations

In this section the analytical method, which was proposed by Marcatili [5], to deal with the three-dimensional optical waveguide, as shown in Fig. 2.10, is described. The important assumption of this method is that the electromagnetic field in the shaded area in Fig. 2.10 can be neglected, since the electromagnetic field of the well-guided mode decays quite rapidly in the cladding region. Then we do not impose the boundary conditions for the electromagnetic field in the shaded area.

We first consider the electromagnetic mode in which $E_x$ and $H_y$ are predominant. According to Marcatili's treatment, we set $H_x = 0$ in Eqs. (2.3) and (2.4).

![Figure 2.8: Power confinement factor of the symmetrical slab waveguide.](image)

**Figure 2.8:** Power confinement factor of the symmetrical slab waveguide.

![Figure 2.9: Dispersion curves for the TE and TM modes in the slab waveguide.](image)

**Figure 2.9:** Dispersion curves for the TE and TM modes in the slab waveguide.

![Figure 2.10: Three-dimensional rectangular waveguide.](image)

**Figure 2.10:** Three-dimensional rectangular waveguide.
Then the wave equation and electromagnetic field representation are obtained as

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + (k^2 n^2 - \beta^2) H_z = 0, \quad (2.40)
\]

\[
H_x = 0 \\
E_x = \frac{\omega \mu_0}{\beta} H_y - \frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial x^2} \\
E_y = \frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial x \partial y} \\
E_z = \frac{\partial H_y}{\omega \varepsilon_0 n^2 \beta} \\
H_z = -\frac{j \beta H_y}{\omega \varepsilon_0 n^2 \beta} \frac{\partial}{\partial y}
\]

(2.41)

On the other hand, we set \( H_z = 0 \) in Eqs. (2.3) and (2.4) to consider the electromagnetic field in which \( E_x \) and \( H_y \) are predominant. The wave equation and electromagnetic field representation are given by

\[
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + (k^2 n^2 - \beta^2) H_x = 0, \quad (2.42)
\]

\[
H_z = 0 \\
E_x = -\frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial x \partial y} \\
E_y = -\frac{\omega \mu_0}{\beta} H_z - \frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_y}{\partial y^2} \\
E_z = \frac{j \beta H_y}{\omega \varepsilon_0 n^2 \beta} \\
H_z = -\frac{j \beta H_y}{\omega \varepsilon_0 n^2 \beta} \frac{\partial}{\partial x}
\]

(2.43)

The modes in Eqs. (2.40) and (2.41) are described as \( E_{pm}^x \) (\( p \) and \( q \) are integers), since \( E_x \) and \( H_y \) are the dominant electromagnetic fields. On the other hand, the modes in Eqs. (2.42) and (2.43) are called \( E_{pq}^p \), since \( E_x \) and \( H_z \) are the dominant electromagnetic fields. In the following section, the solution method of the dispersion equation for the \( E_{pq}^p \) mode is described in detail, and only the results are shown for the \( E_{pq}^p \) mode.

### 2.2.2 Dispersion equations for \( E_{pq}^p \) and \( E_{pq}^q \) modes

Since the rectangular waveguide shown in Fig. 2.10 is symmetrical with respect to the \( x \)- and \( y \)-axes, we analyze only regions (1)–(3). We first express the solution fields, which satisfy the wave equation (2.40), as

\[
E_y = \begin{cases} 
A \cos(k_x x - \phi) \cos(k_y y - \psi) & \text{region (1)} \\
A \cos(k_x a - \phi) e^{-\gamma v^a x} \cos(k_y y - \psi) & \text{region (2)} \\
A \cos(k_x x - \phi) e^{-\gamma v^a y} \cos(k_y d - \psi) & \text{region (3)} 
\end{cases} \quad (2.44)
\]

where the transverse wavenumbers \( k_x, k_y, \gamma_x, \) and \( \gamma_y \) and the optical phases \( \phi \) and \( \psi \) are given by

\[
\begin{align*}
(-k_x^2 - k_y^2 + k^2 n_0^2 - \beta^2) &= 0 & \text{region (1)} \\
(\gamma_x^2 - k_y^2 + k^2 n_0^2 - \beta^2) &= 0 & \text{region (2)} \\
(-k_x^2 + \gamma_y^2 + k^2 n_0^2 - \beta^2) &= 0 & \text{region (3)} \\
\end{align*}
\]

(2.45)

and

\[
\begin{align*}
\phi &= (p - 1) \frac{\pi}{2} & (p = 1, 2, \ldots) \\
\psi &= (q - 1) \frac{\pi}{2} & (q = 1, 2, \ldots) \\
\end{align*}
\]

(2.46)

We should note here that the integers \( p \) and \( q \) start from 1 because we follow the mode definition by Marcatili. To the contrary, the mode number \( m \) in Eq. (2.12) for the slab waveguides starts from zero. By the conventional mode definition, the lowest mode in the slab waveguide is the \( \text{TE}_{m=0} \) mode [Fig. 2.7(a)], which has one electric field peak. On the other hand, the lowest mode in the rectangular waveguides is \( E_{p=1,q=1} \) or \( E_{p=1,q=1} \) mode [Fig. 2.11], which has only one electric field peak along both the \( x \)- and \( y \)-axis directions. Therefore in the mode definition by Marcatili, integers \( p \) and \( q \) represent the number of local electric field peaks along the \( x \)- and \( y \)-axis directions. When we apply the boundary conditions that the electric field \( E_x \propto (1/n^2) \partial H_y/\partial x \) should be continuous at \( x = a \) and the magnetic field \( H_z \propto \partial H_y/\partial y \) should be continuous at \( y = d \), we obtain the following dispersion equations:

\[
k_x a = (p - 1) \frac{\pi}{2} + \tan^{-1}\left(\frac{n_2^2 \gamma_x}{n_0^2 k_x}\right) \quad (2.47a)
\]

\[
k_y d = (q - 1) \frac{\pi}{2} + \tan^{-1}\left(\frac{\gamma_y}{k_y}\right) \quad (2.47b)
\]

Transverse wavenumbers \( k_x, k_y, \gamma_x, \) and \( \gamma_y \) are related, by Eq. (2.45), as

\[
\gamma_x^2 = k^2 n_0^2 - n_0^2 k_x^2 \quad (2.48)
\]

\[
\gamma_y^2 = k^2 n_0^2 - n_0^2 k_y^2 \quad (2.49)
\]
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at $y = d$, we obtain the following dispersion equations:

\[ k_x d = (p - 1) \frac{\pi}{2} + \tan^{-1} \left( \frac{\xi_x}{k_x} \right), \]  
\[ k_y d = (q - 1) \frac{\pi}{2} + \tan^{-1} \left( \frac{\xi_y}{k_y} \right). \]  

(2.52)  
(2.53)

2.2.3 Kumar's method

In Marcatili's method, electromagnetic fields and the boundary conditions in the shaded area in Fig. 2.10 are not strictly satisfied. In other words, the hybrid modes in the rectangular waveguides are approximately analyzed by separating into two independent slab waveguides, as shown in Fig. 2.12. It is well understood when we compare the dispersion Eqs. (2.47a) and (2.47b) for the $E_{xy}$ mode with the slab dispersion Eqs. (2.12) and (2.37). Equation (2.47a) corresponds to the TM mode dispersion equation of the symmetric slab waveguide [Fig. 2.12(b)], and Eq. (2.47b) corresponds to the TE mode dispersion equation [Fig. 2.12(c)], respectively.

Kumar et al. proposed an improvement in accuracy for Marcatili's method by taking into account the contribution of the fields in the shaded area in Fig. 2.10 [6]. We call this Kumar's method and describe an example of it by analyzing the $E_{xy}$ mode in rectangular waveguides.

In Kumar's method, the refractive-index distribution of the rectangular waveguide is expressed by

\[ n^2(x, y) = N_2^2(x) + N_1^2(y) + O(n_1^2 - n_3^2) \]  

(2.54)

where

\[ N_2^2(x) = \begin{cases} n_1^2/2 & |x| \leq a \\ n_2^2 - n_1^2/2 & |x| > a \end{cases} \]  

(2.55a)

\[ N_1^2(y) = \begin{cases} n_1^2/2 & |y| \leq d \\ n_2^2 - n_1^2/2 & |y| > d \end{cases} \]  

(2.55b)

In order to calculate the dispersion equation for the $E_{xy}$ mode, we express the magnetic field $H_y$ as

\[ H_y = \begin{cases} A \cos(k_x x - \phi) \cos(k_y y - \psi) & \text{region (1)} \\ A \cos(k_x a - \phi)e^{-\tau(x-a)} \cos(k_y y - \psi) & \text{region (2)} \\ A \cos(k_x x - \phi)e^{-\tau(y-d)} \cos(k_y d - \psi) & \text{region (3)} \end{cases} \]  

(2.51)

Applying the boundary conditions that the magnetic field $H_y \propto \partial H_y/\partial x$ should be continuous at $x = a$ and the electric field $E_z \propto (1/\mu_0)\partial H_y/\partial y$ should be continuous

Figure 2.12: Rectangular waveguide and its equivalent, two independent slab waveguides, in Marcatili's method.
where small quantities of the order of $O(n_2^2 - n_0^2)$ have been neglected. Dividing Eq. (2.58) by $XY$, it can be separated into two terms: one is dependent on variable $x$ and the other is dependent on variable $y$, respectively, as

$$ \frac{1}{X} \frac{d^2X}{dx^2} + \frac{k^2N_2(x)}{Y} \frac{d^2Y}{dy^2} + k^2N_2(x) = \beta^2. \tag{2.59} $$

The conditions necessary for Eq. (2.59) to be satisfied for arbitrary values of $x$ and $y$ are

$$ \frac{1}{X} \frac{d^2X}{dx^2} + k^2N_2(x) = \beta_x^2, \tag{2.60a} $$

$$ \frac{1}{Y} \frac{d^2Y}{dy^2} + k^2N_2(y) = \beta_y^2, \tag{2.60b} $$

where $\beta_x$ and $\beta_y$ are constants that are independent from $x$ and $y$. We then have two independent wave equations:

$$ \frac{d^2X}{dx^2} + [k^2N_2(x) - \beta_x^2]X(x) = 0 \tag{2.61a} $$

$$ \frac{d^2Y}{dy^2} + [k^2N_2(y) - \beta_y^2]Y(y) = 0, \tag{2.61a} $$

From Eqs. (2.59) and (2.60) we can derive an equation to determine the propagation constant:

$$ \beta^2 = \beta_x^2 + \beta_y^2. \tag{2.62} $$

The solution fields of Eqs. (2.61a) and (2.61b) are expressed in a similar manner as previously in Section 2.2.2 by

$$ X(x) = \begin{cases} A \cos(k_x x - \phi) & (0 \leq x \leq a) \\
A \cos(k_x a - \phi) e^{-\gamma_x (x - a)} & (x > a) \end{cases} \tag{2.63} $$

$$ Y(y) = \begin{cases} B \cos(k_y y - \psi) & (0 \leq y \leq d) \\
B \cos(k_y d - \psi) e^{-\gamma_y (y - d)} & (y > d) \end{cases} \tag{2.64} $$

where only the first quadrant is considered due to the symmetry of the waveguide, and transverse wavenumbers $k_x$, $\gamma_x$, $k_y$, and $\gamma_y$ are related with $\beta_x$ and $\beta_y$ by

$$ \gamma_x^2 = k_x^2(n_1^2 - n_0^2) - \beta_x^2, \tag{2.65a} $$

$$ \gamma_y^2 = k_y^2(n_1^2 - n_0^2) - \beta_y^2. \tag{2.65b} $$
\[ \beta_s^2 = \frac{k^2 n_1^2}{\mu} - k_s^2, \]  
\[ \beta_p^2 = \frac{k^2 n_1^2}{\mu} - k_p^2, \]  
and optical phases are expressed by
\[
\begin{align*}
\phi &= (p - 1) \frac{\pi}{2} \quad (p = 1, 2, \ldots) \\
\psi &= (q - 1) \frac{\pi}{2} \quad (q = 1, 2, \ldots)
\end{align*}
\]

When we apply the boundary conditions that the electric field
\[ E_x \propto (1/n)^2 \partial H_y / \partial x = (Y/n^2) \partial X / \partial x \]
should be continuous at \( x = a \) and the magnetic field \( H_y \propto \partial H_y / \partial y = X \partial Y / \partial y \)
should be continuous at \( y = d \), we obtain the following dispersion equations:
\[
\begin{align*}
k_x a &= (p - 1) \frac{\pi}{2} + \tan^{-1} \left( \frac{n_1 \gamma_x}{\gamma_x x} \right), \\
k_x d &= (q - 1) \frac{\pi}{2} + \tan^{-1} \left( \frac{\gamma_x}{k_x} \right).
\end{align*}
\]

The propagation constant is obtained, from Eqs. (2.62) and (2.66), by
\[ \beta^2 = k^2 n_1^2 - (k_s^2 + k_p^2). \]  

Equations (2.68)–(2.70) for the dispersion and the propagation constant are known to be the same as Eqs. (2.47a), (2.47b), and (2.50), by Marcadut’s method. But in Kumar’s method, an improvement in the accuracy of the propagation constant can be obtained by the perturbation method with respect to the shaded area in Fig. 2.13. We rewrite the refractive-index distribution for the rectangular waveguide as
\[ n^2(x, y) = N_1^2(x) + N_2^2(y) + \delta \cdot \eta(x, y), \]
where \( \delta \) is a small quantity and \( \delta \cdot \eta(x, y) \) denotes the perturbation term, which is expressed by
\[ \delta \cdot \eta(x, y) = \begin{cases} (n_1^2 - n_0^2) & |x| > a \text{ and } |y| > d \\ 0 & |x| \leq a \text{ or } |y| \leq d \end{cases} \]

Generally the wave equation is expressed by
\[ \nabla^2 f + (k^2 n^2 - \beta^2) f = 0, \]  
where \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \). The solution field \( f \) and the eigenvalue \( \beta^2 \) of this equation are expressed in first-order perturbation form as
\[ f = f_0 + \delta \cdot f_1, \]  
\[ \beta^2 = \beta_0^2 + \delta \cdot \beta_1^2 \]

Substituting Eqs. (2.74) and (2.75) into Eq. (2.73) and comparing the terms for each order of \( \delta \), the following equations are obtained:
\[
\begin{align*}
\nabla^2 f_0 + [k^2(N_1^2 + N_2^2) - \beta_0^2] f_0 &= 0, \\
\nabla^2 f_1 + [k^2(N_1^2 + N_2^2) - \beta_0^2] f_1 + k^2 \eta f_0 - \beta_1^2 f_0 &= 0.
\end{align*}
\]

Here we consider the integral
\[ \int_0^1 [\text{Eq. (2.76)} f_1 \cdot \text{Eq. (2.77)} f_0^*] \, dx \, dy \]
in region \( D \). Then we have
\[ \beta_1^2 \int_D |f_0|^2 \, dx \, dy = \int_D \left[ f_0 \nabla^2 f_1 - f_1 \nabla^2 f_0^* \right] \, dx \, dy + 2k \int_D \eta |f_0|^2 \, dx \, dy. \]  

The first term in the right-hand-side of this equation is rewritten, by using Green’s theorem (refer to Chapter 10), as
\[ \int_D \left[ f_0 \nabla^2 f_1 - f_1 \nabla^2 f_0^* \right] \, dx \, dy = \int_{\partial D} \left[ f_0 \frac{\partial f_1}{\partial n} - f_1 \frac{\partial f_0^*}{\partial n} \right] \, ds, \]  

where \( \partial / \partial n \) represents the differentiation along the outside normal direction on the periphery of the integration region, and \( 4 \partial D \) represents the line integral along the periphery. The line integral of Eq. (2.79) becomes zero when region \( D \) is enlarged to infinity. Therefore we have
\[ \beta_1^2 = \frac{k^2 \int_{\partial D} \eta |f_0|^2 \, ds}{\int_{\partial D} |f_0|^2 \, ds}. \]  

Equation (2.73) corresponds to Eq. (2.40), and Eq. (2.76) corresponds to Eq. (2.58). Then \( \beta_0 \) is the eigenvalue for the dispersion equations (2.68)–(2.70) and \( f_0(x, y) \) is the field distribution given by Eqs. (2.57), (2.63), and (2.64). The
where the eigenvalue is given by the first-order perturbation is therefore expressed, from Eqs. (2.72), (2.75), and (2.80), by

\[
\beta^2 = \beta_0^2 + \frac{k^2}{\beta_0^2} \left[ \int_0^\infty \left( \int_0^\infty |X(x)Y(y)|^2 \, dx \, dy \right) \, dx \, dy \right]^{-1} \int_0^\infty |X(x)Y(y)|^2 \, dx \, dy
\]

\[
= (k^2 \eta_0^2 - k_0^2 - k_\perp^2) + \frac{k^2 (\eta_0^2 - \eta_0^2) \left( \int_0^\infty |X(x)|^2 \, dx \int_0^\infty |Y(y)|^2 \, dy \right)}{\left( \int_0^\infty |X(x)|^2 \, dx \right) \left( \int_0^\infty |Y(y)|^2 \, dy \right)} \cos^2(k_x a - \phi) \cos^2(k_y d - \psi) \frac{1}{(1 + \gamma_0 a)(1 + \gamma_0 d)}. \quad (2.81)
\]

In the second term of Eq. (2.81), we approximated \( \eta_0^2 \eta_0^2 \approx 1 \). The normalized propagation constant is obtained, from Eqs. (2.17), (2.18), and (2.81), as

\[
b = 1 - \frac{k_\perp^2}{k^2 (\eta_0^2 - \eta_0^2)} + \frac{\cos^2(k_x a - \phi) \cos^2(k_y d - \psi)}{(1 + \gamma_0 a)(1 + \gamma_0 d)}. \quad (2.82)
\]

Figure 2.14 shows the dispersion curves for the rectangular waveguides with core aspect ratios \( a/d = 1 \) and \( a/d = 2 \), which are calculated by using different analysis methods for the scalar wave equations [6]. Analysis by the point matching method [7] gives the most accurate value among four of the analyses and is used as the standard for the comparison of accuracy. The effective index method will be described in the following section. It is clear from Fig. 2.14 that Kumar's method gives more accurate results than Marcatili's method. The accuracy of the effective index method is almost the same as that of Marcatili's method; but the effective index method gives a larger estimation than the accurate solution, whereas Marcatili's method gives a lower estimation than the accurate solution, respectively. For practicality, however, the effective index method is a very important way to analyze, for example, ridge waveguides, which require a numerical method such as the finite element method.

2.2.4 Effective index method

The ridge waveguide, such as shown in Fig. 2.15, is difficult to analyze by Marcatili's method or Kumar's method, since the waveguide structure is too complicated to deal with by the division of waveguide. In order to analyze ridge waveguides, we should use numerical methods, such as the finite element method and the finite difference method. The effective index method [8–9] is an analytical method applicable to complicated waveguides such as ridge waveguides and diffused waveguides in LiNbO₃. In the following, the effective index method of analysis is described, taking as example the \( E_{m} \) mode in the ridge waveguide.

The wave equation for the \( E_{m} \) mode is given, by Eq. (2.40), as

\[
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + [k^2 n^2(x, y) - \beta^2] H_y = 0. \quad (2.83)
\]

Therefore if the assumption of separation of variables is not accurate, due to the waveguide structure or the wavelength of light, the accuracy of the method itself becomes very poor. Substituting Eq. (2.84) into Eq. (2.83) and dividing by \( XY \), we obtain

\[
1 \frac{d^2 X}{X \, dx^2} + 1 \frac{d^2 Y}{Y \, dy^2} + [k^2 n^2(x, y) - \beta^2] = 0. \quad (2.85)
\]

Here we add to, and subtract from, Eq. (2.85) the \( y \)-independent value of \( k^2 n^2_{m0}(x) \)
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The height of the rib, which takes the following values, depending on the position $x$:

$$s = \begin{cases} h & 0 \leq |x| \leq a \\ t & |x| > a \end{cases}$$  \hfill (2.87)

From the boundary condition that $H_y \propto \partial H_y/\partial y$ should be continuous at $y = 0, d$, and $d + s$, we have the continuity condition for $dY/\partial y$ at the foregoing boundaries. The dispersion equations for the four-layer slab waveguide shown in Fig. 2.16 is given by

$$\sin(kd - 2\phi) = \sin(kd)e^{-2\cos\psi},$$  \hfill (2.88)

where the parameters are

$$\phi = \tan^{-1}\left(\frac{\gamma}{\sigma}\right),$$  \hfill (2.89a)

$$\psi = \tanh^{-1}\left(\frac{\kappa}{\gamma}\right),$$  \hfill (2.89b)

$$\kappa = k\sqrt{n_t^2 - n_{eff}^2},$$  \hfill (2.89c)

$$\sigma = k\sqrt{n_{eff}^2 - n_t^2},$$  \hfill (2.89d)

$$\gamma = k\sqrt{n_s^2 - n_{eff}^2}.$$  \hfill (2.89e)

The solution of Eq. (2.88) with $s = h$ ($0 \leq |x| \leq a$) gives the effective index $n_{eff}(h)$ for $0 \leq |x| \leq a$, and the solution of Eq. (2.88) with $s = t$ ($|x| > a$) gives the effective index $n_{eff}(t)$ for $|x| > a$, respectively. Then the effective index distribution $n_{eff}(x)$ is obtained, see Fig. 2.17. The solution of the wave equation (2.86b) is calculated by solving the three-layer symmetrical slab waveguide. The boundary condition is that $E_z \propto (1/n_t^2)\partial H_y/\partial x$ should be continuous at $x = \pm a$. Therefore $(1/n_t^2)X$ should be continuous at $x = \pm a$. Under the foregoing boundary condition, the
dispersion equation is obtained as

\[ u \tan(u) = \frac{n_{2r}(h)}{n_{2r}(t)} w, \quad (2.90) \]

where

\[ u = k a \sqrt{\frac{n_{2r}(h)}{n_{2r}(t)} - \left( \frac{p}{k} \right)^2}, \quad (2.91a) \]

\[ w = k a \sqrt{\left( \frac{p}{k} \right)^2 - n_{2r}(t)}. \quad (2.91b) \]

The dispersion equations for the \( E_{r0} \) mode are obtained in a similar manner:

\[ \sin(kd - 2\phi) = \sin(kd)e^{-2(\sigma + \psi)}, \quad (2.92) \]

where

\[ \sigma = \frac{\tau_{r2}}{\tau_{r1}} \quad \text{and} \quad \psi = \frac{\tau_{r2}}{\tau_{r1}}, \quad (2.94a) \]

\[ \phi = \tan^{-1} \left( \frac{\tau_{r2}}{\tau_{r1}} \right). \quad (2.94b) \]

2.3 Radiation field from a waveguide

The radiation field from an optical waveguide into free space propagates divergently. The radiation field is different from the field in the waveguide. Therefore, it is important to know the profile of the radiation field for efficiently coupling the light between two waveguides or between a waveguide and an optical fiber. In this section, we describe the derivation of the radiation field pattern from the rectangular waveguide.

2.3.1 Fresnel and Fraunhofer regions

We consider the coordinate system shown in Fig. 2.18, where the endface of the waveguide is located at \( z = 0 \) and the electromagnetic field is radiated into the free space with refractive index \( n \). The electric field at the endface of the waveguide is denoted by \( E(x, y, 0) \), and the electric field distribution on the observation plane at distance \( z \) is expressed as \( f(x, y, z) \). By the Fresnel–Kirchhoff diffraction formula \[2] \) (see Chapter 10), the radiation pattern \( f(x, y, z) \) is related to the endface field \( g(x_0, y_0, 0) \) as

\[ f(x, y, z) = \frac{jk}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_0, y_0, 0) e^{-jpk} \frac{dx_0 dy_0}{r} \quad (2.95) \]

Figure 2.18: Coordinate system for the waveguide endface \((z = 0)\) and observation plane \((z = z)\).

where \( k \) is the free-space wavenumber \( k = 2\pi/\lambda \) and the distance \( r \) between \( Q \) and \( P \) is given by

\[ r = \left( (x - x_0)^2 + (y - y_0)^2 + z^2 \right)^{1/2}. \quad (2.96) \]

When the distance of the observation plane \( z \) is very large compared with \( |x - x_0| \) and \( |y - y_0| \), Eq. (2.96) is approximated by

\[ r = z \left[ 1 + \frac{(x - x_0)^2 + (y - y_0)^2}{z^2} \right]^{1/2} = z + \frac{(x - x_0)^2 + (y - y_0)^2}{2z} + \cdots \]

\[ = z + \frac{r^2}{2} - \frac{x^2 + y^2}{z} \frac{x_0^2 + y_0^2 + z_0^2}{2z} + \cdots. \quad (2.97) \]

The number of expansion terms to approximate \( r \) accurately depends on the distance \( z \) between the endface of the waveguide and the observation plane. Generally, the electromagnetic field in the waveguide is confined in a small area of the order of 10 \( \mu \)m. Therefore if \( z \) is longer than, for example, 1 mm, any term higher than the fourth term in the right-hand side of Eq. (2.97) can be neglected. The radiation field where this condition is satisfied is called the far-field region or Fraunhofer region. On the other hand, when \( z \) is not so large, we should take into account up to the fourth term in Eq. (2.97). The radiation field in which this condition holds is called the near-field region or Fresnel region. However, we should note that even the Fresnel approximation is not satisfied in the region close to the waveguide endface. The fourth term in the extreme right of Eq. (2.97) determines which approximation we should adopt. Generally, the contribution to \( kbr \) by the fourth term, \( km(x_0^2 + y_0^2)/2z \), determines whether the Fresnel or Fraunhofer approximation should be used. The measure for the judgment is \( km(x_0^2 + y_0^2)/2z = \pi/2 \). If, for example, the optical field is confined in the rectangular region with square core area \( D^2 \), then \( km(x_0^2 + y_0^2)/2z = \pi/2 \) at \( z = nD^2/\lambda \), and
we have the following criteria:
\[
\begin{cases}
  z > \frac{nD^2}{\lambda} & \text{Fraunhofer region} \\
  z < \frac{nD^2}{\lambda} & \text{Fresnel region}
\end{cases}
\]
(2.98)

When we apply the Fraunhofer approximation to \( r \), Eq. (2.95) reduces to
\[
f(x, y, z) = \frac{jkn}{2\pi z} \exp \left\{ -jkn \left[ z + \frac{x^2 + y^2}{2z} \right] \right\} \\
\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_0, y_0, 0) \exp \left\{ -jkn \left[ \frac{\sqrt{x_0^2 + y_0^2}}{z} \right] \right\} dx_0 dy_0.
\]
(2.99)

It is known from this equation that the Fraunhofer pattern \( f(x, y, z) \) is a spatial Fourier transformation of the field profile at the waveguide endface \( g(x_0, y_0, 0) \).

2.3.2 Radiation pattern of Gaussian beam

It has been described that the radiation pattern from the waveguide is expressed by Eq. (2.95) or (2.99). The accurate electromagnetic field distribution in the rectangular waveguide \( g(x_0, y_0, 0) \) is determined numerically by, for example, the finite element method, as described in Chapter 6. An analytical method such as Marcatili's method does not give the accurate field distribution, especially for the cladding region. Even though the accuracy of the eigenvalue is improved by Kuma's method, the field distribution is not accurate, since Eq. (2.77) is difficult to solve to obtain the perturbation field \( f_1 \). Therefore it is not easy to calculate the radiation pattern from the rectangular waveguide analytically. Here we approximate the electric field distribution in the rectangular waveguide by a Gaussian profile to obtain the radiation pattern analytically. The Gaussian electric field profile in the waveguide is expressed by
\[
g(x_0, y_0, 0) = A \exp \left\{ -\frac{\left[ \frac{x_0^2}{w_1^2} + \frac{y_0^2}{w_2^2} \right]}{1} \right\},
\]
(2.100)

where \( w_1 \) and \( w_2 \) are the spot size of the field (the position at which electric field \( |E| \) becomes \( 1/e \) to the peak value) along the \( x_0 \) and \( y_0 \) axis, respectively, and \( A \) is a constant. Substituting Eqs. (2.97) and (2.100) into Eq. (2.95) we obtain
\[
f(x, y, z) = \frac{jkn}{2\pi z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_0, y_0, 0) \exp \left\{ -jkn \left[ z + \frac{(x-x_0)^2 + (y-y_0)^2}{2z} \right] \right\} dx_0 dy_0
\]
(2.101)

where the Fresnel approximation to \( r \) has been used. Since the integral in Eq. (2.101) for \( x_0 \) and \( y_0 \) has the same form, only the detailed calculation for \( x_0 \) will be described.

When we define the parameter \( p \) as
\[
p = \frac{1}{w_1^2} + \frac{\pi n}{\lambda z},
\]
the integral with respect to \( x_0 \) in Eq. (2.101) becomes
\[
\int_{-\infty}^{\infty} \exp \left\{ -j\frac{\pi n}{\lambda z} x^2 - \frac{\pi n^2 z^2}{\lambda^2} \right\} dx_0
\]
\[
= \sqrt{\frac{\pi}{p}} \exp \left\{ -j\frac{\pi n}{\lambda z} x^2 - \frac{\pi n^2 z^2}{\lambda^2} \right\}.
\]
(2.102)

We further introduce new variables, whose physical meanings are explained later, as
\[
W_1(z) = w_1 \sqrt{1 + \left( \frac{\lambda z}{\pi n w_1^2} \right)^2},
\]
(2.104a)
\[
R_1(z) = z \left[ 1 + \left( \frac{\pi n w_1^2}{\lambda z} \right)^2 \right],
\]
(2.104b)
\[
\Theta_1(z) = \tan^{-1} \left( \frac{\lambda z}{\pi n w_1^2} \right).
\]
(2.104c)

Parameter \( p \) of Eq. (2.102) can be rewritten, by using Eq. (2.104), as
\[
p = \frac{\pi n W_1(z)}{\lambda z w_1} e^{-j\Theta_1}.
\]
(2.105)

Substituting Eq. (2.105) into Eq. (2.101), Eq. (2.103) is finally expressed as
\[
\int_{-\infty}^{\infty} \exp \left\{ -j\frac{\pi n}{\lambda z} x^2 - \frac{\pi n^2 z^2}{\lambda^2} \right\} dx_0
\]
\[
= \sqrt{\frac{\lambda z}{j n W_1(z)}} \exp \left\{ -\left[ \frac{1}{W_1^2(z)} + \frac{jk n}{2 R_1(z)}\right] x^2 + j\Theta_1(z) \right\}.
\]
(2.106)
Similarly, the integral with respect \( y_0 \) in Eq. (2.101) is given by

\[
\int_{-\infty}^{\infty} \exp \left\{ -\frac{y_0^2}{w_1^2} - j \frac{kn}{2z} (y - y_0)^2 \right\} dy_0 = \frac{\lambda w_2 z}{\sqrt{2 \pi n w_1^2}} \exp \left\{ -\frac{1}{2} \left[ \frac{1}{W_1^2(z)} + \frac{j kn}{2 R_2(z)} \right] y^2 + j \frac{\Theta_2(z)}{2} \right\},
\]

where the parameters \( W_1, R_2, \) and \( \Theta_2 \) are defined by

\[
W_1(z) = w_1 \sqrt{1 + \left( \frac{\lambda z}{\pi n w_1} \right)^2},
\]

\[
R_2(z) = z \left[ 1 + \left( \frac{\pi n w_2^2}{\lambda z} \right)^2 \right],
\]

\[
\Theta_2(z) = \tan^{-1} \left( \frac{\lambda z}{\pi n w_2^2} \right).
\]

Substituting Eqs. (2.106) and (2.107) into Eq. (2.101), the radiation pattern from the rectangular waveguide \( f(x, y, z) \) is expressed by

\[
f(x, y, z) = \frac{w_1 w_2}{\sqrt{W_1 W_2}} \cdot A \exp \left\{ -\frac{x^2}{W_1^2} - \frac{y^2}{W_2^2} \right\} - j kn \left[ \frac{x^2}{2 R_1} + \frac{y^2}{2 R_2} + z \right] + j \left( \Theta_1 + \Theta_2 \right),
\]

(2.109)

It is known from this equation that \( W_1(z) \) and \( W_2(z) \) represent the spot sizes of the radiation field, and \( R_1(z) \) and \( R_2(z) \) represent the radii of curvature of the wavefronts, respectively. If the observation point \( P \) is sufficiently far from the endface of the waveguide, and the following conditions \( z \gg \pi n w_1^2/\lambda \) and \( \pi n w_2^2/\lambda \) are satisfied in the Fraunhofer region, then Eqs. (2.104) and (2.108) are approximated as

\[
\begin{align*}
W_1(z) &\approx \frac{\lambda z}{\pi n w_1}, \\
W_2(z) &\approx \frac{\lambda z}{\pi n w_2}, \\
R_1(z) &\approx R_2(z) \approx z.
\end{align*}
\]

(2.110)

In this Fraunhofer region, the divergence angles \( \theta_1 \) (Fig. 2.19) and \( \theta_2 \) of the radiation field along the \( x \)- and \( y \)-axis directions are expressed by

\[
\begin{align*}
\theta_1 &= \tan^{-1} \left( \frac{W_1(z)}{z} \right) = \tan^{-1} \left( \frac{\lambda}{\pi n w_1} \right), \\
\theta_2 &= \tan^{-1} \left( \frac{W_2(z)}{z} \right) = \tan^{-1} \left( \frac{\lambda}{\pi n w_2} \right).
\end{align*}
\]

(2.111)

Let us calculate the divergence angles of the radiation field from a semiconductor laser diode operating at \( \lambda = 1.55 \mu m \) and having the active layer (core) refractive index \( n_1 = 3.5 \), cladding index \( n_0 = 3.17 \), and core width and thickness of \( 2a = 1.5 \mu m \) and \( 2d = 0.15 \mu m \), respectively. The electric field distribution of the waveguide is calculated by using finite element method waveguide analysis (which will be described in Chapter 6) and is Gaussian fitted to obtain the spot sizes \( w_1 \) and \( w_2 \) along the \( x \)- and \( y \)-axis directions. Gaussian-fitted spot sizes \( w_1 \) and \( w_2 \) are

\[
\begin{align*}
w_1 &= 0.88 \mu m, \\
w_2 &= 0.55 \mu m.
\end{align*}
\]

(2.112)

The divergence angles \( \theta_1 \) and \( \theta_2 \) are then obtained, by Eqs. (2.111) and (2.112), as

\[
\begin{align*}
\theta_1 &= 0.51 \text{(rad.)} = 29.4 \text{(degrees)}, \\
\theta_2 &= 0.95 \text{(rad.)} = 54.4 \text{(degrees)}.
\end{align*}
\]

(2.113)

It is known from this result that the radiation field from the semiconductor laser diode has an elliptical shape and that the divergence angle along the thin active-layer direction (\( y \)-axis direction) is much larger than that along the wide active-layer direction (\( x \)-axis direction).

References


