Chapter 10

Several Important Theorems and Formulas

In this chapter, several important theorems and formulas [1–4] are described that are the bases for the derivation of various equations throughout the book. Gauss's theorem, Green's theorem, and Stokes' theorem are foundations for electromagnetic theory. The integral theorem of Helmholtz and Kirchhoff and the Fresnel–Kirchhoff diffraction formula are basic theories for solving diffraction problems.

10.1 Gauss's theorem

We consider the function \( f(x, y, z) \) in a volume \( V \) enclosed by a smooth surface \( S \). Function \( f(x, y, z) \) and its derivatives \( \partial f/\partial x, \partial f/\partial y, \) and \( \partial f/\partial z \) are assumed to be continuous in volume \( V \) and on surface \( S \). Let us consider the volume integral of the form

\[
\iiint_{V} \frac{\partial f}{\partial z} \, dx \, dy \, dz.
\]  
(10.1)

When volume \( V \) is penetrated by a column \( dV \) that is parallel to the \( z \)-axis, as shown in Fig. 10.1, we obtain

\[
\iiint_{V} \frac{\partial f}{\partial z} \, dx \, dy \, dz = \iint_{S} dx \, dy \, \int_{z_{1}}^{z_{2}} \frac{\partial f}{\partial z} \, dz,
\]  
(10.2)

where \( z_{1} \) and \( z_{2} \) denote the \( z \)-axis coordinates at which column \( dV \) penetrates surface \( S \) and \( G \) is a projection of volume \( V \) onto the \( x-y \) plane. Partial integration of Eq. (10.2) with respect to \( z \) gives

\[
\iiint_{V} \frac{\partial f}{\partial z} \, dx \, dy \, dz = \iint_{S} f(x, y, z_{2}) \, dx \, dy - \iint_{S} f(x, y, z_{1}) \, dx \, dy.
\]  
(10.3)

Figure 10.1: A volume \( V \) enclosed by a surface \( S \). Volume \( V \) is penetrated by a column \( dV \) that is parallel to the \( z \)-axis.

We express angles of unit vector \( \mathbf{n} \) normal to the incremental surface \( dS_{1} \) and \( dS_{2} \) by \( \gamma_{1} \) and \( \gamma_{2} \). Here \( \gamma_{1} \) and \( \gamma_{2} \) are measured from the positive \( z \)-axis. Then each term of the right-hand side of Eq. (10.3) can be rewritten as

\[
\iint_{S} f(x, y, z_{2}) \, dx \, dy = \iint_{S} f(x, y, z_{2}) \cos \gamma_{2} \, dS_{2},
\]  
(10.4a)

\[
\iint_{S} f(x, y, z_{1}) \, dx \, dy = \iint_{S} f(x, y, z_{1}) \cos(\pi - \gamma_{1}) \, dS_{1}
\]  
(10.4b)

\[
= - \iint_{S} f(x, y, z_{1}) \cos \gamma_{1} \, dS_{1},
\]  
(10.4b)

Substituting Eqs. (10.4a) and (10.4b) into Eq. (10.3), we obtain

\[
\iiint_{V} \frac{\partial f}{\partial z} \, dx \, dy \, dz = \iint_{S} f(x, y, z_{1}) \cos \gamma_{1} \, dS_{1} + \iint_{S} f(x, y, z_{2}) \cos \gamma_{2} \, dS_{2}.
\]  
(10.5)

The first and second terms of Eq. (10.5) represent the surface integral on the
upper and lower surfaces of $S$, respectively. The generalized expression for Eq. (10.5) becomes

$$\iiint_V df \, dx \, dy \, dz = \iiint_S f(x, y, z) \cos \gamma \, dS.$$  \hspace{1cm} (10.6)

Similar expressions for the volume integral of $\partial f / \partial x$ and $\partial f / \partial y$ are obtained:

$$\iiint_V \frac{\partial f}{\partial x} \, dx \, dy \, dz = \iiint_S f(x, y, z) \cos \alpha \, dS,$$  \hspace{1cm} (10.7)

$$\iiint_V \frac{\partial f}{\partial y} \, dx \, dy \, dz = \iiint_S f(x, y, z) \cos \beta \, dS.$$  \hspace{1cm} (10.8)

Here $\alpha$ denotes the angle between the vector normal to the incremental surface area $dS$ and the $x$-axis and $\beta$ denotes the angle between the vector normal to the incremental surface area $dS$ and the $y$-axis.

When we replace $f(x, y, z)$ in Eqs. (10.7), (10.8), and (10.6) by $X(x, y, z)$, $Y(x, y, z)$, and $Z(x, y, z)$ and add them together, we obtain

$$\iiint_V \left[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right] \, dx \, dy \, dz = \iiint_S \left[ X \cos \alpha + Y \cos \beta + Z \cos \gamma \right] \, dS.$$  \hspace{1cm} (10.9)

Moreover, if functions $X$, $Y$, and $Z$ denote $x$, $y$, and $z$ components of vector $A$, the divergence of $A$ is given by

$$\text{div} \ A = \nabla \cdot A = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$  \hspace{1cm} (10.10)

where $\nabla$ is called nabla. Nabla represents the following differential operator:

$$\nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z},$$  \hspace{1cm} (10.11)

where $u_x$, $u_y$, and $u_z$ denote unit vectors along the $x$, $y$, and $z$-axis directions.

The expression $\nabla \cdot A$ in Eq. (10.10) is confirmed by the fact that the scalar product of $\nabla$ and vector $A$ is given by

$$\nabla \cdot A = \left[ u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right] \cdot \left[ X u_x + Y u_y + Z u_z \right] = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$  \hspace{1cm} (10.12)

Substituting Eq. (10.10) in Eq. (10.9), we obtain

$$\iiint_V \nabla \cdot A \, dx \, dy \, dz = \iiint_S A \cdot n \, dS,$$  \hspace{1cm} (10.13)

where $n$ is an outward vector normal to the incremental surface area $dS$, which is given by

$$n = u_x \cos \alpha + u_y \cos \beta + u_z \cos \gamma.$$  \hspace{1cm} (10.14)

Equation (10.13) is called Gauss's theorem, which states that the summation of the divergence of vector $A$ in a volume space is equal to the sum of the outward normal components of $A$ on the surface enclosing the space.

Next, let us consider the special case where the shape of volume $V$ is columnlike, with its upper and lower surfaces parallel to the $x$-$y$ plane. Assuming $Z = 0$ and $X$ and $Y$ are independent of $z$, Eq. (10.9) reduces to

$$\iiint_S \left[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right] \, dx \, dy = \oint_C (X \cos \alpha + Y \cos \beta) \, dl.$$  \hspace{1cm} (10.15)

Here surface area $S$ and contour $C$ enclosing $S$ are as shown in Fig. 10.2. $\oint C \, dl$ represents the line integral along contour $C$. When we introduce a two-dimensional differential operator in the $x$-$y$ plane

$$\nabla_i = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y},$$  \hspace{1cm} (10.16)

Figure 10.2: Surface area $S$ and contour $C$ enclosing $S$ in two-dimensional Gauss's theorem.
Equation (10.15) is rewritten as

\[ \int \int \nabla \cdot \mathbf{A} \, d\mathbf{v} = \oint_{c} \mathbf{A} \cdot \mathbf{n} \, d\mathbf{l}. \]  

(10.17)

10.2 Green's theorem

Let the functions \( X, Y, \) and \( Z \) in Eq. (10.9) be expressed by

\[ X = F \frac{\partial G}{\partial x}, \quad Y = F \frac{\partial G}{\partial y}, \quad Z = F \frac{\partial G}{\partial z}, \]  

(10.18)

where \( F \) and \( G \) are functions of \( x, y, \) and \( z \). Substituting Eq. (10.18) in Eq. (10.10), we obtain

\[ \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = F \nabla^2 G + \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial G}{\partial z}. \]  

(10.19)

Here \( \nabla^2 \) is a Laplacian operator defined by

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \]  

(10.20)

Since \( \mathbf{n} \) is an outward unit normal vector perpendicular to the incremental surface area \( dS \), angles \( \alpha, \beta, \) and \( \gamma \) between \( \mathbf{n} \) and the \( x, y, \) and \( z \)-axes, respectively, are given by

\[ \cos \alpha = \frac{\partial x}{\partial n}, \quad \cos \beta = \frac{\partial y}{\partial n}, \quad \cos \gamma = \frac{\partial z}{\partial n}. \]  

(10.21)

\[ X \cos \alpha + Y \cos \beta + Z \cos \gamma \]  

in Eq. (10.9) is then expressed, by using Eqs. (10.18) and (10.21), as

\[ X \cos \alpha + Y \cos \beta + Z \cos \gamma = F \left[ \frac{\partial G}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial n} + \frac{\partial G}{\partial z} \frac{\partial z}{\partial n} \right] = F \frac{\partial G}{\partial n}. \]  

(10.22)

Substituting Eqs. (10.19) and (10.22) in Eq. (10.9), we obtain

\[ \int \int \int_{V} F \nabla^2 G \, d\mathbf{v} + \int \int \left[ \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial G}{\partial z} \right] \, d\mathbf{v} = \oint_{S} F \frac{\partial G}{\partial n} \, dS. \]  

(10.23)

Exchanging \( F \) and \( G \) in the last equation, we have

\[ \int \int \int_{V} G \nabla^2 F \, d\mathbf{v} + \int \int \left[ \frac{\partial G}{\partial x} \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} \frac{\partial F}{\partial z} \right] \, d\mathbf{v} = \oint_{S} G \frac{\partial F}{\partial n} \, dS. \]  

(10.24)

Subtracting Eq. (10.24) from Eq. (10.23), we obtain

\[ \int \int \int_{V} \left( F \nabla^2 G - G \nabla^2 F \right) \, d\mathbf{v} = \oint_{S} \left[ F \frac{\partial G}{\partial n} - G \frac{\partial F}{\partial n} \right] \, dS. \]  

(10.25)

This is called Green's theorem.

When we substitute Eq. (10.21) into Eq. (10.14), the outward normal vector \( \mathbf{n} \) can be expressed as

\[ \mathbf{n} = \mathbf{u}_x \frac{\partial x}{\partial n} + \mathbf{u}_y \frac{\partial y}{\partial n} + \mathbf{u}_z \frac{\partial z}{\partial n}. \]  

(10.26)

We notice here that the gradient of function \( G \) is expressed by

\[ \text{grad} G = \nabla G = \mathbf{u}_x \frac{\partial G}{\partial x} + \mathbf{u}_y \frac{\partial G}{\partial y} + \mathbf{u}_z \frac{\partial G}{\partial z}. \]  

(10.27)

The scalar product of \( \nabla G \) and \( \mathbf{n} \) is then given by

\[ \nabla G \cdot \mathbf{n} = \mathbf{u}_x \frac{\partial G}{\partial x} \frac{\partial x}{\partial n} + \mathbf{u}_y \frac{\partial G}{\partial y} \frac{\partial y}{\partial n} + \mathbf{u}_z \frac{\partial G}{\partial z} \frac{\partial z}{\partial n}. \]  

(10.28)

Green's theorem can be rewritten, by using Eq. (10.28), as

\[ \int \int \int_{V} \left( F \nabla^2 G - G \nabla^2 F \right) \, d\mathbf{v} = \oint_{S} \left( F \nabla G - G \nabla F \right) \cdot \mathbf{n} \, dS. \]  

(10.29)

Next let us consider the special case where the shape of volume \( V \) is columnlike, with its upper and lower surfaces parallel to the \( x-y \) plane. Using a similar two-dimensional coordinate as shown in Fig. 10.2, we obtain a two-dimensional expression of Green's theorem:

\[ \int \int_{S} \left( F \nabla^2 G - G \nabla^2 F \right) \, dS = \oint_{c} \left( F \nabla G - G \nabla F \right) \cdot \mathbf{n} \, d\mathbf{c}. \]  

(10.30)
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10.3 Stokes' theorem

Let us consider the line integral of vector $\mathbf{A}$ along contour $C$ (Fig. 10.3) of the form

$$\int_C \mathbf{A} \cdot ds,$$

where $ds$ denotes the line increment vector along contour $C$. The direction of line integration on $C$ is defined by the right-hand screw with respect to the vector $n$ normal to surface $S$, as shown in Fig. 10.3. When we divide surface $S$ into many small triangular surfaces $\Delta S$, Eq. (10.31) is given by the sum of the line integrals along contour $C$:

$$\int_C \mathbf{A} \cdot ds = \sum \int_{C'} \mathbf{A} \cdot ds.$$  

(10.32)

The validity of the last expression is confirmed as follows. Consider, for example, the case in which contour $C$ in Fig. 10.3 is divided into two sections by line $ab$, as shown in Fig. 10.4. Line integrals along $C_1$ and $C_2$ cancel each other out, since the line increment vector $ds$ is opposite on line $ab$. Therefore, it is known that sum of the line integrals along $C_1$ and $C_2$ in Fig. 10.4 is equal to the line integral along $C$ in Fig. 10.3. By the same principle, the sum of the line integrals along contour $C'$ of subdivided triangular $\Delta S$ becomes equal to the line integral along $C$ in Fig. 10.3.

We then take triangular region $PQR$ in Fig. 10.5 as $\Delta S$ and calculate the line integral:

$$\int_{C'} \mathbf{A} \cdot ds.$$  

(10.33)

Figure 10.3: Coordinates to describe Stokes' theorem.

Figure 10.4: Division of the line integral.

As just described, the line integral along $PQRP$ is equal to the sum of the line integrals along $OPQO$, $OQRO$, and $ORPO$. First let us consider the line integral along $OPQO$, as shown in Fig. 10.6. Since the line increment vector $ds$ on $OP$ is directed along the $x$-axis, only the $x$-component of $\mathbf{A}$ is important.

Values of $\mathbf{A}$ at point $O$ and $P$ are expressed as

$$\left( A_x, A_y, A_z \right)$$

point $O$

$$\left( A_x + \frac{\partial A_x}{\partial x} \Delta x, A_y + \frac{\partial A_y}{\partial x} \Delta x, A_z + \frac{\partial A_z}{\partial x} \Delta x \right)$$

point $P$

(10.34)

The line integral on $OP$ is then given by

$$A_x + \left( A_x + \frac{\partial A_x}{\partial x} \Delta x \right) \frac{\Delta x}{2}.$$  

(10.35a)

Figure 10.5: Small triangular region $\Delta S$ for calculating the line integral.
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Therefore, Eq. (10.37) is expressed as

\[
\int_{OPQO} A \cdot ds = \left[ \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} \right] \Delta S \cos \gamma. \quad \text{(10.38)}
\]

In a similar manner, the line integrals along \(OQRO\) and \(ORPO\) are obtained:

\[
\int_{OQRO} A \cdot ds = \left[ \frac{\partial A_x}{\partial y} - \frac{\partial A_z}{\partial z} \right] \Delta S \cos \alpha; \quad \text{(10.39)}
\]

\[
\int_{ORPO} A \cdot ds = \left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_y}{\partial x} \right] \Delta S \cos \beta. \quad \text{(10.40)}
\]

The line integral of Eq. (10.33) is then given as the sum of the line integrals in Eqs. (10.38)–(10.40):

\[
\oint_{C} A \cdot ds = \left[ \frac{\partial A_x}{\partial y} - \frac{\partial A_z}{\partial z} \right] \cos \alpha + \left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_y}{\partial x} \right] \cos \beta + \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y} \right] \cos \gamma \right] \Delta S. \quad \text{(10.41)}
\]

We introduce here vector \(B\), which is defined by

\[
B = \left[ \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z} \right] u_x + \left[ \frac{\partial A_z}{\partial z} - \frac{\partial A_x}{\partial x} \right] u_y + \left[ \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} \right] u_z. \quad \text{(10.42)}
\]

Since the value in the braces in Eq. (10.41) is given by the scalar product of \(B\) and the normal vector \(n\) [see Eq. (10.14)], Eq. (10.41) is expressed as

\[
\oint_{C} A \cdot ds = B \cdot n \Delta S. \quad \text{(10.43)}
\]

Vector \(B\) defined by Eq. (10.41) is called the rotation of vector \(A\) and is expressed by

\[
B = \text{rot } A = \nabla \times A = \left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] u_x + \left[ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] u_y + \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] u_z. \quad \text{(10.44)}
\]

We note here that the vector product of \(E\) and \(H\) is expressed by

\[
E \times H = \left[ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right] = (E_x E_y E_z) \left[ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right] = (E_x H_y - E_y H_x) u_x + (E_y H_z - E_z H_y) u_y + (E_z H_x - E_x H_z) u_z. \quad \text{(10.45)}
\]
When we replace \( E = V \) [see Eq. (10.11)] and \( H = A \) in the last equation, the formal expression \( V \times A \) coincides with \( B \) in Eq. (10.42). Substituting Eq. (10.43) in Eq. (10.32), we obtain an expression for Stokes' theorem:

\[
\sum_{c} A \cdot ds = \int_{S} B \cdot n dS = \int_{S} (V \times A) \cdot n dS. \quad (10.46)
\]

### 10.4 Integral theorem of Helmholtz and Kirchhoff

Let us consider a volume \( V \) enclosed by a surface \( S \), as shown in Fig. 10.7. The volume does not include the singular point \( P \). According to Green's theorem, the following relationship holds for the given functions \( F \) and \( G \) inside a volume \( V \) enclosed by the surface \( S \):

\[
\int_{V} \left( \nabla^{2} G - G \nabla^{2} F \right) dv = \int_{S} (F \nabla G - G \nabla F) \cdot n dS, \quad (10.47)
\]

where \( n \) is the outward vector normal to surface \( S \). In order to investigate the diffraction of light, we consider \( F \) to be a wave function of light, which satisfies Helmholtz equation:

\[
\nabla^{2} F + k^{2} F = 0. \quad (10.48)
\]

Here we assumed the refractive index in volume \( V \) is \( n = 1 \). Next, \( G \) is considered to be a spherical wave originating at point \( P \). Function \( G \) is expressed by

\[
G = \frac{1}{r} e^{-\alpha r} \quad (10.49)
\]

where \( r \) is measured from point \( P \). Since \( G \to \infty \) for \( r \to 0 \), point \( P \) is a singular point. In order to apply Green's theorem to functions \( F \) and \( G \), the singular point should be excluded from volume \( V \). Then a small sphere centered at \( P \) with radius \( \varepsilon \) is excluded from volume \( V \). The removal of this sphere creates a new surface \( S' \), and the total surface of the volume becomes \( S + S' \), where \( S' \) is the external surface as shown in Fig. 10.7. Spherical wave \( G \) satisfies the Helmholtz equation in spherical coordinates (Fig. 10.8):

\[
\nabla^{2} G + k^{2} G = \frac{1}{r^{2}} \frac{\partial}{\partial r} \left( r^{2} \frac{\partial G}{\partial r} \right) + \frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial G}{\partial \theta} \right) + \frac{1}{r^{2} \sin^{2} \theta} \frac{\partial^{2} G}{\partial \phi^{2}} + k^{2} G = 0. \quad (10.50)
\]

Substituting Eqs. (10.48) and (10.50) in Eq. (10.47), we obtain

\[
\int_{S} (F \nabla G - G \nabla F) \cdot n dS = 0. \quad (10.51)
\]

![Figure 10.7](image1.png) A volume \( V \) enclosed by a surface \( S \). The volume does not include the singular point \( P \).

![Figure 10.8](image2.png) A spherical coordinate system.
Substitution of Eq. (10.49) into Eq. (10.51) gives

\[
\oint_S \left[ F\left(\frac{e^{-jk\rho}}{r}\right) - \frac{e^{-jk\rho}}{r} \nabla F \right] \cdot \mathbf{n} \, dS + \oint_{S'} \left[ F\left(\frac{e^{-jk\rho}}{r}\right) - \frac{e^{-jk\rho}}{r} \nabla F \right] \cdot \mathbf{n} \, dS = 0.
\]

(10.52)

Since the unit vectors \( \mathbf{u}_r \) and \( \mathbf{n} \) are parallel but in opposite directions on the spherical surface with radius \( r \), the value of \( \mathbf{u}_r \cdot \mathbf{n} = -1 \). Hence the value of \( \nabla (e^{-jk\rho}/r) \cdot \mathbf{n} \) on the surface of sphere \( S'(r = \epsilon) \) is given by

\[
\left[ \nabla \left(\frac{e^{-jk\rho}}{r}\right) \right] \cdot \mathbf{n} = \left[ \mathbf{u}_r \cdot \frac{d}{dr} \left(\frac{e^{-jk\rho}}{r}\right) \right] \cdot \mathbf{n} = \left(-\frac{jk}{r} - \frac{1}{r} \right) e^{-jk\rho} \mathbf{u}_r \cdot \mathbf{n} = \left(jk + \frac{1}{r} \right) \frac{e^{-jk\rho}}{r}.
\]

(10.53)

Putting this last value into the second term on the left-hand side of Eq. (10.52) and taking the limit as \( \epsilon \to 0 \), and noting that \( dS = 4\pi\epsilon^2 \) on the surface of \( S' \), we obtain

\[
\lim_{\epsilon \to 0} \oint_{S'} \left[ F\left(\frac{e^{-jk\rho}}{r}\right) - \frac{e^{-jk\rho}}{r} \nabla F \cdot \mathbf{n} \right] \, dS = 4\pi F(P).
\]

(10.54)

Substitution of Eq. (10.54) into Eq. (10.52) gives

\[
F(P) = \frac{1}{4\pi} \oint_S \left[ e^{-jk\rho} \nabla F - F\left(\frac{e^{-jk\rho}}{r}\right) \right] \cdot \mathbf{n} \, dS.
\]

(10.55)

The last equation is called the integration theorem of Helmholtz and Kirchhoff. Using this theorem, the amplitude of the light at an arbitrary observation point \( P \) can be obtained by knowing the field distribution of light \( F \) and \( \partial F/\partial n \) on the surface enclosing the observation point.

10.5 Fresnel–Kirchhoff diffraction formula

The integration theorem of Helmholtz and Kirchhoff will be used to find the diffraction pattern of an aperture when illuminated by a point source and projected onto a screen. Let us consider a domain of integration enclosed by a masking screen \( S_c \), a surface \( S_a \) bridging the aperture, and a semisphere \( S_R \) centered at the observation point \( P \) with radius \( R \), as shown in Fig. 10.9. Since the light amplitude on the surface of the masking screen \( S_c \) is zero, the surface integral of Eq. (10.55) becomes

\[
F(P) = \frac{1}{4\pi} \iint_{S_a} \left[ e^{-jk\rho} \nabla F - F\left(\frac{e^{-jk\rho}}{r}\right) \right] \cdot \mathbf{n} \, dS.
\]

(10.56)

The integral over semisphere \( S_R \) is expressed as

\[
\iint_{S_R} \left[ e^{-jk\rho} \nabla F - F\left(\frac{e^{-jk\rho}}{r}\right) \right] \cdot \mathbf{n} \, dS,
\]

where \( \mathbf{u}_r \) is a unit vector pointing from \( P \) to the point on \( S_R \). When \( R \) is very large, the integral over \( S_R \) can be approximated as

\[
\iint_{S_R} \frac{e^{-jk\rho}}{R} \left( \nabla F \cdot \mathbf{n} + jk F R^2 \right) d\Omega,
\]

(10.57)

where \( \Omega \) is the solid angle from \( P \) to \( S_R \). Since the directions of \( \mathbf{u}_r \) and \( \mathbf{n} \) are identical, \( \mathbf{u}_r \cdot \mathbf{n} \) is unity. If the condition

\[
\lim_{R \to \infty} R \left( \nabla F \cdot \mathbf{n} + jk F R^2 \right) = 0
\]

(10.58)

is satisfied, the integral of Eq. (10.57) vanishes. This condition is called the...
Sommerfeld radiation condition. If function $F$ is a spherical wave expressed by Eq. (10.49), then Eq. (10.58) is indeed satisfied as

$$\lim_{R \to \infty} R[\nabla F \cdot n + jkF] = \lim_{R \to \infty} \left( -\frac{e^{-jkR}}{R} \right) = 0. \quad (10.59)$$

Actually, a light wave entering the aperture is a spherical wave or a summation of spherical waves. Therefore, the Sommerfeld radiation condition is generally satisfied.

It is then seen that in order to calculate $F(P)$ we should consider the integral just over $S_A$. Taking $s$ as the distance from the source $P_0$ to point $Q$, the amplitude of a spherical wave on the aperture is given by

$$F = A \frac{e^{-ik_s}}{s}, \quad (10.60)$$

where $A$ is a constant. Since the direction of vector $\nabla F$ is such that the change in $F$ with respect to a change in location is a maximum, $\nabla F$ is in the same direction as $u_s$. Therefore, $\nabla F$ is expressed as

$$\nabla F = u_s A \frac{d}{ds} \left( \frac{e^{-ik_s}}{s} \right) = -u_s A \left( jk + \frac{1}{s} \right) \frac{e^{-ik_s}}{s}. \quad (10.61)$$

Also we have

$$\nabla \left( \frac{e^{-ikr}}{r} \right) = -u_s \left( jk + \frac{1}{r} \right) \frac{e^{-ikr}}{r}. \quad (10.62)$$

When $s$ and $r$ are much longer than a wavelength, the second term inside the parentheses on the right-hand side of Eqs. (10.61) and (10.62) can be ignored when compared to the first term inside those parentheses. Substituting all the foregoing results in Eq. (10.56) we obtain

$$F(P) = \frac{Ak}{4\pi} \int_{S_A} \frac{e^{-j(n \cdot r)}}{sr} [u_s \cdot n - u_s \cdot n] \, dS. \quad (10.63)$$

Expressing the angle between unit vectors $u_s$ and $n$ as $(u_s, n)$ and expressing the angle between $u_s$ and $n$ as $(u_s, n)$, the preceding equation is rewritten as

$$F(P) = \frac{Ak}{4\pi} \int_{S_A} \frac{e^{-j(n \cdot r)}}{sr} [\cos(u_s, n) - \cos(u_s, n)] \, dS. \quad (10.64)$$

Equation (10.64) is called the Fresnel–Kirchhoff diffraction formula.

Among the factors in the integrand of Eq. (10.64), the term

$$\cos(u_s, n) - \cos(u_s, n) \quad (10.65)$$

is the obliquity factor, which relates to the incident and transmission angles. For the special case in which the light source is located approximately on the center with respect to the aperture, the obliquity factor becomes $(1 + \cos \psi)$, where $\psi$ is the angle between $u_s$ and line $QP$, as shown in Fig. 10.10. Then Eq. (10.64) is expressed as

$$F(P) = \frac{Ak}{4\pi} \int_{S_A} \frac{e^{-j(n \cdot r)}}{sr} (1 + \cos \psi) \, dS. \quad (10.66)$$

When both $s$ and $r$ are nearly perpendicular to the mask screen, the angle $\psi \approx 0$. Then Eq. (10.66) becomes

$$F(P) = \frac{Ak}{2\pi} \int_{S_A} \frac{e^{-j(n \cdot r)}}{sr} \, dS. \quad (10.67)$$

If we express the amplitude of the spherical wave on the aperture by

$$g = A \frac{e^{-ik_r}}{r}, \quad (10.67)$$

then $F(P)$ can be expressed by

$$F(P) = \frac{k}{2\pi} \int_{S_A} g \frac{e^{-jkr}}{r} \, dS. \quad (10.68)$$

Figure 10.10: Case where source $P_0$ is located near the center of the aperture.
This last equation states that if the amplitude distribution \( g \) of the light across the aperture is known, then the field \( F(P) \) at the point of observation can be obtained by Eq. (10.68). It can be interpreted as a mathematical formulation of Huygens' principle. The integral can be thought of as a summation of contributions from innumerable small spherical sources of amplitude \( g \, dS \) lined up along aperture \( S_a \).

### 10.6 Formulas for vector analysis

Here, unit vectors directed along the \( x \)-, \( y \)- and \( z \)-axis directions are denoted by \( \mathbf{u}_x \), \( \mathbf{u}_y \), and \( \mathbf{u}_z \), respectively. First an important equation related to the vector product will be explained.

Consider a scalar product of \( \mathbf{A} \) with \( \mathbf{B} \times \mathbf{C} \). Referring to the expression of the vector product in Eq. (10.45), we can express

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x)
= A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x),
\]

\[
\left| \begin{array}{ccc}
A_x & A_y & A_z \\
B_x & B_y & B_z \\
C_x & C_y & C_z
\end{array} \right|.
\] (10.69)

The value of the determinant in Eq. (10.69) remains the same when \( A \), \( B \), and \( C \) are rotated in the manner of \( A \rightarrow B \rightarrow C \rightarrow A \). Then we have the equality of

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).
\] (10.70)

Next, important equations relating to the divergence [see Eq. (10.10)] and gradient [see Eq. (10.27)] of vector \( \mathbf{A} \) will be derived.

We first consider the meaning of \( \nabla \times (\nabla \times \mathbf{A}) \). The \( x \)-component of \( \nabla \times (\nabla \times \mathbf{A}) \) is given by

\[
[\nabla \times (\nabla \times \mathbf{A})]_x = \frac{\partial}{\partial y} (\nabla \times \mathbf{A})_z - \frac{\partial}{\partial z} (\nabla \times \mathbf{A})_y
= \frac{\partial}{\partial y} \left[ \frac{\partial A_x}{\partial x} + \frac{\partial A_z}{\partial z} \right] - \frac{\partial}{\partial z} \left[ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right]
= \frac{\partial}{\partial x} \left[ \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right] - \frac{\partial}{\partial x} \left[ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} \right].
\] (10.71)

Here \( \frac{\partial^2 A_z}{\partial x^2} \) was added to and subtracted from the equation to obtain the rightmost expression. Similarly, the \( y \)- and \( z \)-components are obtained:

\[
[\nabla \times (\nabla \times \mathbf{A})]_y = \frac{\partial}{\partial z} \left[ \frac{\partial A_y}{\partial z} + \frac{\partial A_x}{\partial x} \right] - \frac{\partial}{\partial x} \left[ \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right]
= \frac{\partial}{\partial z} \left[ \frac{\partial^2 A_y}{\partial z^2} + \frac{\partial^2 A_z}{\partial y^2} \right],
\]

\[
[\nabla \times (\nabla \times \mathbf{A})]_z = \frac{\partial}{\partial x} \left[ \frac{\partial A_z}{\partial x} + \frac{\partial A_y}{\partial y} \right] - \frac{\partial}{\partial y} \left[ \frac{\partial A_z}{\partial z} + \frac{\partial A_x}{\partial x} \right]
= \frac{\partial}{\partial x} \left[ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} \right].
\] (10.73)

Noticing Eqs. (10.71)–(10.73), \( \nabla \times (\nabla \times \mathbf{A}) \) is expressed in operator form as

\[
\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}).
\] (10.74)

The formal expression of \( \nabla^2 \mathbf{A} \) is used, since the operator

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

is applied to \( A_x \), \( A_y \), and \( A_z \) in Eqs. (10.71)–(10.73). We should note that the expression of Eq. (10.74) is applicable to \( \nabla \times (\nabla \times \mathbf{A}) \) only in Cartesian coordinates.

Next, let us consider about \( \nabla \times (\nabla \times \mathbf{A}) \). According to the formulas for a vector product and the divergence of a vector, \( \nabla \cdot (\mathbf{A} \times \mathbf{B}) \) is written as

\[
\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \frac{\partial}{\partial x} (A_y B_z - A_z B_y) + \frac{\partial}{\partial y} (A_z B_x - A_x B_z) + \frac{\partial}{\partial z} (A_x B_y - A_y B_x)
= B_x \left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] + B_y \left[ \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right] + B_z \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]
= \nabla \cdot (\nabla \times \mathbf{A}) + \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})
= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).
\] (10.75)

Letting \( \psi \) be a scalar function, we obtain two expressions relating to the divergence and rotation of \( \psi \mathbf{A} \):

\[
\nabla \cdot (\psi \mathbf{A}) = \frac{\partial}{\partial x} (\psi A_x) + \frac{\partial}{\partial y} (\psi A_y) + \frac{\partial}{\partial z} (\psi A_z)
= \left[ \frac{\partial \psi}{\partial x} A_x + \frac{\partial \psi}{\partial y} A_y + \frac{\partial \psi}{\partial z} A_z \right] + \psi \left[ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] = \nabla \psi \cdot \mathbf{A} + \psi \nabla \cdot \mathbf{A}
\] (10.76)
\[ \nabla \times (\psi \mathbf{A}) = u_x \left( \frac{\partial}{\partial y} (\psi A_y) - \frac{\partial}{\partial z} (\psi A_z) \right) + u_y \left( \frac{\partial}{\partial z} (\psi A_z) - \frac{\partial}{\partial x} (\psi A_x) \right) + u_z \left( \frac{\partial}{\partial x} (\psi A_x) - \frac{\partial}{\partial y} (\psi A_y) \right) \]
\[ = u_x \left[ \frac{\partial \psi}{\partial y} A_y - \frac{\partial \psi}{\partial z} A_z \right] + u_y \left[ \frac{\partial \psi}{\partial z} A_z - \frac{\partial \psi}{\partial x} A_x \right] + u_z \left[ \frac{\partial \psi}{\partial x} A_x - \frac{\partial \psi}{\partial y} A_y \right] \]
\[ + \psi \left[ \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial z} \right] + u_y \left[ \frac{\partial A_z}{\partial z} - \frac{\partial A_x}{\partial x} \right] + u_z \left[ \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} \right] \]
\[ = \nabla \psi \times \mathbf{A} + \psi \nabla \times \mathbf{A}. \quad (10.77) \]

Other important formulas for vector analyses are
\[ \nabla \cdot (\nabla \psi) = \nabla^2 \psi \quad (10.78) \]
\[ \nabla \times (\nabla \psi) = 0 \quad (10.79) \]
\[ \nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (10.80) \]

10.7 Formulas in cylindrical and spherical coordinates

In this section, formulas in cylindrical and spherical coordinates are summarized.

10.7.1 Cylindrical coordinates
\[ (\nabla \psi)_r = \frac{\partial \psi}{\partial r}, \quad (\nabla \psi)_\theta = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad (\nabla \psi)_z = \frac{\partial \psi}{\partial z} \quad (10.81) \]
\[ \begin{cases} 
(\nabla \times \mathbf{A})_r = \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \\
(\nabla \times \mathbf{A})_\theta = \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \\
(\nabla \times \mathbf{A})_z = \frac{1}{r} \left( \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) 
\end{cases} \quad (10.82) \]
\[ \nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \quad (10.83) \]
\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \quad (10.84) \]

10.7.2 Spherical coordinates
\[ \begin{align*}
(\nabla \psi)_r &= \frac{\partial \psi}{\partial r}, \quad (\nabla \psi)_\theta = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad (\nabla \psi)_\phi = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \\
(\nabla \times \mathbf{A})_r &= \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta A_\theta) - \frac{\partial A_\phi}{\partial \phi} \right) \\
(\nabla \times \mathbf{A})_\theta &= \frac{1}{r \sin \theta} \frac{\partial A_z}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\
(\nabla \times \mathbf{A})_\phi &= \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla^2 \psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial \phi^2} 
\end{align*} \quad (10.85, 10.86, 10.87, 10.88) \]

References