4.2 The central emission frequency is
\[ \nu_0 = c/\lambda_0 = (3 \times 10^8 \text{ m s}^{-1}) / (632.8 \times 10^{-9} \text{ m}) = 4.74 \times 10^{14} \text{ s}^{-1}. \]

The FWHM width of the frequencies \( \Delta \nu_{1/2} \) observed will be given by Eq. (3)
\[
\Delta \nu_{1/2} = 2 \nu_0 \sqrt{\frac{2k_B T \ln(2)}{MC^2}} = 2(4.748 \times 10^{14}) \sqrt{\frac{(1.38 \times 10^{-23})(130 + 273) \ln(2)}{(3.35 \times 10^{-30})(3 \times 10^8)^2}} = 1.515 \text{ GHz}
\]

To get FWHM wavelength width \( \Delta \lambda_{1/2} \), differentiate \( \lambda = c/\nu \)
\[
\frac{d\lambda}{d\nu} = c/\nu^2 = \frac{\lambda}{\nu^2}
\]
so that
\[
\Delta \lambda_{1/2} \approx \Delta \nu_{1/2} \frac{\lambda}{\nu^2} = (1.515 \times 10^9 \text{Hz})(632.8 \times 10^{-6} \text{ m}) / (4.74 \times 10^{14} \text{ s}^{-1})
\]
or
\[
\Delta \lambda_{1/2} \approx 2.02 \times 10^{-13} \text{ m or 0.0020} \text{nm}
\]

This width is between the half-points of the spectrum.

b For \( \lambda = \lambda_0 = 632.8 \text{ nm} \), the corresponding mode number \( m_0 \) is,
\[ m_0 = 2L / \lambda_0 = (2 \times 0.5 \text{ m}) / (632.8 \times 10^{-9} \text{ m}) = 1580278.1 \]
and actual \( n_0 \) has to be the closest integer value to 1580278.1, that is 1580278

Consider the minimum and maximum wavelengths corresponding to the extremes of the spectrum at the half-power points:
\[ \lambda_{\text{min}} = \lambda_0 - \frac{1}{2} \Delta \lambda = 632.798987 \]
and
\[ \lambda_{\text{max}} = \lambda_0 + \frac{1}{2} \Delta \lambda = 632.801012 \]

One can now increase and decrease \( m_0 \) step by step and calculate the corresponding wavelength \( \lambda_m \) for each choice of \( m \), and examine whether it is within \( \lambda_{\text{min}} < \lambda_m < \lambda_{\text{max}} \), as in Table 4Q2.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \lambda_{\text{nn}} )</th>
<th>( \lambda_{\text{nm}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1580275</td>
<td>632.801229</td>
<td>632.800556</td>
</tr>
<tr>
<td>1580276</td>
<td>632.801259</td>
<td>632.800582</td>
</tr>
<tr>
<td>1580277</td>
<td>632.801252</td>
<td>632.800572</td>
</tr>
<tr>
<td>1580278 = ( m_0 )</td>
<td>632.800000</td>
<td>632.799851</td>
</tr>
<tr>
<td>1580279</td>
<td>632.799882</td>
<td>632.799803</td>
</tr>
<tr>
<td>1580280</td>
<td>632.799849</td>
<td>632.799849</td>
</tr>
<tr>
<td>1580281</td>
<td>632.799849</td>
<td>632.799849</td>
</tr>
<tr>
<td>1580282</td>
<td>632.799849</td>
<td>632.799849</td>
</tr>
</tbody>
</table>

Five modes are possible.

e The frequency separation \( \Delta \nu_m \) of two consecutive modes is
\[ \Delta \nu_m = \nu_{m+1} - \nu_m = \frac{c}{\lambda_{m+1}} - \frac{c}{\lambda_m} = \frac{c}{2L} (m+1) - \frac{c}{2L} m = \frac{c}{2L} \]
or
\[ \Delta \nu_m = \frac{c}{2L} = \frac{3 \times 10^8}{2(0.5)} = 3 \times 10^6 \text{ Hz.} \]
The wavelength separation of two consecutive modes is
\[ \Delta \lambda_m = \frac{c^2}{2L} \left( \frac{632.8 \times 10^{-9}}{2(0.5)} \right)^2 = 4.004 \times 10^{-13} \text{ m or 0.4004 pm}. \]

d Consider a given mode \( \lambda_m \), \( m \) constant, and
\[ m \frac{\lambda}{2L} = L \]
so that
\[ m \frac{\delta \lambda}{\lambda} = \delta L \]
\[ \frac{\delta \lambda}{\lambda} = \frac{2 \delta L}{L} \]
\[ \frac{\delta \lambda}{L} = \frac{2 \delta L}{L \delta L} \]
\[ \frac{\delta \lambda}{\delta L} = \frac{2 \delta L}{L \delta L} \]
\[ \frac{\delta \lambda}{\delta L} = (632.8 \times 10^{-9} \text{ m})(1 \times 10^{-6} \text{ K}^{-1}) = 0.6328 \text{ pm K}^{-1} \]
For \( \delta T = 130 - 20 = 110 \text{ °C change, } \delta \lambda_m = (110 \text{ K})(0.6328 \text{ pm K}^{-1}) = 69.6 \text{ pm} > \Delta \lambda_{1/2} = 2.02 \text{ pm}. \]
As the tube warms up, the modes sweep across the output spectrum. This is called mode sweeping.
e Consider the mode separation in frequency,
\[ \Delta \nu_m = \frac{c}{2L} \]
\[ \frac{d \Delta \nu_m}{dT} = -\frac{c}{2L} \frac{\nu}{dT} = -\frac{c}{2L} \frac{dL}{dT} = -\frac{c}{2L} \alpha \]
since \( \alpha = L^2/dLdT \).
Thus,
\[ \frac{d \Delta \nu_m}{dT} = -\frac{3 \times 10^6 \text{ m s}^{-1}}{2 \times 0.5 \text{ m}} (1 \times 10^{-6} \text{ K}^{-1}) = -300 \text{ Hz K}^{-1} \]
For \( \delta T = 110 \text{ K}, \)
\[ \delta \Delta \nu_m = (300 \text{ Hz K}^{-1})(110 \text{ K}) = 33 \text{ kHz} \]
Change in the mode separation is 0.014%.

e Consider the mode separation in wavelength,
\[ \Delta \lambda_m = \frac{2L}{m} \]
\[ \frac{d \Delta \lambda_m}{dT} = \frac{2L}{m+1} \frac{dL}{dT} = \frac{2L}{m+1} \frac{2L}{m} \alpha \]
since \( \alpha = L^2/dLdT \).
Thus,
\[ \frac{d \Delta \lambda_m}{dT} = \Delta \lambda_m \alpha = (4.004 \times 10^{-13} \text{ m})(1 \times 10^{-6} \text{ K}^{-1}) = 4.004 \times 10^{-19} \text{ m K}^{-1} \]
For \( \delta T = 110 \text{ K}, \)
\[ \delta \Delta \lambda_m = (4.004 \times 10^{-19} \text{ m K}^{-1})(110 \text{ K}) = 4.41 \times 10^{-17} \text{ m} \]
Change in the mode wavelength separation is 0.011%.

Increase the power input (to excite more atoms), decrease tube diameter to encourage collisions with walls to increase the rate of return to the ground state, and increase the active volume by increasing length (not by increasing the tube diameter).
4.6

\[ g_n = \gamma - \frac{1}{2L} \ln(R_R R_L) = 0.05 \text{ m}^{-1} - \frac{1}{2(0.4 \text{ m})} \ln(1 \times 0.99) = 0.077 \text{ m}^{-1}. \]

The emission frequency \( \nu_e = c / \lambda_e = (3 \times 10^8 \text{ m/s}) / (632.8 \times 10^{-9} \text{ m}) = 4.74 \times 10^{15} \text{ s}^{-1}. \) From laser characteristics,

\[ \Delta N_{e} = g_n \frac{8 \pi \nu_e^2 \tau_e \Delta \nu}{c^2} \]

and

\[ = (0.077 \text{ m}^{-1}) \frac{8 \pi (4.74 \times 10^{15} \text{ s}^{-1})^2}{(3 \times 10^8 \text{ m/s})^2} (10^{-9} \text{ s}) (1.5 \times 10^{-9} \text{ s}) \]

\[ = 2.1 \times 10^{13} \text{ m}^{-3}. \]

**Note:** The number of Ne atoms per unit volume \( n_{Ne} \) can be found from the Gas law using the partial pressure of Ne:

\[ P_{Ne} V = \frac{N_{Ne}}{N_a} RT \]

\[ \therefore \quad n_{Ne} = \frac{N_{Ne}}{V} = \frac{N_a P_{Ne} \ln(6.02 \times 10^{20} \text{ mol}^{-1} \text{K}^{-1} \text{L}^{-1} \text{ mol}^{-1} \text{L}^{-1}) (300 \text{ K})}{(8.315 \times 10^{-3} \text{ J K}^{-1} \text{mol}^{-1})(300 \text{ K})} \]

\[ \approx 1.3 \times 10^{21} \text{ m}^{-3}. \]

b \[ g_n = \gamma - \frac{1}{2L} \ln(R_R R_L) = 0.1 \text{ m}^{-1} - \frac{1}{2(1 \text{ m})} \ln(0.999 \times 0.95) = 0.126 \text{ m}^{-1}. \]

The emission frequency \( \nu_e = c / \lambda_e = (3 \times 10^8 \text{ m/s}) / (488 \times 10^{-9} \text{ m}) = 6.14 \times 10^{15} \text{ s}^{-1}. \) From laser characteristics,

\[ \Delta N_{e} = g_n \frac{8 \pi \nu_e^2 \tau_e \Delta \nu}{c^2} \]

and

\[ = (0.126 \text{ m}^{-1}) \frac{8 \pi (6.14 \times 10^{15} \text{ s}^{-1})^2}{(3 \times 10^8 \text{ m/s})^2} (10^{-9} \text{ s}) (3 \times 10^{-9} \text{ s}) \]

\[ = 4.0 \times 10^{14} \text{ m}^{-3}. \]

c \[ g_n = \gamma - \frac{1}{2L} \ln(R_R R_L) = 1000 \text{ m}^{-1} - \frac{1}{2(5 \times 10^{-10} \text{ m})} \ln(0.32 \times 0.32) \]

\[ = 2.4 \times 10^{14} \text{ m}^{-3}. \]

Substantially larger than a gas laser!

4.7

a \ The beam diameter \( 2w \) increases linearly with distance \( z \) as shown in Figure 4Q7-1. The increase in beam diameter \( 2w \) with \( z \) makes an angle \( 2\theta \) at \( O \), as shown in Figure 4Q7-1 which is called the beam divergence. The greater the waist, the narrower the divergence. The two are related by

\[ 2\theta = \frac{4z}{\pi (2w_z)} = \frac{4(632.8 \times 10^{-9} \text{ m})}{\pi (1.5 \times 10^{-3} \text{ m})} = 0.000537 \text{ rad} \]

From Figure 4Q7-2,

\[ \tan \theta = \frac{\Delta r}{L} \quad \therefore \quad \Delta r = L \tan \theta = (20 \text{ m}) \tan(0.000537/2) = 5.37 \text{ mm} \]

so that the beam diameter

\[ 2r = D + 2\Delta r = 1.5 \text{ mm} + 2(5.37 \text{ mm}) = 12.2 \text{ mm} \]
and \[ I = \frac{P}{dW} = \frac{he^2N_{\text{at}}}{2nl} (1 - R) \]
where \( R \) is the reflectance of the crystal face.

b Consider one round trip through the cavity. The length \( L \) is traversed twice and there is one reflection at each end. The overall attenuation of the coherent radiation after one-round trip is \( RR \exp(-\alpha(2L)) \)
where \( R \) is the reflectance of the crystal end.

Equivalently we can represent this reduction by an effective or a total loss coefficient \( \alpha \) such that after one round trip, the reduction factor is \( \exp(-\alpha(2L)) \)

Evaluating the two,
\[ RR \exp(-\alpha(2L)) = \exp(-\alpha(2L)) \]
and rearranging,
\[ \alpha = \frac{\alpha + \frac{1}{2L} \ln \left(\frac{1}{R} \right)} \]

c The reflectance is \[ R = \left(\frac{n-1}{n+1}\right)^2 = \left(\frac{3.5-1}{3.5+1}\right)^2 = 0.309 \]
The total loss coefficient is \( \alpha = \alpha + \frac{1}{2L} \ln \left(\frac{1}{R} \right) = 1000 \text{ m}^{-1} + \frac{1}{60 \times 10^{-4} \text{ m}} \ln \left(\frac{1}{0.309}\right) = 2.06 \times 10^4 \text{ m}^{-1} \).

\[ \tau_{\text{el}} = \frac{n}{e\alpha} = \frac{3.5}{(3 \times 10^4 \text{ m}^{-3})(2.06 \times 10^4 \text{ m}^{-1})} = 5.7 \times 10^{-12} \text{ s} \](0.57 ps).

Coherent radiation is lost from the cavity after, on average, 0.57 ps.

For the above device, threshold current density \( J_{\text{th}} \approx 500 \text{ A cm}^{-2} \) and \( \tau_{\text{el}} \approx 10 \text{ ps}, d \approx 0.25 \text{ mm} \),

From \( J_{\text{th}} = \frac{n_e \tau_{\text{el}}}{e} \),
we have, \( n_e = \frac{J_{\text{th}} \tau_{\text{el}}}{e d} = \frac{(500 \times 10^4 \text{ A cm}^{-2})(10^{-10} \text{ A cm}^{-2})}{(1.6 \times 10^{-19} \text{ C})(0.25 \times 10^{-6} \text{ m})} = 1.25 \times 10^{24} \text{ m}^{-3} \) or \( 1.2 \times 10^{18} \text{ cm}^{-3} \).

Now, the current density corresponding to \( I = 30 \text{ mA} \) is \( J = I/(\pi W) = (0.05 \text{ A})(10^6 \times 10^6 \text{ m}^2) = 833 \times 10^4 \text{ A m}^2 \).

And, \( N_{\text{at}} \approx \tau_{\text{el}} (J - J_{\text{th}}) = \frac{(5.7 \times 10^{-12})}{(1.6 \times 10^{-19} \text{ C})(0.25 \times 10^{-6} \text{ m})} \times (833 - 500) \times 10^4 \text{ A m}^2 \times (1 - 0.309) \approx 4.7 \times 10^{13} \text{ photons m}^{-3} \)

The optical power is \[ P = \frac{he^2N_{\text{at}}dW}{2nl} \]
\[ = (6.62 \times 10^{-34} \text{ J s}) \times (3 \times 10^8 \text{ m s}^{-1})(4.7 \times 10^{13} \text{ m}^3)(0.25 \times 10^{-4} \text{ m})(10 \times 10^{-6} \text{ m}) \times (1 - 0.309) \approx 0.00053 \text{ W or 0.53 mW} \]

Intensity = Optical Power / Area = \( P_e / (dW) \]
\[ = \frac{(0.00053)(0.25 \times 10^{-10} \text{ mm}^3)}{} \]
\[ = 223 \text{ W mm}^{-2} \]

This intensity is right at the crystal face over the optical cavity cross section. As the beam diverges, the intensity decreases away from the laser diode.

4.12

a The external quantum efficiency \( \eta_{\text{QE}} \), of a laser diode is defined as
\[ \eta_{\text{QE}} = \frac{\text{Number of output photons from the diode (per unit second)}}{\text{Number of injected electrons into diode (per unit second)}} \]
\[ \eta_{\text{QE}} = \frac{\text{Optical Power}/h \nu}{\text{Diode Current} / e} = \frac{P_e/E_{e}}{I/e} \]

The external differential quantum efficiency, \( \eta_{\text{DEQ}} \), of a laser diode is defined as
\[ \eta_{\text{DEQ}} = \frac{\text{Increase in number of output photons from diode (per unit second)}}{\text{Number of injected electrons into diode (per unit second)}} \]
\[ \eta_{\text{DEQ}} = \frac{\text{(Change in Optical Power) } / h \nu}{\text{(Change Diode Current) } / e} = \frac{e}{E_{e}} \frac{\epsilon}{\Delta I} \]

The external power efficiency, \( \eta_{\text{PE}} \), of the laser diode is defined by
\[ \eta_{\text{PE}} = \frac{\text{Optical output power}}{\text{Electrical input power}} = \frac{P_e}{I} = \eta_{\text{QE}} (E_{e}/e) \]

b 670 nm laser diode
\[ E_{e} = h c / \lambda = (6.626 \times 10^{-34})(3 \times 10^8)(670 \times 10^{-9})(1.6 \times 10^{-19}) = 1.85 \text{ eV} \]
so that
\[ \eta_{\text{QE}} = \frac{P_e}{E_{e} I} \]
\[ \eta_{\text{DEQ}} = \frac{(1.6 \times 10^{-19} \text{ C})(2 \times 10^{-3} \text{ Js}^{-1})}{(80 \times 10^{-19} \text{ A})(1.85 \text{ eV} \times 1.6 \times 10^{-19} \text{ eV})} = 0.0135 \text{ or 1.35%} \]
\[ \eta_{\text{DEQ}} = \frac{\epsilon}{E_{e}} \frac{\Delta I}{\Delta I} = \frac{(1.6 \times 10^{-19} \text{ C})}{(1.85 \text{ eV} \times 1.6 \times 10^{-19} \text{ eV})} \frac{2 \times 10^{-3} - 2 \times 10^{-3} \text{ Js}^{-1}}{82 \times 10^{-3} - 80 \times 10^{-3} \text{ A}} \]
\[ = 0.270 \text{ or 27%} \]
\[ \eta_{\text{PE}} = \frac{P_e}{2 \times 10^{-3} \text{ W}} = \frac{0.009}{2.3 \text{ V}} = 0.009 \text{ or 1.09%} \]

c 1310 nm laser diode
\[ E_{e} = h c / \lambda = (6.626 \times 10^{-34})(3 \times 10^8)(1310 \times 10^{-9})(1.6 \times 10^{-19}) = 0.9464 \text{ eV} \]
so that
\[ \eta_{\text{QE}} = \frac{P_e}{I} \]
\[ \eta_{\text{DEQ}} = \frac{(1.6 \times 10^{-19} \text{ C})(3 \times 10^{-3} \text{ Js}^{-1})}{(40 \times 10^{-13} \text{ A})(0.9464 \text{ eV} \times 1.6 \times 10^{-19} \text{ eV})} = 0.079 \text{ or 7.9%} \]
\[ \eta_{\text{PE}} = \frac{\epsilon}{E_{e}} \frac{\Delta I}{\Delta I} = \frac{(0.9464 \text{ eV} \times 1.6 \times 10^{-19} \text{ C})}{45 \times 10^{-3} - 40 \times 10^{-3} \text{ A}} \]
\[ = 0.211 \text{ or 21%} \]
and \[ \eta_{RPE} = \frac{P}{IV} = \frac{3 \times 10^{-3}}{(4 \times 10^{-3} \text{ A})(1.4 \text{ V})} = 0.0535 \text{ or } 5.3\% \]

4.14
Bragg wavelength \( \lambda_b \) is given by the condition for in-phase interference,

\[ \frac{q \lambda_b}{n} = 2\Lambda \]

\[ q = 1 \]
\[ \Lambda = q \frac{\lambda_b}{2n} = 1 \frac{1.550 \text{ \mu m}}{2(3.4)} = 0.23 \text{ \mu m} \]

Number of corrugations is

\[ N_{\text{corrugations}} = \frac{\text{Cavity length}}{\text{Corrugation period}} = \frac{L}{\Lambda} = \frac{20 \text{ \mu m}}{0.23 \text{ \mu m}} = 87 \]

\[ q = 2 \]
\[ \Lambda = q \frac{\lambda_b}{2n} = 2 \frac{1.550 \text{ \mu m}}{2(3.4)} = 0.46 \text{ \mu m} \]

Number of corrugations is

\[ N_{\text{corrugations}} = \frac{\text{Cavity length}}{\text{Corrugation period}} = \frac{L}{\Lambda} = \frac{20 \text{ \mu m}}{0.45 \text{ \mu m}} = 43 \]

The corrugation dimension (\( \Lambda \)) is larger and number of corrugations smaller in \( q = 2 \) and would be “easier” to fabricate.

**NOTE** The allowed DFB modes are not exactly at Bragg wavelengths but are symmetrically placed about \( \lambda_b \). If \( \lambda_m \) is an allowed DFB lasing mode then

\[ \lambda_m = \lambda_b + \frac{\lambda_b}{2nL} (m + 1) \]

Putting \( m = 0 \), \( \lambda_m = 1.55 \text{ \mu m} \), we find, \( \lambda_m = 1.5319 \) and \( 1.5681 \), that is \( 1.2\% \) different. We can then recalculate the corrugation period \( \Lambda \) and the number of corrugations but the differences will be small since the wavelength change is only \( 1.2\% \).