Solutions to Homework Chap 2 for Introduction to Optoelectronics

2.9
a. Given \( n_1 = 1.475 \), \( n_2 = 1.455 \), \( 2a = 8 \times 10^{-6} \) m or \( a = 4 \) \( \mu \)m and \( \lambda = 1.3 \) \( \mu \)m. The \( V \)-number is,
\[
V = \frac{2 \pi n_1}{\lambda} (n_1^2 - n_2^2)^{1/2} = \frac{2 \pi (4 \ \mu \text{m})(1.468^2 - 1.464^2)^{1/2}}{(1.3 \ \mu \text{m})} = 2.694
\]

b. Since \( V < 2.405 \), this is a single mode fiber. The fiber becomes multimode when
\[
\frac{2 \pi n_1}{\lambda} (n_1^2 - n_2^2)^{1/2} > 2.405
\]

or
\[
\lambda < \frac{2 \pi n_1 (n_1^2 - n_2^2)^{1/2}}{2.405} = \frac{2 \pi (4 \ \mu \text{m})(1.468^2 - 1.464^2)^{1/2}}{2.405} = 1.13 \ \mu \text{m}
\]

For wavelengths shorter than 1.13 \( \mu \)m, the fiber is a multi-mode waveguide.

c. The numerical aperture \( NA \) is
\[
NA = \left[ n_1^2 - n_2^2 \right]^{1/2} = (1.468^2 - 1.464^2)^{1/2} = 0.108
\]

d. If \( \alpha_{\text{max}} \) is the maximum acceptance angle, then,
\[
\alpha_{\text{max}} = \arcsin \left( \frac{NA}{n_2} \right) = \arcsin(0.108/1) = 6.2^\circ
\]

so that the total acceptance angle is 12.4\(^\circ\).

e. At \( \lambda = 1.3 \) \( \mu \)m, from the figure, \( D_w \approx 7.5 \text{ ps km}^{-1} \text{ nm}^{-1} \), \( D_m \approx 5 \text{ ps km}^{-1} \text{ nm}^{-1} \).
\[
\frac{\Delta n(x)}{\Delta \lambda} = \frac{|D_m + D_w|}{2} \lambda^{1/2}
\]

\[
= \frac{-7.5 - 5 \text{ ps km}^{-1} \text{ nm}^{-1}}{10 \text{ ps km}^{-1}} = 0.025 \text{ ps km}^{-1}
\]

Obviously materials dispersion is 15 ps km\(^{-1}\) and waveguide dispersion is 10 ps km\(^{-1}\).

The maximum bit-rate distance product is then
\[
BR = \frac{0.59L}{\Delta n(x)/\Delta \lambda} = \frac{0.59}{0.025} \text{ m km}^{-1} = 23.6 \text{ Gb} \text{ c}^{-1} \text{ km}.
\]

2.10
Given
\( 2a = 9 \times 10^{-6} \) m or \( a = 4.5 \) \( \mu \)m. From Ch 1, Question 1.3, the Sellmeier equation is,
\[
n^2 - 1 = \frac{G_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{G_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{G_3 \lambda^2}{\lambda^2 - \lambda_3^2}
\]

where \( G_1 \), \( G_2 \), \( G_3 \) and \( \lambda_1 \), \( \lambda_2 \), \( \lambda_3 \) are constants given below where \( \lambda_s, \lambda_1, \lambda_2, \lambda_3 \) are in \( \mu \)m.

<table>
<thead>
<tr>
<th>Sellmeier constants</th>
<th>( G_1 )</th>
<th>( G_2 )</th>
<th>( G_3 )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SiO(_2)+13.5%GeO(_2)</td>
<td>4.71040</td>
<td>0.45385</td>
<td>0.70408</td>
<td>0.044760</td>
<td>0.129408</td>
<td>0.9425478</td>
</tr>
</tbody>
</table>

The fiber is to operate at \( \lambda_s = 1.3 \) \( \mu \)m, thus, using the Sellmeier equation above with the constants in the table we find
\( n_1 = 1.4682 \).

The \( V \)-number is,
\[
V = \frac{2 \pi n_1}{\lambda} = \frac{2 \pi (4.5 \ \mu \text{m})}{(1.3 \ \mu \text{m})} = 3.175
\]

Apply
\[
NA = \left[ n_1^2 - n_2^2 \right]^{1/2} = (2n_1^2)^{1/2}
\]

or
\[
0.1 = \left[ 2(1.4682)^2 \Delta n(x)/\Delta \lambda ight]^{1/2}
\]

to obtain
\[
\Delta = 0.02329
\]

Apply
\[
\Delta = n_1 - n_2, \quad \text{i.e.} \quad 0.0232 = 1.4681 - n_2
\]

Thus, the required cladding refractive index is
\( n_2 = 1.4648 \).

Pure silica has \( n_s = 1.4473 \), SiO\(_2\)+13.5 \%GeO\(_2\) has \( n_2 = 1.4682 \), by linear interpolation the composition corresponding to \( n_s = 1.4682 \) is 11.3 \%GeO\(_2\). Note, the refractive index \( n_s(x) \) of SiO\(_2\)-x mol\%GeO\(_2\), assuming a linear relationship, can be written as
\[
n_s(x) = n(0) + (n(13.5) - n(0))(x/13.5)
\]

where \( n(0) = 1.4473; n(13.5) = 1.4682 \). Substituting \( n_s(8) = 1.4682 \) gives \( x = 11.3 \).

2.11
From Ch. 1 we know that
\[
\frac{dN_p}{dx} = \alpha - \lambda \frac{dn}{d\lambda}
\]

Differentiate \( \tau \) with respect to wavelength \( \lambda \) using the above relationship between \( N_p \) and \( n \).
\[
\tau = \frac{L}{\nu_p} = \frac{L N_p}{c}
\]

\[
\frac{d\tau}{d\lambda} = \frac{L \frac{dN_p}{dx}}{c} \frac{\nu_p}{d\lambda} + \frac{\frac{d}{d\lambda} (\frac{dn}{d\lambda})}{c} \frac{dN_p}{dx}
\]

Thus,
\[
\frac{d\tau}{d\lambda} = \frac{L}{c} \frac{dn}{d\lambda}
\]

(1)
From Ch. 1 we know that the Sellmeier equation is

$$n^2 - 1 = \frac{G_1 \lambda^2}{\lambda^2 - \lambda_1^2} + \frac{G_2 \lambda^2}{\lambda^2 - \lambda_2^2} + \frac{G_3 \lambda^2}{\lambda^2 - \lambda_3^2}$$  \(\text{(2)}\)

where \(G_1, G_2, G_3, \lambda_1, \lambda_2, \lambda_3\) are constants; given in Table 2Q11 (called Sellmeier coefficients) that are determined by fitting this expression to the experimental data.

We can substitute Eq. (2) in Eq. (1) to obtain \(D_n\) and plot \(D_n\) vs. \(\lambda\) as shown in Figure 2Q11. The substitution, differentiation and the plot were done on Mathview but almost any other math-software package can do the same. Thus,

At \(\lambda = 1.55\ \mu m\), \(D_n = -14\ \text{ps km}^{-1}\ \text{nm}^{-1}\)

### Table 2Q11

The Sellmeier coefficients for SiO\(_2\)-13.5%GeO\(_2\).

<table>
<thead>
<tr>
<th>(SiO_2-13.5%GeO_2)</th>
<th>(G_1)</th>
<th>(G_2)</th>
<th>(G_3)</th>
<th>(\lambda_1)</th>
<th>(\lambda_2)</th>
<th>(\lambda_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.711040</td>
<td>0.455885</td>
<td>0.704048</td>
<td>0.0642700</td>
<td>0.129408</td>
<td>9.425478</td>
<td></td>
</tr>
</tbody>
</table>

2.14

The normalized refractive index difference \(\Delta = (n - n_i)/(n_i - (1.474 - 1.453)/1.474 - 0.01425)\)

Modal dispersion for 1 km of graded index fiber is

$$\sigma_{\text{atm noise}} = \frac{\lambda n_i^2}{2n_i^3} \Delta^2 = \frac{(100\ \mu m)(1.474)}{2n_i^3(3 \times 10^6)}$$

$$= 2.9 \times 10^{-11} \text{ s or 0.629 ns}$$

The material dispersion (FWHM) is

$$\Delta\tau_{\text{mat.2}} = L D_n \Delta\lambda_{-2} = (1000 \ \text{m} \times 5 \ \text{ps nm}^{-1} \ \text{km}^{-1}(3 \ \text{nm})$$

$$= 0.015 \text{ ns}$$

Assuming a Gaussian output light pulse shape, the material dispersion is,

$$\sigma_n = 0.425\Delta\tau_{\text{mat.2}} = 0.425(0.015 \text{ ns}) = 0.00638 \text{ ns}$$

Total dispersion is

$$\sigma_{\text{tot}} = \sqrt{\sigma_{\text{atm noise}}^2 + \sigma_n^2} = \sqrt{0.029^2 + 0.00638^2} = 0.0295 \text{ ns}$$

so that

$$B = 0.25/\sigma_{\text{tot}} = 8.5 \text{ Gb}$$

If this were a multimode step-index fiber with the same \(n_i\) and \(\Delta\), then the rms dispersion would roughly be

$$\Delta\tau = \frac{(n - n_i)}{c} = \frac{1.474 - 1.453}{(2 \times 10^8 \text{ m s}^{-1})}$$

$$= 70 \text{ ps m}^{-1} \text{ or 70 ns km}^{-1}$$

Maximum bit-rate is

$$M = 2.5 \sqrt{\frac{B L}{\sigma_{\text{atm noise}}} = \frac{0.25L}{(0.28)(70 \text{ ns km}^{-1})}}$$

i.e.

$$B L = 12.8 \text{ Mb s}^{-1} \text{ km} \text{ (only an estimate) }$$

The corresponding \(B\) for 1 km would be around 13 Mb s\(^{-1}\)

With LED excitation, again assuming a Gaussian output light pulse shape, rms material dispersion is

$$\sigma_n = (0.425/\Delta\tau_{\text{mat.2}} = (0.425/LD_n \Delta\lambda_{-1})$$

$$= (0.425)(1.000 \ m)(5 \ \text{ ps nm}^{-1} \ \text{km}^{-1}(30 \ \text{nm})$$

$$= 0.17 \text{ ns}$$

Total dispersion is

$$\sigma_{\text{tot}} = \sqrt{\sigma_{\text{atm noise}}^2 + \sigma_n^2} = \sqrt{0.029^2 + 0.17^2 - 0.172 \text{ ns}}$$

so that

$$B = 0.25/\sigma_{\text{tot}} = 1.48 \text{ Gb}$$

The effect of material dispersion now dominates intermode dispersion.

2.15

a The velocity of the ray along \(O\) to \(A\) is \((c/n_i)\) but the distance from \(O\) to \(A\) is \((1/2)\delta/\cos \theta\), so that the time it takes to travel \(O\) to \(A\) is

$$t_{OA} = \frac{(\delta/2)}{\cos \theta}$$

However, the ray \(A\) suffer TIR at \(A\) at the critical angle,

$$\sin \theta_i = n_i/n \Rightarrow \cos \theta = \sqrt{1 - (\sin \theta_i)^2} = \sqrt{1 - (n_i/n)^2}$$

$$\Rightarrow t_{OA} = \frac{(\delta/2)}{\cos \theta} = \frac{(\delta/2)n_i}{\sqrt{1 - (n_i/n)^2}}$$

b The ray has to first travel from \(O\) to \(B\) with a velocity \(c/n_i\), and then from \(B\) to \(B'\) with a velocity \(c/n_2\). The distance \(OB\) is \((1/2)\delta/\cos \theta\) and \(B\) to \(B'\) is \(\delta \cos \theta\).

$$t_{OB} = \frac{(\delta/2)}{\cos \theta} \Rightarrow t_{OB} = \frac{\delta}{\cos \theta}$$

$$t_{B'B} = \frac{\delta}{\cos \theta}$$
Apply Snell's law at $\beta$ and $\beta'$, and use critical angle for $\theta_c$ at the $n_i/n_f$ boundary.

Thus,
\[
\sin \theta_c = \frac{n_f}{n_i} \quad \therefore \quad \cos \theta_c = \left[ 1 - \left( \frac{n_f}{n_i} \right)^2 \right]^{1/2}
\]
and
\[
\frac{n_f}{n_i} \sin \theta_{\beta'} = \frac{n_f}{n_i} \sin \theta_{\beta} \quad \therefore \quad \sin \theta_{\beta'} = \frac{n_f}{n_i} \sin \theta_{\beta} \quad \therefore \quad \cos \theta_{\beta'} = \left[ 1 - \left( \frac{n_f}{n_i} \right)^2 \right]^{1/2}
\]
so that
\[
t_{\text{core}} = \left[ \frac{1}{2} \frac{\pi}{n} \right]^2 \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2} + \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2}
\]
\[
(2)
\]
\[
\text{c} \quad \text{Let the step variations of } n \text{ at } y = \beta 2, 3\beta 2, \ldots \text{ obey}
\]
\[
n_i^2 - n_f^2 = 1 - 2 \Delta \left[ \frac{\gamma}{a} \right]^{\gamma}
\]
\[
(3)
\]
where $\Delta$ is some constant (less than unity), and $\gamma$ is an index which describes the profile of the refractive index along $y$. Obviously, at $y = 0$, $n - n_i$ as we expect. Then at $y = \beta 2$, then
\[
n_i^2 - n_f^2 = 1 - 2 \Delta \left[ \frac{\delta}{2a} \right]^{\gamma}
\]
\[
\therefore \quad n_i = n_i(1 - \epsilon)
\]
\[
(4)
\]
where $e = 2 \Delta \left[ \frac{\delta}{2a} \right]^{\gamma}$

Similarly, at $y = 3\beta 2$,
\[
n_i^2 = n_i \left[ 1 - 2 \Delta \left( \frac{\delta}{2a} \right)^{3\gamma} \right]
\]
\[
\therefore \quad n_i = n_i(1 - \epsilon(3\gamma))
\]
\[
(5)
\]
\[
\text{d} \quad \text{When } t_{\text{core}} - t_{\text{core}} = 0, \text{ the two rays } \lambda \text{ and } \beta \text{ arrive at the same time. From Eq. (2),}
\]
\[
t_{\text{core}} - t_{\text{core}} = \left[ \frac{1}{2} \frac{\pi}{n} \right]^2 \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2} + \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2} = 0
\]
Substitute for $n_f$ and $n_i$ from Eqs. (4) and (5)
\[
\frac{1}{2} \frac{n_f}{n_i} \left[ 1 - \left( \frac{n_f}{n_i} \right) \right]^{2\gamma/2} + \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2} + \frac{\delta n_1}{1 - \left( \frac{n_f}{n_i} \right)^2} = 0
\]
\[
\therefore \quad \left[ \epsilon(3\gamma)^{2/2} + \frac{1}{1 - \left( \frac{n_f}{n_i} \right)^2} \right] - \left[ 1 - \left( \frac{n_f}{n_i} \right)^2 \right]^{2/2} = 0
\]
\[
(6)
\]
As the layer thickness $\delta$ becomes small we have $\epsilon \to 0$.

As $\epsilon \to 0$, Eq. (6) becomes
\[
\frac{2(1 - \epsilon)}{\left( 3\gamma - 1 \right)^{2/2}} - \frac{1}{3\gamma - 1} = 0
\]

When $\gamma = 2.067$,
\[
\frac{1}{3\gamma - 1} = \frac{2}{\left( 2.067 \right)^{2/2}} - 1 = \frac{1}{3(2.067)^{2/2}} - 1 = 0
\]

Indeed, if we plot LHS -- $y$ as a function of $y$, as shown, the curve crosses the zero-axis at $y = 2.067$ as shown in Figure 2Q15-2.

2.17 Possible Answers

a) It is essential to control the refractive index profile, core radius, and minimize variations in the refractive index due to variations in doping.

b) Minimize impurities. Reduce scattering by reducing density and hence refractive index $n$ fluctuations (may not be possible). Use a low index material with a lower glass transition temperature so that the frozen $n$-variations are smaller.