Ray Optics

Quantum optics
E-M wave optics
Wave optics
Ray optics

Fundamentals of Photonics

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Chapter 1

Fermat's Principle
Simple components
Graded-index optics
ABCD matrix

From lenses说起

Positive lens
Biconvex
Planocconcave
Convexo-convex (positive meniscus lens)

Negative lens
Biconcave
Planocconcave
Convexo-concave (negative meniscus lens)
Ray Optics

- Transverse effects on spherical or plane waves
- Change Amplitude and Phase

Fermat’s Principle

Optical Path length = \int_A^B n(r) ds

- Fermat’s Principle
  - Optical rays traveling between two points A and B, follow a path such that the time of travel (or the optical path length) between the two points is an extremum relative to neighboring paths.
  
  \[ \delta \int_A^B n(r) ds = 0 \]

- Light ray travels along the path of least time.

Ray Optics

- Fermat’s Principle
- Simple components
- Graded-index optics
- ABCD matrix

\[
\frac{d}{ds} \left( n \frac{dr}{ds} \right) = \nabla n \tag{1.3-2}
\]

\[
\mathbf{S_1} \quad \mathbf{n} \quad \mathbf{S_2} \quad \mathbf{S_3}
\]

\[
n_3 = n_1 \quad \hat{n} \times \hat{S}_3 = \hat{n} \times \hat{S}_1 \quad \sin \theta_3 = \sin \theta_1 \]

\[
\theta_3 = \theta_1 \quad (1.1-2)
\]

M. Born and E. Wolf, *Principle of Optics*
Snell’s Law

\[ n_2 \sin \theta_2 = n_1 \sin \theta_1 \]  (1.1-1)

**EXERCISE 1.1-1**

*Proof of Snell’s Law.* The proof of Snell’s law is an exercise in the application of Fermat’s principle. Referring to Fig. 1.1-4, we seek to minimize the optical path length \( n_1 AB + n_2 BC \) between points \( A \) and \( C \). We therefore have the following optimization problem: Find \( \theta_1 \) and \( \theta_2 \) that minimize \( n_1 d_1 \sec \theta_1 + n_2 d_2 \sec \theta_2 \), subject to the condition \( d_1 \tan \theta_1 + d_2 \tan \theta_2 = d \). Show that the solution of this constrained minimization problem yields Snell’s law.

**Figure 1.1-4** Construction to prove Snell’s law.

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**MIRRORS**

The reflected ray lies in the plane of incidence; the angle of reflection equals the angle of incidence.

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**Spherical Mirrors**

- Fermat’s Principle
- Simple components
- Graded-index optics
- ABCD matrix

\[
\begin{align*}
\theta_2 + \theta_1 &= \frac{2y}{-R} \\
\frac{1}{z_1} + \frac{1}{z_2} &= \frac{2}{-R} \\
\end{align*}
\]  (1.2-1, 1.2-2)

Focal length of a spherical mirror:

\[ f \equiv -\frac{R}{2} \]  (1.2-3)

Image equation for paraxial rays:

\[ \frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \]  (1.2-4)
Spherical Mirrors

**Exercise 1.2-1**

*Image Formation by a Spherical Mirror.* Show that within the paraxial approximation, rays originating from a point $P_1 = (y_1, z_1)$ are reflected to a point $P_2 = (y_2, z_2)$, where $z_1$ and $z_2$ satisfy (1.2.4) and $y_2 = -y_1/z_2/z_1$ (Fig. 1.2-7). This means that rays from each point in the plane $z = z_1$ meet at a single corresponding point in the plane $z = z_2$, so that the mirror acts as an image-forming system with magnification $-z_2/z_1$. Negative magnification means that the image is inverted.

![Image 1.2-7 Image formation by a spherical mirror.](image)

Total Internal Refraction

**Critical Angle**

$$\theta_c = \sin^{-1} \frac{n_2}{n_1} \quad (1.2-5)$$

![Figure 1.2-9 (a) Total internal reflection at a planar boundary. (b) The reflecting prism. If $n_1 > \sqrt{2}$ and $n_2 = 1$ (air), then $\theta_c < 45^\circ$; since $\theta_1 = 45^\circ$, the ray is totally reflected. (c) Rays are guided by total internal reflection from the internal surface of an optical fiber.](image)

Refracton at Plane Boundaries

![Figure 1.2-8 Relation between the angles of refraction and incidence.](image)

Prisms

**Deviation Angle**

$$\theta_d = \theta - \alpha + \sin^{-1} \left( \frac{\sin \alpha \cdot \sqrt{n_1^2 - \sin^2 \theta} - \sin \theta \cdot \cos \alpha}{n_2/n_1} \right) \quad (1.2-6)$$

**Thin Prism**

$$\theta_d = (n-1) \cdot \alpha \quad (1.2-7)$$

![Figure 1.2-10 Ray deflection by a prism. The angle of deflection $\theta_d$ as a function of the angle of incidence $\theta$ for different apex angles $\alpha$ when $n = 1.5$. When both $\alpha$ and $\theta$ are small $\theta_d = (n-1)\alpha$, which is approximately independent of $\theta$. When $\alpha = 45^\circ$ and $\theta = 0^\circ$, total internal reflection occurs, as illustrated in Fig. 1.2-9(b).](image)
Beam Splitters & Beam Combiners

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{beam_splitter.png}
\caption{Figure 1.2-11 Beam splitters and combiners: (a) partially reflective mirror; (b) thin glass plate; (c) beam combiner.}
\end{figure}

\textbf{Imaging}

\textbf{EXERCISE 1.2-2}

Image Formation. Derive (1.2-8). Prove that paraxial rays originating from $P_1$ pass through $P_2$ when (1.2-9) and (1.2-10) are satisfied.

\textbf{EXERCISE 1.2-3}

Aberration-Free Imaging Surface. Determine the equation of a convex aspherical (nonspherical) surface between media of refractive indices $n_1$ and $n_2$ such that all rays (not necessarily paraxial) from an axial point $P_1$ at a distance $z_1$ to the left of the surface are imaged onto an axial point $P_2$ at a distance $z_2$ to the right of the surface [Fig. 1.2-12(a)].

\textit{Hint:} In accordance with Fermat's principle the optical path lengths between the two points must be equal for all paths.

\textbf{Refraction at Spherical Boundaries}

\begin{align}
\theta_2 & \approx \frac{n_1 \theta_1 - n_2 - n_1}{n_2 R} y \quad (1.2-8) \\
\frac{n_1 + n_2}{z_1} & \approx \frac{n_2 - n_1}{R} \quad (1.2-9) \\
y_2 & = \frac{n_1}{n_2} \frac{z_2}{z_1} y_1 \quad (1.2-10)
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spherical_refraction.png}
\caption{Figure 1.2-12 Refraction at a convex spherical boundary ($R > 0$).}
\end{figure}

\textbf{Lenses}

\begin{align}
\theta_2 & = \theta_1 - \frac{y}{f} \quad (1.2-11) \\
\frac{1}{f} & = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad (1.2-12) \\
\frac{1}{z_1} + \frac{1}{z_2} & = \frac{1}{f} \quad (1.2-13) \\
y_2 & = -\frac{z_2}{z_1} y_1 \quad (1.2-14)
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lens.png}
\caption{Figure 1.2-13 A biconvex spherical lens.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lens_examples.png}
\caption{Figure 1.2-14 (a) Ray bending by a thin lens. (b) Image formation by a thin lens.}
\end{figure}
Thick Lens (Non-paraxial rays)

**EXERCISE 1.2-4**

**Proof of the Thin Lens Formulas.** Using (1.2-8), prove (1.2-11), (1.2-12), and (1.2-13).

![Diagram of light rays and lens](image)

**Figure 1.2-15** Nonparaxial rays do not meet at the paraxial focus. The dashed envelope of the refracted rays is called the caustic curve.

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Numerical Aperture of an Optical Fiber

**EXERCISE 1.2-5**

**Numerical Aperture and Angle of Acceptance of an Optical Fiber.** An optical fiber is illuminated by light from a source (e.g., a light-emitting diode, LED). The refractive indices of the core and cladding of the fiber are $n_1$ and $n_2$, respectively, and the refractive index of air is $1$ (Fig. 1.2-18). Show that the angle $\theta_a$ of the cone of rays accepted by the fiber (transmitted through the fiber without undergoing refraction at the cladding) is given by

$$NA = \sin \theta_a = \sqrt{n_1^2 - n_2^2}$$

(1.2-15)

![Diagram of optical fiber](image)

**Figure 1.2-18** Acceptance angle of an optical fiber.

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Light Guides

**EXERCISE 1.2-6**

**Light Trapped in a Light-Emitting Diode**

(a) Assume that light is generated in all directions inside a material of refractive index $n$ cut in the shape of a parallelepiped (Fig. 1.2-19). The material is surrounded by air with refractive index 1. This process occurs in light-emitting diodes (see Chap. 16). What is the angle of the cone of light rays (inside the material) that will emerge from each face? What happens to the other rays? What is the numerical value of this angle for GaAs ($n = 3.6$)?

![Diagram of light guides](image)

**Figure 1.2-16** Guiding light: (a) lenses; (b) mirrors; (c) total internal reflection.

![Diagram of optical fiber](image)

**Figure 1.2-19** Trapping of light in a parallelepiped of high refractive index.

(b) Assume that when light is generated isotropically the amount of optical power associated with the rays in a given cone is proportional to the solid angle of the cone. Show that the ratio of the optical power that is extracted from the material to the total generated optical power is $\frac{n^2}{(n^2 - 1)^{3/2}}$, provided that $n > \sqrt{3}$. What is the numerical value of this ratio for GaAs?
Fermat's Principle

Ray Equation

From Fermat's Principle, we have

\[ \delta \int_{A}^{B} n(r) \sqrt{dx^2 + dy^2 + dz^2} = 0 \]  (1.2-16)

\[ \frac{d}{ds} \left( n \frac{dx}{ds} \right) = \frac{dn}{dx}, \quad \frac{d}{ds} \left( n \frac{dy}{ds} \right) = \frac{dn}{dy}, \quad \frac{d}{ds} \left( n \frac{dz}{ds} \right) = \frac{dn}{dz} \]

Paraxial Ray Equation

It can also be derived using Snell's law:

\[ \theta(y) = \frac{dy}{dz} \]

\[ n(y) \cos \theta(y) = n(y + \Delta y) \cos \theta(y + \Delta y) \]

\[ = \left[ n(y) + \frac{dn}{dy} \Delta y \right] \left[ \cos \theta(y) + \frac{d\theta}{dy} \Delta y \sin \theta(y) \right] \]

\[ \frac{dn}{dy} = n \tan \theta \frac{d\theta}{dy} \]  (1.3-6)
**EXAMPLE 1.3-1. Slab with Parabolic Index Profile.** An important particular distribution for the graded refractive index is

\[
n^2(y) = n_0^2(1 - \alpha^2y^2) \tag{1.3-7}
\]

![Figure 1.3-4](image)

**GRIN Lens**

**EXERCISE 1.3-1.** The GRIN Slab as a Lens. Show that a SELFOC slab of length \( d < \pi/2\alpha \) and refractive index given by (1.3-7) acts as a cylindrical lens (a lens with focusing power in the \( yz \) plane) of focal length

\[
f \approx \frac{1}{n_0\alpha \sin \alpha d} \tag{1.3-11}
\]

Show that the principal point (defined in Fig. 1.3-6) lies at a distance from the slab edge \( AB = (1/n_0\alpha)\tan(\alpha d/2) \). Sketch the ray trajectories in the special cases \( d = \pi/\alpha \) and \( \pi/2\alpha \).

![Figure 1.3-6](image)

**SELFOC**

\[
d^2y/dz^2 = -\alpha^2y \tag{1.3-8}
\]

\[
y(z) = y_0 \cos \alpha z + \frac{\theta_0}{\alpha} \sin \alpha z \tag{1.3-9}
\]

\[
\theta(z) = \frac{dy}{dz} = -y_0 \alpha \sin \alpha z + \frac{\theta_0}{\alpha} \cos \alpha z \tag{1.3-10}
\]

![Figure 1.3-5](image)

**Graded Index Fibers**

\[
n(x, y) = n_0^2\left[1 - \alpha^2(x^2 + y^2)\right] \tag{1.3-12}
\]

\[
d^2x/dz^2 \approx -\alpha^2x, \quad d^2y/dz^2 \approx -\alpha^2y \tag{1.3-13}
\]

If \( \theta y_0 = 0, \theta y_0 = \alpha y_0 \)

\[
x(z) = \frac{\theta_0}{\alpha} \sin \alpha z \quad \gamma(z) = \gamma_0 \cos \alpha z \tag{1.3-14}
\]

\[
x(z) = y_0 \sin \alpha z \quad y(z) = y_0 \cos \alpha z \tag{1.3-15}
\]

![Figure 1.3-7](image)
Numerical Aperture of a Graded Index Fiber

**EXERCISE 1.3-2**

**Numerical Aperture of the Graded-Index Fiber.** Consider a graded-index fiber with the index profile in (1.3-12) and radius $a$. A ray is incident from air into the fiber at its center, making an angle $\theta_0$ with the fiber axis (see Fig. 1.3-8). Show, in the paraxial approximation, that the numerical aperture is

$$NA = \sin \theta_0 \approx n_i \alpha$$

(1.3-16)

![Figure 1.3-8](image)

Acceptance angle of a graded-index optical fiber.

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Matrix Optics

**Figure 1.4-1** A ray is characterized by its coordinate $y$ and its angle $\theta$.

$$y_2 = Ay_1 + B\theta_1$$  
$$\theta_2 = Cy_1 + D\theta_1$$

(1.4-1)  
(1.4-2)

**Figure 1.4-2** A ray enters an optical system at position $y_1$ and angle $\theta_1$ and leaves at position $y_2$ and angle $\theta_2$.

---

**Ray Matrix (ABCD Matrix)**

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

**EXERCISE 1.4-1**

**Special Forms of the Ray-Transfer Matrix.** Consider the following situations in which one of the four elements of the ray-transfer matrix vanishes:

(a) Show that if $A = 0$, all rays that enter the system at the same angle leave at the same position, so that parallel rays in the input are focused to a single point at the output.

(b) What are the special features of each of the systems for which $B = 0$, $C = 0$, or $D = 0$?
Ray Matrices of Simple Optical Components

n Free-Space Propagation

\[ M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \] \hspace{1cm} (1.4-3)

n Reflecting at a planar boundary

\[ M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_1 & 0 \\ 0 & n_2 & 1 \end{bmatrix} \] \hspace{1cm} (1.4-4)

n Reflecting at a spherical boundary

\[ M = \begin{bmatrix} \frac{1}{n_2 - n_1} & 0 & \frac{n_1}{n_2 R} \\ 0 & n_1 & 0 \\ \frac{n_1}{n_2} & 0 & n_2 \end{bmatrix} \] \hspace{1cm} (1.4-5)

Convex, \( R > 0 \); concave, \( R < 0 \)

n Transmission through a thin lens

\[ M = \begin{bmatrix} 1 + \frac{1}{f} & 0 \\ -
\end{bmatrix} \] \hspace{1cm} (1.4-6)

Convex, \( f > 0 \); concave, \( f < 0 \)

Matrices of Cascaded Optical Components

\[ M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \ldots \rightarrow M_n \]

\[ M = M_N \cdot M_{n-1} \cdot M_1 \] \hspace{1cm} (1.4-9)

EXERCISE 1.4-2

A Set of Parallel Transparent Plates. Consider a set of \( N \) parallel planar transparent plates of refractive indices \( n_1, n_2, \ldots, n_N \) and thicknesses \( d_1, d_2, \ldots, d_N \), placed in air \( (n = 1) \) normal to the \( z \) axis. Show that the ray-transfer matrix is

\[ M = \begin{bmatrix} 1 + \sum_{i=1}^{N} \frac{d_i}{n_i} \\ 0 \end{bmatrix} \] \hspace{1cm} (1.4-10)
Matrices of Cascaded Optical Components

EXERCISE 1.4-3

A Gap Followed by a Thin Lens. Show that the ray-transfer matrix of a distance d of free space followed by a lens of focal length f is

\[
M = \begin{bmatrix}
\frac{1}{f} & d \\
-1 & -d f
\end{bmatrix}
\]

(1.4-11)

EXERCISE 1.4-4

Imaging with a Thin Lens. Derive an expression for the ray-transfer matrix of a system consisting of free space/thin lens/free space, as shown in Fig. 1.4-3. Show that if the imaging condition \((d_1 + 1/d_2 = 1/f)\) is satisfied, all rays originating from a single point in the input plane reach the output plane at the single point \(y_2\), regardless of their angles. Also show that if \(d_2 = f\), all parallel incident rays are focused by the lens onto a single point in the output plane.

Figure 1.4-3 Single-lens imaging system.

Imaging with a Thick Lens

EXERCISE 1.4-5

Imaging with a Thick Lens. Consider a glass lens of refractive index \(n\), thickness \(d\), and two spherical surfaces of equal radius \(R\). (Fig. 1.4-4). Determine the ray-transfer matrix of the system between the two planes at distances \(d_1\) and \(d_2\) from the vertices of the lens. The lens is placed in air (refractive index \(= 1\)). Show that the system is an imaging system (i.e., the input and output planes are conjugate).

\[
\begin{aligned}
\frac{1}{z_1} + \frac{1}{z_2} &= \frac{1}{f} & \text{or} & \quad s_1s_2 &= f^2 \\
\end{aligned}
\]

(1.4-12)

where

\[
\begin{aligned}
z_1 &= d_1 + h \\
\frac{1}{z_1} &= \frac{1}{f} + \frac{1}{d_1} \\
\end{aligned}
\]

(1.4-13)

\[
\begin{aligned}
z_2 &= d_2 + h \\
\frac{1}{z_2} &= \frac{1}{f} + \frac{1}{d_2} \\
\end{aligned}
\]

(1.4-14)

The points \(F_1\) and \(F_2\) are known as the front and back focal points, respectively. The points \(P_1\) and \(P_2\) are known as the first and second principal points, respectively. Show the importance of these points by tracing the trajectories of rays that are incident parallel to the optical axis.

Figure 1.4-4 Imaging with a thick lens. \(P_1\) and \(P_2\) are the principal points and \(F_1\) and \(F_2\) are the focal points.

Periodic Optical Systems

\[
\begin{bmatrix}
y_n \\
\theta_n
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
y_0 \\
\theta_0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\theta_{n+1} \\
\theta_{n+2}
\end{bmatrix} = \frac{y_{n+1} - Ay_n}{B} - \frac{y_{n+2} - Ay_{n+1}}{B}
\]

Fig. 1.4-5 A cascade of identical optical components.

Periodic Optical Systems

n Stable Condition

\[
|b| \leq 1 \quad \text{or} \quad \frac{|A + D|}{2} \leq 1
\]

Fig. 1.4-6 Examples of trajectories in periodic optical systems: (a) unstable trajectory \((b > 1)\); (b) stable and periodic trajectory \((\varphi = 0\pi/11); \) period \(= 11\) stages; (c) stable but nonperiodic trajectory \((\varphi = 1.5)\).
Equally Spaced Identical Lenses

**EXAMPLE 1.4.1. A Sequence of Equally Spaced Identical Lenses.** A set of identical lenses of focal length \( f \) separated by distance \( d \), as shown in Fig. 1.4-7, may be used to relay light between two locations. The unit system, a distance \( d \) of free space followed by a lens, has a ray-transfer matrix given by (1.4-11): \( A = 1, B = d, C = -1/f, D = 1 - d/f \).

The parameter \( b = (A + D)/2 = 1 - d/2f \) and the determinant is unity. The condition for a stable ray trajectory, \( |b| \leq 1 \) or \( -1 \leq b \leq 1 \), is therefore

\[
0 \leq d \leq 4f
\]

\[
y_{m} = y_{\text{max}} \sin(m\varphi + \varphi_{0})
\]

\[
\varphi = \cos^{-1}\left(1 - \frac{d}{2f}\right)
\]

- Case 1: \( d = 2f \)
- Case 2: \( d = f \)

\[
0 \leq \left(1 - \frac{d}{2f_{1}}\right)\left(1 - \frac{d}{2f_{2}}\right) \leq 1
\]

(1.4-33)

**Optical Resonator**

**EXERCISE 1.4-7**

An Optical Resonator. Paraxial rays are reflected repeatedly between two spherical mirrors of radii \( R_{1} \) and \( R_{2} \) separated by a distance \( d \) (Fig. 1.4-10). Regarding this as a periodic system whose unit system is a single round trip between the mirrors, determine the condition of stability of the ray trajectory. Optical resonators will be studied in detail in Chap. 9.

![Figure 1.4-10](image)

Equally Spaced Identical Lenses

**EXERCISE 1.4-6**

A Periodic Set of Pairs of Different Lenses. Examine the trajectories of paraxial rays through a periodic system composed of a set of lenses with alternating focal lengths \( f_{1} \) and \( f_{2} \) as shown in Fig. 1.4-9. Show that the ray trajectory is bounded (stable) if

![Figure 1.4-9](image)

Optical Systems

- Projector

![Figure 1.4-11](image)
Optical Systems

Microscope

Telescope

Objective
eyepiece

Keplerian

GeElean