INTRODUCTION TO SYMPLECTIC GEOMETRY

FOR NOVEMBER 4

1. ON THE COADJOUNT ORBIT

Let $G$ be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. The adjoint representation of $G$ on $\mathfrak{g}$ is defined by

$$\text{Ad}_g(Y) = \left. \frac{d}{dt} \right|_{t=0} (g e^{tY} g^{-1}).$$  \hspace{1cm} (1.1)

The coadjoint action of $G$ on $\mathfrak{g}^*$ is characterized by

$$\langle \text{Ad}_g^*(\xi), Y \rangle = \langle \xi, \text{Ad}_{g^{-1}}(Y) \rangle$$ \hspace{1cm} (1.2)

for any $Y \in \mathfrak{g}$. Here, $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the dual pairing. The purpose of this note is to explain that there is a canonical symplectic form on the orbits of the coadjoint action. Part of the material here is taken from [CdS1, Homework 17].

1.1. The vector fields associated to the adjoint and coadjoint action. For any $X \in \mathfrak{g}$, it induces a vector field on $\mathfrak{g}$ by the adjoint action:

$$\text{The vector field at } Y = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{e^{tX}}(Y)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} e^{tX} e^{sY} e^{-tX}$$

$$= \left. \frac{d}{dt} \right|_{t=0} (e^{-tX})_*(Y|_{e^{tX}}) = [X,Y].$$ \hspace{1cm} (1.3)

The computation relies on the following facts:

- If we think $Y$ as a left invariant vector field, $Y|_g = \left. \frac{d}{ds} \right|_{s=0} ge^{sY}$.
- On $G$, the flow generated by $X$ for time $t$ is tantamount to the right multiplication by $e^{tX}$. This is basically the same as the previous fact.
- The equality (1.3) is an equivalent definition for the Lie derivative.

For any $X \in \mathfrak{g}^*$, denote by $X^\sharp$ the vector field on $\mathfrak{g}^*$ induced by the coadjoint action. Its value at $\xi$ is characterized by

$$\langle X^\sharp|_\xi, Y \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle \text{Ad}_{e^{tX}}^* \xi, Y \rangle$$

$$= \left. \frac{d}{dt} \right|_{t=0} \langle \xi, \text{Ad}_{e^{-tX}} Y \rangle = \langle \xi, [Y,X] \rangle.$$ \hspace{1cm} (1.4)
1.2. The skew-symmetric bilinear form. For any \( \xi \in g^* \), define a skew-symmetric bilinear form on \( g \) by
\[
\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle .
\] (1.5)

We now examine the kernel of this skew-symmetric bilinear form. If \( X \in \ker(\omega_\xi) \), we apply (1.4) to find that
\[
0 = \omega_\xi(X, Y) = \langle X^\dag | \xi, Y \rangle
\]
for any \( Y \in g \). Therefore, \( X^\dag \) vanishes at \( \xi \). It follows that \( \ker(\omega_\xi) \) is the Lie algebra of the stabilizer of the action (1.2) at \( \xi \).

We denote the stabilizer by \( G_\xi \), and its Lie algebra by \( g_\xi \). Since \( \ker(\omega_\xi) = g_\xi \), \( \omega_\xi \) induces a non-degenerate skew-symmetric bilinear form on \( g / g_\xi \). This quotient space is identified with the tangent space of the orbit of the coadjoint action at \( \xi \).

Hence, \( \omega \) induces a non-degenerate 2-form on the orbit of the coadjoint action. It is defined by
\[
\omega_\xi(X^\dag | \xi, Y^\dag | \xi) = \langle \xi, [X, Y] \rangle .
\] (1.6)

Notice that (1.5) is not a 2-form on \( g^* \). The inputs are elements of \( g \), but not \( g^* \).

1.3. Exterior derivative of \( \omega \). We calculate the exterior derivative of (1.6). Due to the above comment, it only makes sense to compute the exterior derivative on the orbit.
\[
(d_\omega)(X^\dag, Y^\dag, Z^\dag) = X^\dag(\omega(Y^\dag, Z^\dag)) + Z^\dag(\omega(X^\dag, Y^\dag)) + Y^\dag(\omega(Z^\dag, X^\dag))
- \omega([Y^\dag, Z^\dag], X^\dag) - \omega([X^\dag, Y^\dag], Z^\dag) - \omega([Z^\dag, X^\dag], Y^\dag) .
\]

For the terms of the first line,
\[
X^\dag(\omega(Y^\dag, Z^\dag)) = X^\dag(\langle \xi, [Y, Z] \rangle)
= \frac{d}{dt} \bigg|_{t=0} \langle \text{Ad}_{e^{-tX}}^* \xi, [Y, Z] \rangle
= \frac{d}{dt} \bigg|_{t=0} \langle \xi, \text{Ad}_{e^{tX}} [Y, Z] \rangle
= -\langle \xi, [Y, Z] \rangle .
\]

For the terms of the second line,
\[
\omega([Y^\dag, Z^\dag], X^\dag) = \omega(\langle [Y, Z] \rangle^\dag, X^\dag)
= \langle \xi, [Y, Z], X \rangle .
\]

By the Jacobi identity, \( d_\omega = 0 \). Thus, \( \omega \) defines a symplectic form on the orbit of the coadjoint action. It is also known as the Kostant–Kirillov symplectic structure.
1.4. The moment map. Surely the Lie group $G$ acts on the coadjoint orbit. It turns out that this group action is Hamiltonian, and the moment map is very simple.

For any $X \in \mathfrak{g}$, we claim that $\iota_X \omega$ is d-exact. By (1.6) and (1.4),

$$\iota_Y \iota_X \omega = \langle \xi, [X, Y] \rangle$$
$$= \langle Y^\sharp \xi, X \rangle = Y^\sharp (\langle \xi, X \rangle)$$

for any $Y \in \mathfrak{g}$. It follow that $\iota_X \omega = d(\langle \xi, X \rangle)$.

Hence, we can just take the moment map to be $\xi$, and it is automatically equivariant with respect to the coadjoint action. In other words, the moment map of the coadjoint orbit is the inclusion map to $\mathfrak{g}^\ast$.

1.5. Remarks.

- In fact, we can apply this framework for $G = \text{SO}(3)$. Then we will get the answer for (6) of the Midterm.
- For Item (1.e) of Homework 8, the symplectic form should be

$$\omega_\xi (X^\sharp, Y^\sharp) = i \text{trace}([X, Y] \xi) .$$

The correct statement of the first fact should be

$$(d\omega)(X^\sharp, Y^\sharp, Z^\sharp) = X^\sharp (\omega(Y^\sharp, Z^\sharp)) + Z^\sharp (\omega(X^\sharp, Y^\sharp)) + Y^\sharp (\omega(Z^\sharp, X^\sharp)) - \omega([Z^\sharp, X^\sharp], Y^\sharp) .$$

For the second fact, we consider the derivative of

$$\xi \mapsto \omega_\xi (X^\sharp, Y^\sharp) = i \text{trace}([X, Y] \xi)$$

along $Z^\sharp$, which is equal to

$$\frac{d}{dt} \bigg|_{t=0} i \text{trace}([X, Y] (e^{tZ} \xi e^{-tZ}) .$$

As mentioned above, the whole business of the symplectic form computation is done on the coadjoint orbit.