Social Choice — Finite Sets

1 Individual Preferences: Assumptions

1. A finite set N of individuals: $i = 1, 2, \ldots, n$, with $n \geq 2$

2. A finite set X of alternatives: $|X| \geq 3$

3. Individual preference $R^i \in \Psi$: a weak order on X
   □ Ψ $\equiv$ Set of all possible weak orders on X (complete, reflexive, transitive)

4. Preference profile $\rho = (R^1, \cdots, R^n) \in \Psi^n$: n-tuple of weak orders.
   □ Ψ^n $\equiv$ Set of all possible preference profiles.

5. Restricted preference profile: $\rho|_S = (R^1|_S, \cdots, R^n|_S)$, $S \subseteq X$

6. Collective decision rule (CDR) $f(\rho)$, denoted R, is a mapping: $\Psi^n \mapsto \Re$
   □ $\Re$ $\equiv$ Set of all *complete* binary relations.
   ▷ f is a preference aggregation rule, yielding a social ranking.
   ▷ Also called social welfare function (SWF) or constitution [Green/Laffont 1979].

7. Some sets of voters:

   - $R(x, y; \rho) \;\equiv\; \{i \in N \mid {}_xR^i{}_y\}$
   - $P(x, y; \rho) \;\equiv\; \{i \in N \mid {}_xP^i{}_y\}$
   - $I(x, y; \rho) \;\equiv\; \{i \in N \mid {}_xI^i{}_y\}$

   Note: $R(x, y; \rho) \neq R(y, x; \rho)$, and $P(x, y; \rho) \neq P(y, x; \rho)$

2 Collective Decision Rules

1. Some examples:

   - Constant rule: $f(\rho) = f(\rho'),\ \forall\, \rho, \rho' \in \Psi^n$
   - Pareto/unanimity rule: ${}_xP_y$ iff ${}_xP^i{}_y, \forall\, i \in N$
   - Majority rule: ${}_xP_y$ iff $|P(x, y; \rho)| > \frac{n}{2}$; ${}_xR_y$ iff $|R(x, y; \rho)| \geq \frac{n}{2}$
   - Plurality rule: ${}_xP_y$ iff $|P(x, y; \rho)| > |P(y, x; \rho)|$; ${}_xR_y$ iff $|R(x, y; \rho)| \geq |R(y, x; \rho)|$
   - Border rule: ${}_xP_y$ iff $\sum_i r_i(x) < \sum_i r_i(y)$; $r_i(\cdot)$ $\equiv$ assigned rank number

2. Properties of CDR $f(\rho)$:

   - Universal Domain (UD): f has full Ψ^n as its domain.

- Non-dictatorial (ND): $\nexists\, i \in N \quad \text{s.t.} \quad \forall\, \rho \in \Psi^n, \forall\, x, y \in X : {}_xP^i{}_y \Rightarrow {}_xP_y$
- Weakly Paretian (WP): $\forall\, \rho \in \Psi^n, \forall\, x, y \in X : {}_xP^i{}_y, \forall\, i \in N \Rightarrow {}_xP_y$
- Independence of irrelevant alternatives (IIA): $\forall\, \rho, \rho' \in \Psi^n, \forall\, x, y \in X$:

$$\rho|_{\{x,y\}} = \rho'|_{\{x,y\}} \quad \Rightarrow \quad f(\rho)|_{\{x,y\}} = f(\rho')|_{\{x,y\}}$$

3. Some examples:
   - Constant rule: ND, IIA, but not WP.
   - Dictatorship: WP, IIA, but not ND.
   - Pareto rule: ND, WP, IIA, but not COMP, not TRAN (only Q-TRAN).
   - Borda rule:[1] ND, WP, but not IIA.
   - Majority rule: ND, WP, IIA, but not ACYC (hence not TRAN).

3 Rationality of Collective Decision Rules

Def 1: A CDR $f(\rho)$ is said to be ACYC/Q-TRAN/TRAN if it is so for any profile $\rho \in \Psi^n$.

Def 2 (Decisiveness) Coalition $L \subseteq N$ is:

- semi-decisive over (x, y): ${}_x\tilde{D}^L{}_y$ if

$$\forall\, \rho : \ (\forall\, i \in L, {}_xP^i{}_y) \,\&\, (\forall\, j \notin L, {}_yP^j{}_x) \quad \Rightarrow \quad {}_xP_y$$

- decisive over (x, y): ${}_xD^L{}_y$ if

$$\forall\, \rho : \ \forall\, i \in L, {}_xP^i{}_y \quad \Rightarrow \quad {}_xP_y$$

- decisive if L is decisive over any pair $(x, y) \in X^2$

Lmm: If f is Q-TRAN, WP, and IIA, then for any coalition $L \in N$:
(1) $\exists\, x, y \in X,\ {}_x\tilde{D}^L{}_y \quad \Rightarrow \quad \forall\, z_{\,(\neq x,y)} \in X : \ {}_xD^L{}_z$ and ${}_yD^L{}_z$
(2) $\exists\, x, y \in X,\ {}_x\tilde{D}^L{}_y \quad \Rightarrow \quad \forall\, r, s \in X : \ {}_rD^L{}_s$ ■

Thm (Arrow 1951) For $|X| \geq 3$ and $n \geq 2$:
If f is UD, TRAN, WP, and IIA, then it must be dictatorial. ■

Def 3: Veto power

- Agent $i \in N$ has a veto for (x, y) if $\forall\, \rho \in \Psi^n,\ {}_xP^i{}_y \Rightarrow \sim {}_yP_x$
- Agent $i \in N$ has a veto if i has a veto for all (x, y).
- f is oligarchic if $\exists\, L\ (\subseteq N)$ that is decisive, and every $i \in L$ has a veto.

Thm (Gibbard 1973) If f is UD, Q-TRAN, WP, and IIA, then it is oligarchic. ■

[1] Or any point voting system.

Def 4: Winning/decisive coalition set $\mathcal{L}(f) \equiv \{L \subseteq N \mid L \text{ is decisive under rule } f\} \subseteq 2^N$

▷ If $L_1, L_2 \in \mathcal{L}(f)$, then $L_1 \cap L_2 \neq \emptyset$. [Otherwise conflict may result.]

▷ If f is WP, then $N \in \mathcal{L}(f)$.

Def 5: Winning coalition set $\mathcal{L}(f)$ is:

- monotonic: $L \in \mathcal{L}(f)$ and $L \subseteq L' \;\Rightarrow\; L' \in \mathcal{L}(f)$
- proper: $L \in \mathcal{L}(f) \;\Rightarrow\; N \setminus L \notin \mathcal{L}(f)$

Lmm: For any f, coalition set $\mathcal{L}(f)$ is monotonic and proper. ■

Def 6 (Collegial) f is collegial (決策核心制) if

$$\bigcap_{L \in \mathcal{L}(f)} L \neq \emptyset$$

▷ Collegium (核心成員) of a collegial f: $K(f) \equiv \bigcap_{L \in \mathcal{L}(f)} L$

▷ The collegium is necessary for any decision, but may not be sufficient.

(eg) Decision rule of the *UN Security Council*

Thm (Brown 1975) If $|X| \geq n$ and f is `ACYC` and `WP`, then it is collegial. ■

Ex: Decision rule of *UN Security Council* is collegial, not oligarchic.[2] □

Thm: If $|X| > n$ and f is `ACYC`, `WP`, and `IIA`, then $\exists\, i \in N$ with a veto over some (x, y). ■

Def 7: Derived rule $f_{\mathcal{L}}$ for $\mathcal{L} \subseteq 2^N$: ${}_xP^{\mathcal{L}}{}_y \;\rightleftharpoons\; \exists\, L \in \mathcal{L}$ s.t. ${}_xP^i{}_y, \forall\, i \in L$

▷ f is simple if $f = f_{\mathcal{L}(f)}$.

▷ f is more resolute than $f_{\mathcal{L}(f)}$, since ${}_xP^{\mathcal{L}(f)}{}_y \Rightarrow {}_xP_y$.

Def 8: For all $\rho, \rho' \in \Psi^n$ and $x, y, a, b \in X$, a rule f is:

- decisive iff $[P(x,y;\rho) = P(x,y;\rho') \;\&\; {}_xP_y] \;\Rightarrow\; {}_xP'_y$
- neutral iff $[P(x,y;\rho) = P(a,b;\rho') \;\&\; P(y,x;\rho) = P(b,a;\rho')] \;\Rightarrow\; [{}_xP_y \rightleftharpoons {}_aP'_b]$
- monotonic iff $[P(x,y;\rho) \subseteq P(x,y;\rho') \;\&\; R(y,x;\rho) \subseteq R(x,y;\rho') \;\&\; {}_xP_y] \;\Rightarrow\; {}_xP'_y$

▷ Neutrality implies that names of agents do not matter.

Thm: f is simple iff f is decisive, neutral, and monotonic. ■

▷ Plurality rule is not decisive, hence not simple.

Def 9: A simple rule f is a q-rule ($q > \frac{n}{2}$) iff $\mathcal{L}(f) = \{L \subseteq N : |L| \geq q\}$.

▷ Pareto rule is a q-rule with $q = n$.

▷ Majority rule is a q-rule with $q = \frac{n+1}{2}$.

[2] It is ACYC, not Q-TRAN.

Def 10: Nakamura number for rule f:

$$v(f) \equiv \min\{\,|\mathcal{L}'|:\ \mathcal{L}' \subseteq \mathcal{L}(f),\ \ \cap_{L\in\mathcal{L}'} L = \emptyset\}$$

Ex: For majority rule:

- $n = 3$: $v(f) = 3$, $\mathcal{L}(f) = \{\underline{\{1,2\}}, \underline{\{1,3\}}, \underline{\{2,3\}}, \{1,2,3\}\}$.
- $n = 4$: $v(f) = 4$, $\mathcal{L}(f) = \{\underline{\{1,2,3\}}, \underline{\{1,2,4\}}, \underline{\{1,3,4\}}, \underline{\{2,3,4\}}, \{1,2,3,4\}\}$. □

Lmm: For any rule f, $v(f) \geq 3$.[3] ■

Lmm: If f is not collegial, then $v(f) \leq n$. ■

Thm (Nakamura) A simple rule f is ACYC iff $|X| < v(f)$. ■

Def 11: A simple rule f is strong iff $\forall L$: $L \notin \mathcal{L}(f) \Rightarrow N \setminus L \in \mathcal{L}(f)$.

▷ Majority rule (with n odd, or n even with a tie-breaker) is strong.

Lmm: If f is collegial and strong, then it is dictatorial. ■

Lmm: If f is non-collegial and strong, then $v(f) = 3$. ■

Lmm: If f is a q-rule with $q \leq n$, then $v(f) = \frac{n}{n-q}$. ■

Ex: For majority rule f: $v(f) = 3$ (if $n \neq 4$) and $v(f) = 4$ (if $n = 4$). □

Cor: A non-collegial, strong simple rule is ACYC iff $|X| < 3$.

Cor: A non-collegial q-rule is ACYC iff $|X| < \frac{n}{n-q}$.

▷ Majority rule with $|X| \geq 3$ (when $n \neq 4$) is not acyclic.

Def 12: Blocking coalition set:

$$\mathcal{B}(f) \equiv \{L \subseteq N \,|\, N \setminus L \notin \mathcal{L}(f)\}$$

Losing coalition set:

$$\mathcal{S}(f) \equiv 2^N - B(f) = \{L \subseteq N \,|\, N \setminus L \in \mathcal{L}(f)\}$$

Non-winning blocking coalitions:

$$\{L \subseteq N \mid L \in B(f),\ L \notin \mathcal{L}(f)\}$$

▷ For strong f, $B(f) = \mathcal{L}(f)$: a coalition either wins or loses.

[3] Two disjoint coalitions cannot both be decisive.

Ex: q-rule with $n = 100$ and $q = 60$:

$$\mathcal{L}(f) = \{L \subseteq N : |L| \geq 60\}$$
$$\mathcal{B}(f) = \{L \subseteq N : |L| > 40\}$$
$$\mathcal{S}(f) = \{L \subseteq N : |L| \leq 40\} \ \square$$

Remark (Fundamental Dilemma) Trade-off among ACYC, Equality, and Resoluteness.

(1) Pareto rule: most EQU, most ACYC, but worst in RES.

(2) Majority rule: most EQU, most RES, but worst in ACYC.

(3) Dictatorship: most ACYC, most RES, but worst in EQU.

Def 13 (Core) $C_f(\rho, X) \equiv M(f(\rho), X)$

– Core of a simple rule f is not empty iff $|X| < v(f)$ [Nakamura Thm]

– Theory predicts only in very restricted situations.

– Core is typically empty!

4 Social Decision Functions (SDF)

Def 14 (Social decision function) An SDF $\delta(\rho)$ is a mapping: $\Psi^n \mapsto X$.

▷ SDFs assign to each preference profile ρ an element in X.

▷ Also known as social choice function (SCF) [Green/Laffont 1979].

Def 15: SDF $\delta(\rho)$ is manipulable at $\rho = (R^1, \cdots, R^n)$ if there exists $R^{i\prime} \in \Psi$ such that:

$$_{[\delta(R^1,\cdots,R^{i\prime},\cdots,R^n)]}P^i{}_{[\delta(R^1,\cdots,R^i,\cdots,R^n)]}$$

Def 16: SDF $\delta(\rho)$ is strongly individually incentive compatible (SIIC) if it is not manipulable at any preference profile ρ.

▷ Truthful revelation of preferences is dominant strategy for all i and for any ρ:

$$\forall \rho \in \Psi^n,\ \forall R^{i\prime} \in \Psi : \ _{[\delta(R^1,\cdots,R^i,\cdots,R^n)]}R^i{}_{[\delta(R^1,\cdots,R^{i\prime},\cdots,R^n)]}$$

▷ Also called cheat-proof, strategy-proof, or straightforward.

NB: SIIC is weaker then *Group-/coalition-nonmanipulability.*

Def 17: If SDF $\delta(\cdot)$ has range $S \subseteq X$, then it is called SDF with range S.

Def 18: SDF δ with range S is dictatorial if $\exists\, i \in N$ (the dictator) such that:

$$\forall \rho \in \Psi^n, \forall x \in S : \ x \neq \delta(\rho) \ \Rightarrow \ _{\delta(\rho)}R^i{}_x$$

▷ There is an agent whose favorite alternative is always the social choice.

Lmm If there exists $i \in N$ such that

$$\delta(R^1, \cdots, R^i, \cdots, R^n) = x, \;\; \delta(R^1, \cdots, R^{i\prime}, \cdots, R^n) = y; \;\; x \neq y$$

and if ${}_xP^i{}_y$ and ${}_xP^{i\prime}{}_y$, then ρ is manipulable at either $(R^1, \cdots, R^i, \cdots, R^n)$ or $(R^1, \cdots, R^{i\prime}, \cdots, R^n)$. ■

(Pf) Obvious by the definition of manipulability. □

Lmm Let δ be `SIIC` with range $S \subseteq X$. If $T \subseteq S$ and $\rho = (R^1, \cdots, R^n)$ is a profile such that

$$\forall\, i \in N, \; \forall\, x, y \in S \text{ with } \; x \in T, y \notin T : \;\; {}_xP^i{}_y,$$

then $\delta(\rho) \in T$. ■

Thm (Gibbard 1973/Satterthwaite 1975) If $|S| \geq 3$, then any SDF with range S satisfying `SIIC` and `UD` is dictatorial. ■

▷ Allowing more complex strategy space does not help!

Generalization Follow-up research:

(1) Domain restriction [Maskin; Kalai/Muller]

(2) Imposed Structure: free disposal, neutral agent

(3) Statistical info about taste distribution [Grandmont]

(4) Random social lottery [Gibbard]