Individual Choice and Preferences

1 Preference Ordering as a Binary Relation on Finite Sets

Def: R-relation ("weakly prefer"): a set $R \subseteq X \otimes X$, ${}_xR_y \rightleftharpoons (x,y) \in R$
P-relation ("stricly prefer"): ${}_xP_y \equiv [\,{}_xR_y \ \underline{\text{and}} \ \sim {}_yR_x\,]$
I-relation ("indifference"): ${}_xI_y \equiv [\,{}_xR_y \ \underline{\text{and}} \ {}_yR_x\,]$
▷ ${}_xP_y \Rightarrow \sim {}_yR_x \Rightarrow \sim {}_yP_x$; ${}_xP_y \Rightarrow \sim {}_xI_y$
▷ ${}_xI_y \rightleftharpoons {}_yI_x$; $\sim {}_xI_y \rightleftharpoons \sim {}_yI_x$
▷ ${}_xR_y \rightleftharpoons [\,{}_xP_y \ \underline{\text{or}} \ {}_xI_y\,]$; ${}_xR_y \Rightarrow \sim {}_yP_x$

2 Properties of Binary Relation R

Def: Reflexive (REFL): ${}_xR_x$, $\forall\, x \in X$
Irreflexive (IRR): $\sim {}_xR_x$, $\forall\, x \in X$
Complete (COMP): ${}_xR_y$ $\underline{\text{or}}$ ${}_yR_x$, $\forall\, x, y \in X$
Symmetric (SYM): ${}_xR_y \rightleftharpoons {}_yR_x$, $\forall\, x, y \in X$
Asymmetric (ASYM): ${}_xR_y \rightleftharpoons \sim {}_yR_x$, $\forall\, x, y \in X$
Transitive (TRAN): ${}_xR_yR_z \Rightarrow {}_xR_z$, $\forall\, x, y, z \in X$
Quasi-transitive (Q-TRAN): ${}_xP_yP_z \Rightarrow {}_xP_z$, $\forall\, x, y, z \in X$
Acyclic (ACYC): ${}_xP_yP_z \Rightarrow \sim {}_zP_x$, $\forall\, x, y, z \in X$

Def: Equivalence: R that is REFL, SYM, and TRAN.
Weak order: R that is REFL, COMP, and TRAN.

Lmm 1: If R is complete, then: $\sim {}_xR_y \rightleftharpoons {}_yP_x$. ■

Lmm 2: If R is transitive, then:
(a) ${}_xP_yP_z \Rightarrow {}_xP_z$ (ie, R is quasi-transitive)
(b) ${}_xI_yI_z \Rightarrow {}_xI_z$
(c) ${}_xR_yP_z \Rightarrow {}_xP_z$; ${}_xP_yR_z \Rightarrow {}_xP_z$ ■

▷ TRAN $\Rightarrow$ Q-TRAN $\Rightarrow$ ACYC.
▷ Q-TRAN $\not\Rightarrow$ TRAN. [Ex] ${}_xP_y$, ${}_yI_z$, ${}_zI_x \Rightarrow {}_yI_zI_x$, but ${}_xP_y$ (not ${}_yR_x$)
▷ ACYC $\not\Rightarrow$ Q-TRAN. [Ex] ${}_xP_y$, ${}_yP_z$, ${}_zI_x \Rightarrow {}_xP_yP_z$, but ${}_xI_z$

3 Rationalizability of Individual Choices

Def: Choice Function $C(\cdot)$, defined on 2^X: $\forall\, S_{(\neq\emptyset)} \subseteq X$

$$C(S) \subseteq S, \quad \text{and} \quad C(S) \neq \emptyset$$

▷ $C(\cdot)$ is resolute if:

$$|C(S)| = 1, \ \forall\, S_{(\neq\emptyset)} \subseteq X$$

Def: Maximal set: for any non-empty $S \subseteq X$

$$M(R,S) \ \equiv \ \{x \in S \,|\, {}_xR_y, \forall\, y \in S\}$$

Lmm 3: Maximal set $M(R,S) \neq \emptyset$ iff R is `REFL, COMP`, and `ACYC`. ■

▷ ACYC is minimal requirement for non-empty maximal sets.

▷ If R is a weak order, then $M(R,S) \neq \emptyset$; but not vice versa.

Def: Choice function $C(\cdot)$ is rationalizable (`RAT`) if $\exists\, R$ s.t.

$$C(S) \ = \ M(R,S), \ \ \forall\, S \subseteq X$$

▷ R that rationalizes $C(\cdot)$ must be `REFL, COMP`, and `ACYC`.

Ex: If $C(\{x,y\}) = \{x\}$ and $C(\{x,y,z\}) = \{y\}$, then $C(\cdot)$ is not rationalizable.[1]

▷ This is IIA pertaining to individual choice.

Def: Base relation R^* for $C(\cdot)$: binary comparison

$${}_xR^*{}_y \ \ \rightleftharpoons \ \ x \in C(\{x,y\}), \ \forall\, x,y \in X$$

Lmm 4 (Uniqueness): $C(\cdot)$ is rationalizable iff $\forall\, S \subseteq X: \ C(S) = M(R^*,S)$. ■

▷ Rationalizing R^* must be `REFL, COMP, ACYC` [by Lmm 3].

▷ Rationalizing R^* may not be `Q-TRAN` or `TRAN`.

Ex: $C(\{x,y,z\}) = \{x\}, \ C(\{x,y\}) = \{x\}, \ C(\{y,z\}) = \{y\}, \ C(\{x,z\}) = \{x,z\}$

$\Rightarrow \ {}_xP^*{}_y, \ {}_yP^*{}_z, \ {}_xI^*{}_z$: $C(\cdot)$ is `RAT`. But R^* is not `Q-TRAN` (only `ACYC`). □

4 Properties of Choice Function $C(\cdot)$

Condition BC (Binary Contraction) $C(\cdot)$ such that $\forall\, S \subseteq X$:

$$x \in C(S) \ \Rightarrow \ \forall\, y \in S, \ x \in C(\{x,y\}) \quad ■$$

▷ $C(\cdot)$ satisfies `BC` iff $C(S) \subseteq M(R^*,S), \ \forall\, S \subseteq X$

▷ Contrapositive: $\forall\, x \in S$: $[\exists\, y \in S, x \notin C(\{x,y\}) \ \Rightarrow \ x \notin C(S)]$

▷ If $C(\cdot)$ satisfies `BC`, then its R^* is `ACYC`.

[1] Proof: From the former: $\sim {}_yR_x$. From the latter: ${}_yR_x$. $\not\rightarrow$ □

Condition BE (Binary Expansion) $C(\cdot)$ such that $\forall S \subseteq X$:

$$\forall y \in S,\ x \in C(\{x,y\}) \ \Rightarrow\ x \in C(S) \quad \blacksquare$$

▷ $C(\cdot)$ satisfies `BE` iff

$$M(R^*, S) \subseteq C(S),\ \forall S \subseteq X$$

▷ Contrapositive: $\forall x \in S$:

$$x \notin C(S) \ \Rightarrow\ \exists y \in S, x \notin C(\{x,y\})$$

▷ $C(\cdot)$ is `RAT` iff it satisfies `BC` and `BE`:

$$C(S) \ =\ M(R^*, S),\ \ \forall S_{(\neq\emptyset)} \subseteq X$$

▷ IF $C(\cdot)$ is resolute, then: `BC` $\Rightarrow$ `BE`

Condition α (Contraction) (Sen 1970) $C(\cdot)$ such that $\forall S, T \subseteq X$:

$$S \subseteq T \ \Rightarrow\ C(T) \cap S \ \subseteq\ C(S) \quad \blacksquare$$

Condition β (Equivalence) $C(\cdot)$ such that $\forall S, T \subseteq X$:

$$S \subseteq T \ \ \&\ \ C(T) \cap C(S) \neq \emptyset \ \Rightarrow\ C(S) \subseteq C(T) \quad \blacksquare$$

Condition γ (Expansion) (Sen 1970) $C(\cdot)$ such that $\forall S, T \subseteq X$:

$$C(S) \cap C(T) \ \subseteq\ C(S \cup T) \quad \blacksquare$$

Lmm 5 (Sen 1970)

(a) If $C(\cdot)$ satisfies α, then it satisfies `BC`: $\alpha \Rightarrow$ BC.

(b) If $C(\cdot)$ satisfies γ, then it satisfies `BE`: $\gamma \Rightarrow$ BE. $\blacksquare$

Thm 1: $C(\cdot)$ is rationalizable iff it satisfies α and γ. $\blacksquare$

Corr 1: $\alpha + \gamma \ \rightleftharpoons\ $ BC + BE $\ \rightleftharpoons\ C(\cdot)$ is RAT. $\blacksquare$

▷ With `BE`, α and `BC` are equivalent.

▷ With `BC`, γ and `BE` are equivalent.

Ex (IIA) Political process of 2 agents:

<Congress> $a \succ b \succ S$ ($S \equiv$ status quo)

<President> $b \succ S \succ a$

<Decision process> Congress proposes a or b; if President vetos, S remains.

$\Longrightarrow$ Choice made with this process is not RAT!

Hint: Condition α is violated: $C(\{a,b,S\}) = \{b\}$, yet $C(\{a,b\}) = \{a\}$. $\square$

Lmm 6: If $C(\cdot)$ satisfies α and β, then it satisfies γ, and R^* is transitive. ■

Thm 2: $C(\cdot)$ is rationalizable by transitive R^* iff it satisfies α and β. ■

Condition PI (Path Independence) $C(\cdot)$ such that $\forall S, T \subseteq X$:

$$C(S \cup T) = C(C(S) \cup C(T)) \quad \blacksquare$$

Condition PPI (Partition Path Independence) $C(\cdot)$ such that $\forall S, T \subseteq X$:

$$S \cap T = \emptyset \implies C(S \cup T) = C(C(S) \cup C(T)) \quad \blacksquare$$

$\triangleright$ `PI` $\Rightarrow$ `PPI`.

Lmm 6a: If $C(\cdot)$ satisfies `PPI`, then it satisfies `BC`, and R^* is quasi-transitive. ■

Thm 2a: $C(\cdot)$ is rationalizable by quasi-transitive R^* iff it is `PPI` and `BE`. ■

Lmm 6b: If $C(\cdot)$ satisfies `BC` and β, then it satisfies `BE`, and R^* is transitive. ■

Thm 2b: $C(\cdot)$ is rationalizable by transitive R^* iff it satisfies `BC` and β. ■

Corr 2: If $C(\cdot)$ is resolute, then: $C(\cdot)$ is `TRAN RAT` $\rightleftharpoons$ it satisfies `BC`. ■

Condition AA (Arrow's Axiom) $C(\cdot)$ such that $\forall S \subseteq T$:

$$C(T) \cap S = \emptyset \quad \text{or} \quad C(T) \cap S = C(S) \quad \blacksquare$$

Condition WARP (Weak Axiom of Revealed Preferences) $C(\cdot)$ such that:

$$(\forall S, T \subseteq X,\ \forall x, y \in S) \quad x \in C(S),\ y \notin C(S),\ \text{ then } \ y \in C(T) \ \Rightarrow\ x \notin T \quad \blacksquare$$

Thm 2c: The following are equivalent:

ⓐ $C(\cdot)$ satisfies `BC` and β.
ⓑ $C(\cdot)$ satisfies `AA`.
ⓒ $C(\cdot)$ satisfies `WARP`. ■