基 本 模 型

1. Definition: Public Goods (PGs)

- Mathimatical definition:

$$U_i(x_i, y), \quad U_j(x_j, y)$$

- "Non-rival" and "non-excludable" goods:

	Excludable	Non-excludable
Rival	Private good	Impure PG
Non-rival	Club good	Pure PG

  - Impure PGs: driving, smoking, pollution

$$U_i(x_i, s_i, S), \quad S \equiv \sum_j s_j$$

  - "Consumptive" v. "productive"
  - "Continuous" v. "binary/threshold/discrete"

- Entension: altruistic preferences
  - One-way:

$$U_P(x_P, x_K), \quad U_K(x_K)$$

  - Bilateral:

$$U_R(x_R, x_J), \quad U_J(x_J, x_R)$$

2. Optimality Condition

2.1. Samuelson FOC

- Utility function:

$$U_A(x_A, y), \;\; U_B(x_B, y)$$

  ▷ *Pure altruism* (v. pure eogoism)

  ▷ Continuous public goods

- Resource allocation:

$$(x_A, x_B, y)$$

- Aggregate budget constraint:

$$P_x\,[x_A + x_B] \; + \; P_y y \; \leq \; W \; \equiv \; I_A + I_B$$

  ▷ Feasible allocation set:

$$F \; \equiv \; \{ \; (x_A, x_B, y) \mid P_x[x_A + x_B] + P_y y \; \leq \; W \; \}$$

- Pareto optimality:

$$\max_{x_A,\, x_B\, ,y} \;\; U_A(x_A, y) \quad \text{s.t.} \quad \begin{cases} U_B(x_B, y) \; \geq \; \bar{U}_B \\ P_x[x_A + x_B] + P_y y \; \leq \; W \end{cases}$$

  <u>Lagrangian</u>:

$$\begin{aligned} L \;\; = \;\; & U_A(x_A, y) \\ & + \lambda_1 \{U_B(x_B, y) - \bar{U}_B\} \\ & - \lambda_2 \{P_x[x_A + x_B] + P_y y - W\} \end{aligned}$$

foc:

$$\frac{\partial U_A}{\partial x_A} = \lambda_2 P_x \quad (1)$$

$$\lambda_1 \frac{\partial U_B}{\partial x_B} = \lambda_2 P_x \quad (2)$$

$$\frac{\partial U_A}{\partial y} + \lambda_1 \frac{\partial U_B}{\partial y} = \lambda_2 P_y \quad (3)$$

Substitute (1) into (2) then into (3):

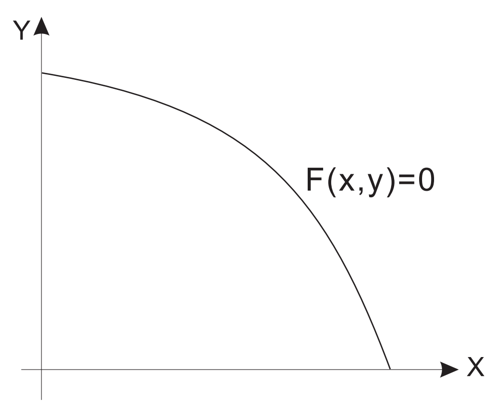
$$\frac{\partial U_A/\partial y}{\partial U_A/\partial x_A} + \frac{\partial U_B/\partial y}{\partial U_B/\partial x_B} = \frac{P_y}{P_x}$$

- Samuelson foc: [Samuelson 1954, 1955, 1958]

$$\sum_i \text{MRS}_i^{y,x} = \frac{P_y}{P_x} = \text{MC}(y)$$

or, with *production possibility frontier* (PPF), $F(x, y) = 0$:

$$\sum_i \text{MRS}_i^{y,x} = \frac{F_y}{F_X} = \text{MRT}^{y,x}$$



N Alternatively, PO allocations can also be formulated as:

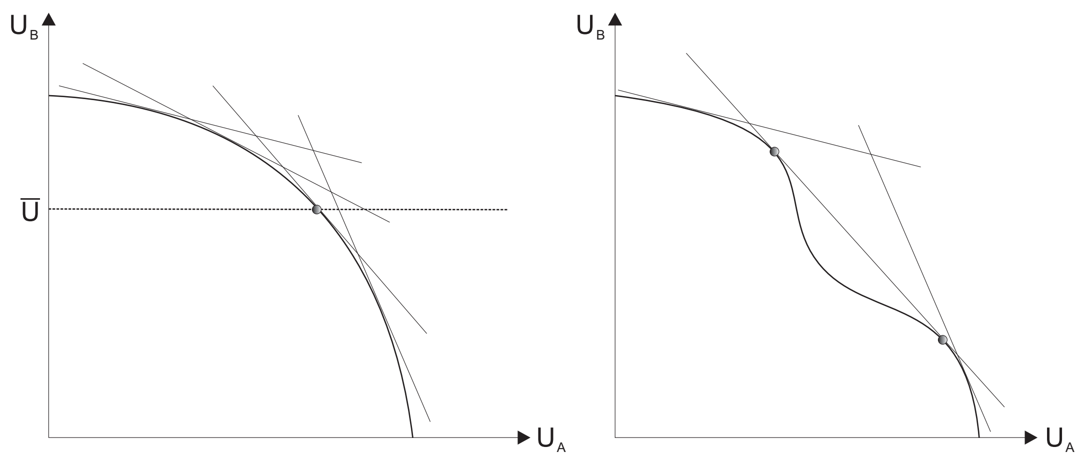
$$\max_{x_A,\, x_B\,, y} \quad \lambda_A U_A(x_A, y) \;+\; \lambda_B U_B(x_B, y)$$

$$\text{s.t.} \quad P_x[x_A + x_B] + P_y y \;\leq\; W$$

for any convex combination of $(\lambda_A \geq 0, \lambda_B \geq 0)$ with

$$\lambda_A \;+\; \lambda_B \;=\; 1$$

as long as the utility possibility set (UPS) is convex.



2.2. Market Equilibrium

- Individual provision incentive:

$$\mathrm{MRS}_i^{y,x} \;=\; \frac{P_y}{P_x}$$

- Samuelson foc is violated
- Private PG provision is inefficient

2.3. Generalization: n-person Economy

- To determine unique optimal allocation:

$$(x_1^*, \cdots, x_n^*, y^*)$$

- Need $(n+1)$ equations: given $W = \sum_i I_i$

1. Samuelson foc:

$$\sum_i \frac{\partial U_i / \partial y}{\partial U_i / \partial x_i} \;=\; \frac{P_y}{P_x}$$

2. Aggregate budget:

$$P_x \sum_i x_i \;+\; P_y y \;=\; W$$

3. $(n-1)$ income distribution rules about $(x_1, \cdots, x_n)$

E Equal consumption:

$$x_1 \;=\; \cdots \;=\; x_n \;=\; \frac{W - P_y y^*}{n P_x} \quad \square$$

- In general, y^* depends on *income distribution/transfer*.

3. Kolm's Triangle

3.1. Graphic Presentation

- Resource allocation:

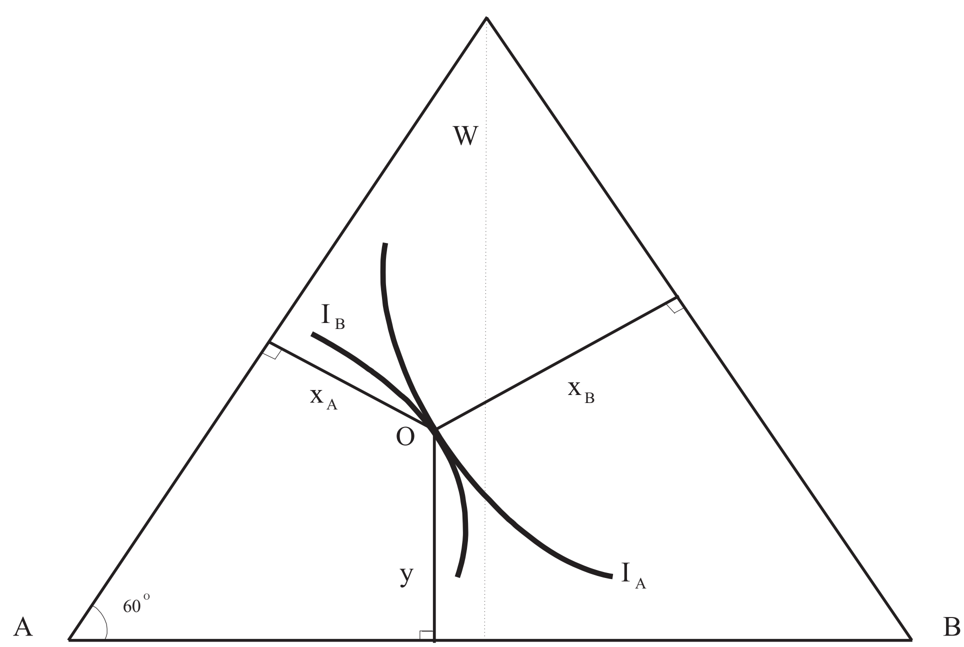
$$\{x_A, x_B, y\}$$

- Assume:

$$p_x = p_y = 1$$

- Feasible set:

$$\{\ (x_A, x_B, y) \mid x_A + x_B + y = W\ \}$$



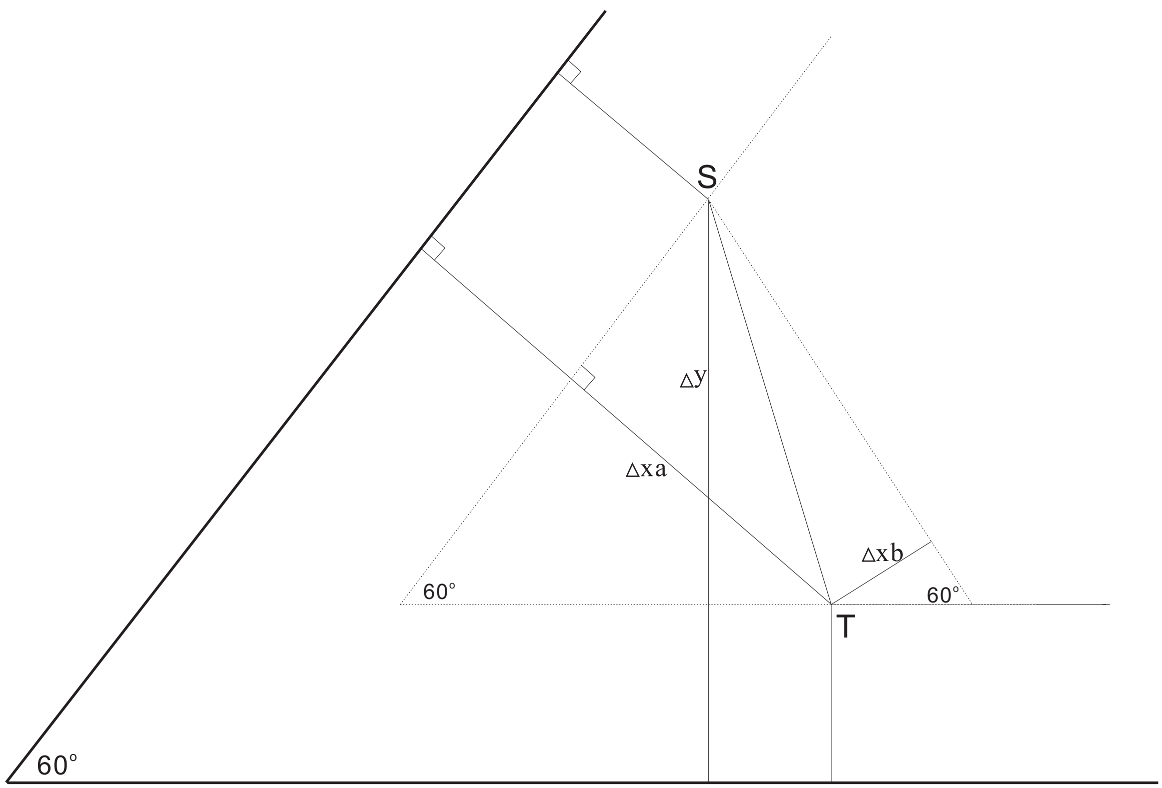
3.2. Optimality

- Consider common tangency at point T
- Move from T to S: $(\Delta x_A, \Delta x_B, \Delta y)$

$$\text{MRS}_A^{y,x} = \frac{\Delta x_A}{\Delta y}, \quad \text{MRS}_B^{y,x} = \frac{\Delta x_B}{\Delta y}$$

▷ Samuelson foc:

$$\text{MRS}_A^{y,x} + \text{MRS}_B^{y,x} = \frac{\Delta x_A}{\Delta y} + \frac{\Delta x_B}{\Delta y} = \frac{\Delta x_A + \Delta x_B}{\Delta y} = 1$$



4. Government Economic Functions

4.1. Richard Musgrave and Peggy Musgrave [1989]

- Resource allocation: efficiency
- Income redistribution: equity
- Economic stability: macro goal

[Q] when is "efficiency rule" independent from "equity rule"?

4.2. Example 1: quasi-linear utility

$$U_C(x_C, y) = x_C + f_C(y); \quad f'_C > 0, \; f''_C < 0$$

$$U_D(x_D, y) = x_D + f_D(y); \quad f'_D > 0, \; f''_D < 0$$

$$\text{MRS}_C^{y,x} = f'_C(y), \quad \text{MRS}_D^{y,x} = f'_D(y)$$

foc for optimal y^*:

$$f'_C(y) + f'_D(y) = P_y$$

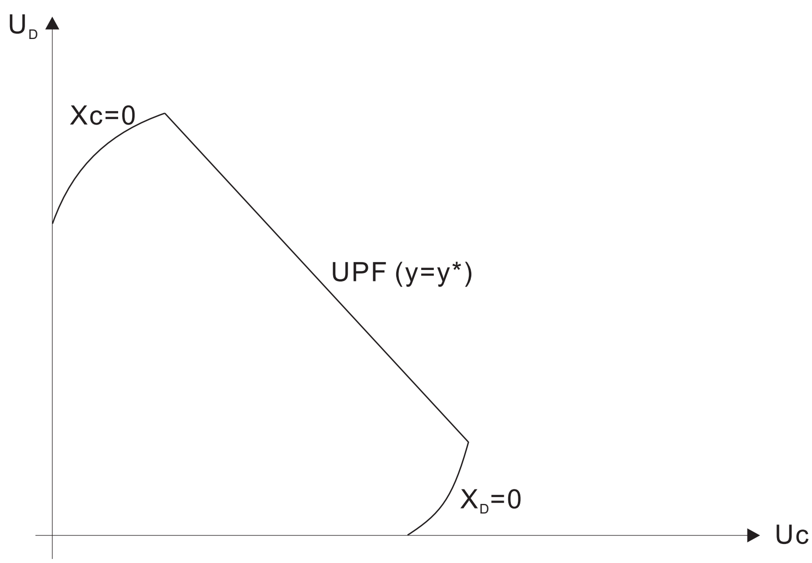
Private consumption:

$$x_C + x_D = W - P_y \cdot y^*$$

Total utility:

$$\begin{aligned} U_C + U_D &= [x_C + f_C(y^*)] + [x_D + f_D(y^*)] \\ &= [W - P_y \cdot y^*] + [f_C(y^*) + f_D(y^*)] \end{aligned}$$

▷ Linear utility possibility frontier (UPF)



4.3. Example 2: C-D utility

$$U_C(x_C, y) \;=\; x_C \cdot y^2; \;\; U_D(x_D, y) \;=\; x_D \cdot y$$

Samuelson foc:

$$\sum_i \text{MRS}_i^{y,x} \;=\; \frac{2x_C + x_D}{y} = 1 \tag{4}$$

with:

$$x_C + x_D + y = W \tag{5}$$

We only know:

$$y^* \;=\; \frac{x_C + W}{2}$$

Assuming $x_C = x_D$:

$$y^* \;=\; \frac{3W}{5}; \;\; x_C^* \;=\; x_D^* \;=\; \frac{W}{5}$$

4.4. Example 3: Identical C-D utility

$$U_i(x_i, y) \;=\; x_i^{\alpha} y^{1-\alpha}, \;\; 0 < \alpha < 1$$

$$\text{MRS}_i^{y,x} \;=\; \frac{[1-\alpha] y^{-\alpha} x_i^{\alpha}}{\alpha y^{1-\alpha} x_i^{\alpha-1}} \;=\; \frac{[1-\alpha] x_i}{\alpha y}$$

Samuelson foc:

$$\sum_i \text{MRS}_i^{y,x} \;=\; \frac{1-\alpha}{\alpha y} \sum_i x_i \;=\; \frac{1-\alpha}{\alpha y} X \;=\; \text{MC}(y) \tag{6}$$

We can then solve for X^* and y^* with aggregate budget:

$$P_x X + P_y y \;=\; W$$

4.5. Bergstrom-Cornes [1981]

- Gorman-form utility:[1]

$$U_i(x_i, y) = A(y)x_i + B_i(y), \ \ \forall\, i$$

- MRS:

$$\mathrm{MRS}_i^{y,x} \ = \ \frac{A'(y)x_i + B_i'(y)}{A(y)} \ = \ \alpha(y)x_i + \beta_i(y)$$

so

$$\sum_i \mathrm{MRS}_i^{y,x} \ = \ \alpha(y)X + \sum_i \beta_i(y) \ = \ f(X, y)$$

▷ Optimal y^* depends only on X $(= W - P_y y^*)$

- UPF(y^*) is a simplex: transferable utility

$$\sum_i U_i \ = \ A(y^*)X \ + \sum_i B_i(y^*) \ = \ K$$

[1] A representative consumer exists and has aggregate demand:

$$X(P, I) \ = \ \sum_i x_i(P, I_i)$$

instead of

$$X(P, I_1, \cdots, I_n) \ = \ \sum_i x_i(P, I_i)$$

when individual consumers have Gorman-form utilitiy functions.

5. Individualized Prices: Lindahl Equilibrium

5.1. Transformation

- Public-good economy:

$$U_C(x_C, y), \;\; U_D(x_D, y)$$

$$x_C \,+\, x_D \,+\, p_y y \;=\; W_C \,+\, W_D$$

- Private good economy *with joint production/consumption*:

$$U_C(x_C, y_C), \;\; U_D(x_D, y_D)$$

$$y_C \;=\; y_D \;\equiv\; \bar{y}$$

$$x_C \,+\, x_D \,+\, P_y \bar{y} \;=\; W_C \,+\, W_D$$

5.2. Equilibrium

$$(x_C^*,\, x_D^*,\, y_C^*,\, y_D^*,\, P_C,\, P_D)$$

- Demand-side: util-max

$$\begin{aligned}(x_C^*, y_C^*) &= \operatorname{argmax}_{x_C, y_C} \;\; U_C(x_C, y_C) \quad \text{s.t.} \quad x_C + P_C \cdot y_C = W_C \\ (x_D^*, y_D^*) &= \operatorname{argmax}_{x_D, y_D} \;\; U_D(x_D, y_D) \quad \text{s.t.} \quad x_D + P_D \cdot y_D = W_D\end{aligned}$$

- Supply-side:

$$P_C \,+\, P_D \;=\; P_y$$

- Equilibrium:

$$y_C^* \;=\; y_D^*$$

5.3. Example

- 2 roommates: C, D

  2 goods: x, y

- Rent share:

$$\tau_C,\ \tau_D(\equiv 1-\tau_C)$$

- Utility-max:

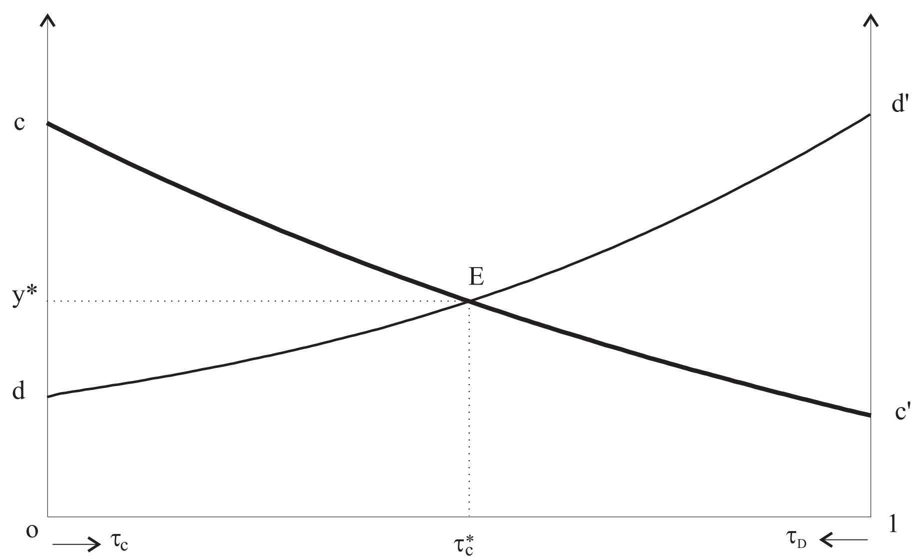
$$\max_{x_C, y_C} \ U_C(x_C, y_C) \quad \text{s.t.} \quad x_C + P_C \cdot y_C = W_C$$

- Demand:

$$x_C^*(P_C),\ y_C^*(P_C)$$

$\triangleright$

$$y_C^* \ = \ y_C^*(P_C) \ = \ y_C^*(\tau_C)$$



- Lindahl equilibrium/prices:

$$y_C^*(\tau_C) \;=\; y_D^*(\tau_D) \;=\; y^*$$

$$\tau_C + \tau_D \;=\; 1$$

- n-consumer generalization:

$$y_1(\tau_1) \;=\; y_2(\tau_2) \;=\; \cdots \;=\; y_n(\tau_n)$$
$$\sum_{i=1}^{n} \tau_i \;=\; 1$$

- Efficiency:

$$\text{MRS}_C^{y,x} \;=\; P_C \;=\; \tau_C P_y$$

$$\text{MRS}_D^{y,x} \;=\; P_D \;=\; \tau_D P_y$$

▷ Samuelson foc:

$$\text{MRS}_C^{y,x} \;+\; \text{MRS}_D^{y,x} \;=\; [\tau_C + \tau_D] P_y \;=\; P_y$$

6. Impure PG: Congestion

6.1. Context

- Example: smoking, driving, littering, having babies

$$U_i(x_i, s_i, S); \quad S \equiv \sum_j s_j$$

▷ *Diminishing marginal (dis-)utility*:

$$\frac{\partial U_i}{\partial s_i} > 0, \quad \frac{\partial^2 U_i}{\partial {s_i}^2} < 0$$

$$\frac{\partial U_i}{\partial S} < 0, \quad \frac{\partial^2 U_i}{\partial S^2} > 0$$

- Cornes-Sandler [1996/2e, Chapter 8]

$$U_j(x_j, d_j, C(D, H)), \quad j = 1, \cdots, n$$

with:

$$U_x^j > 0, \; U_d^j > 0, \; U_C^j < 0$$

and congestion effect:

$$C_D > 0, \quad C_H < 0$$

6.2. Optimality v. Equilibrium

- $(n+1)$ Samuelson FOCs:

$$(H)\quad \sum_{j=1}^{n} \frac{\partial U_j/\partial H}{\partial U_j/\partial x_j} = P_H$$

$$(d_i)\quad \sum_{j=1}^{n} \frac{\partial U_j/\partial d_i}{\partial U_j/\partial x_j} = P_D;\quad i = 1, \ldots, n$$

or:

$$(H)\quad \left[\sum_{j} \frac{U_C^j}{U_x^j}\right] C_H \;=\; P_H$$

$$(d_i)\quad \frac{U_d^i + U_C^i C_D}{U_x^i} + \left[\sum_{j\neq i}^{n} \frac{U_C^j}{U_x^j}\right] C_D \;=\; P_D;\quad i = 1, \ldots, n$$

- Define

$$\eta \;\equiv\; \sum_{j} \frac{U_C^j}{U_x^j} \;=\; \sum_{j} \mathrm{MRS}_j^{C,x} \quad (< 0)$$

and

$$\eta_{-i} \;\equiv\; \sum_{j\neq i} \mathrm{MRS}_j^{C,x} \quad (< 0)$$

we have:

$$(H)\quad \eta C_H = P_H$$

$$(d_i)\quad \frac{U_d^i + U_C^i C_D}{U_x^i} = P_D - \eta_{-i} C_D$$

- Optimal driving d_i^*:

$$\mathrm{MRS}_i^{d,x} \;\equiv\; \frac{U_d^i + U_C^i C_D}{U_x^i} \;=\; P_D - \frac{\eta_{-i}}{\eta} \cdot \frac{C_D P_H}{C_H} \;>\; P_D$$

! Market outcome: individual choice

$$\text{MRS}_i^{d,x} \;=\; P_D$$

▷ Too much driving without government regulation

6.3. Government Intervention

- Driving tax for efficiency:

$$\text{MRS}_i^{d,x} \;=\; P_D + T_i$$

▷ Individual-specific tax rate:

$$T_i^* \;=\; -\frac{\eta_{-i}}{\eta} \cdot \frac{C_D \cdot P_H}{C_H} \quad (>0)$$

- Identical-consumer symmetric equilibrium: uniform tax rate

$$T^* \;=\; \frac{-[n-1]}{n} \cdot \frac{C_D P_H}{C_H} \;\approx\; \frac{-C_D P_H}{C_H} \;>\; 0 \quad (\text{as } \; n \to \infty)$$

6.4. Government Budget

- Budget balance/surplus/deficit:

$$T^* \cdot D^* \gtreqless P_H \cdot H^*$$

- Assume: $C(D, H)$ is a homogeneous function of degree k:

$$C(tD, tH) = t^k \cdot C(D, H)$$

  - If $k = 0$:

$$C(tD, tH) = C(D, H)$$

  - If $k > 0$:

$$C(tD, tH) > C(D, H)$$

  - If $k < 0$:

$$C(tD, tH) < C(D, H)$$

- Euler equation:

$$D \cdot C_D + H \cdot C_H = k \cdot C(D, H)$$

▷

$$D \cdot C_D \gtrless -H \cdot C_H \iff k \gtrless 0$$

- At T^*:

$$R \equiv T^* \cdot D = \frac{-D \cdot C_D \cdot P_H}{C_H} \gtrless \frac{-[-H \cdot C_H] P_H}{C_H} = P_H \cdot H \rightleftharpoons k \gtrless 0$$

  - $k = 0$: budget balance
  - $k > 0$: budget surplus
  - $k < 0$: budget deficit

7. Information Asymmetry

- Consumer preference:

$$U_i(\, x_i,\, y \mid \alpha_i\,) \tag{7}$$

▷ Goct knows U_i, but not α_i[2]

- Self-reporting: $\hat{\alpha}_i$, tax price $q_i(\hat{\alpha}_i)$

- Consumer util-max:

$$U_i^*(\hat{\alpha}_i) \;\equiv\; \max_{x_i,\, y} \; U_i(x_i, y) \quad \text{s.t.} \quad x_i + q_i(\hat{\alpha}_i)y = W_i \tag{8}$$

▷

$$x_i(\hat{\alpha}_i), \;\; y(\hat{\alpha}_i), \;\; U_i^*(\hat{\alpha}_i) = U_i(x_i(\hat{\alpha}_i), y(\hat{\alpha}_i))$$

- Incentive:

$$\begin{aligned} \frac{dU_i^*}{d\hat{\alpha}_i} &= \frac{\partial U_i}{\partial x_i} \cdot \frac{dx_i}{d\hat{\alpha}_i} + \frac{\partial U_i}{\partial y} \cdot \frac{dy}{d\hat{\alpha}_i} \\ &= \frac{\partial U_i}{\partial x_i} \cdot \frac{d[W_i - q_i(\hat{\alpha}_i)y(\hat{\alpha}_i)]}{d\hat{\alpha}_i} + \frac{\partial U_i}{\partial y} \cdot \frac{dy}{d\hat{\alpha}_i} \\ &= \frac{\partial U_i}{\partial x_i} \cdot \left[-q_i \frac{dy}{d\hat{\alpha}_i} - y\frac{dq_i}{d\hat{\alpha}_i}\right] + \frac{\partial U_i}{\partial y} \cdot \frac{dy}{d\hat{\alpha}_i} \\ &= \left[\frac{-\partial U_i}{\partial x_i} y\right] \frac{dq_i}{d\hat{\alpha}_i} + \left[\frac{\partial U_i}{\partial y} - q_i \frac{\partial U_i}{\partial x_i}\right] \frac{dy}{d\hat{\alpha}_i} \end{aligned} \tag{9}$$

- By consumer foc:

$$\text{MRS}_i^{y,x} \;=\; \frac{\partial U_i/\partial y}{\partial U_i/\partial x_i} \;=\; q_i$$

▷

$$\frac{dU_i^*}{d\hat{\alpha}_i} \;=\; \left[\frac{-\partial U_i}{\partial x_i} y\right] \frac{dq_i}{d\hat{\alpha}_i} \tag{10}$$

▷ $dU_i^*/d\hat{\alpha}_i$ and $dq_i/d\hat{\alpha}_i$ have opposite sign

[2] For example:

$$U_i(x_i, y) \;=\; x_i^{1-\alpha_i} y^{\alpha_i}$$