Simple Two-Period Model

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General principles for specifying a model (1)

- agents
- decisions
- constraints
- information
- how agents interact with each other
General principles for specifying a model (2)

Types of decision-makers:

- households – preferences; endowment (over commodities)
- firms – (production) technology
- government – policy instruments
  positive v.s. normative analysis

equilibrium concept:
how agents perceive their power to affect market prices
A simple intertemporal model

- a static intertemporal model with one nonstorable good
- Agents live for two periods.
- $U(c_1, c_2) = U(c_1) + \beta U(c_2)$
- $\beta$ is the discount factor.
  $\beta = 1/(1 + \rho), \rho > 0$ is the rate of time preference.
- $p$: price of 1 unit of consumption good tomorrow

$$p = \frac{1}{1 + r}$$
Maximization problem

\[
\max_{c_1,c_2} U(c_1) + \beta U(c_2)
\]

s.t. \[
c_1 + \frac{c_2}{1 + r} = e_1 + \frac{e_2}{1 + r}
\] (1)

\( s \): saving

- Period 1 b.c. \( s = e_1 - c_1 \) \hspace{1cm} (2)
- Period 2 b.c. \( c_2 = e_2 + (1 + r)s \) \hspace{1cm} (3)

(2) and (3) imply (1).
First order conditions

\[ L = U(c_1) + \beta U(c_2) + \lambda \left[ e_1 + \frac{e_2}{1+r} - c_1 - \frac{c_2}{1+r} \right] \]

F.O.C.s

\[ U'(c_1) = \lambda \]
\[ \beta U'(c_2) = \frac{\lambda}{1+r} \]
\[ e_1 + \frac{e_2}{1+r} = c_1 + \frac{c_2}{1+r} \]

⇒ Euler Equation \[ U'(c_1) = (1 + r) \beta U'(c_2). \]
Example 1: Maximization

\[ u(c_1, c_2) = \ln c_1 + \beta \ln c_2 \]

F.O.C.s \rightarrow solve for \( c_1, c_2 \) in terms of \( r, e_1, e_2 \):

\[ c_1 = \frac{1}{1 + \beta} (e_1 + \frac{e_2}{1 + r}) \tag{4} \]

\[ c_2 = \frac{(1 + r)\beta}{1 + \beta} (e_1 + \frac{e_2}{1 + r}) \]
Example 1 (cont’d)

- market clearing conditions:

\[ c_1 = e_1 \quad (5) \]
\[ c_2 = e_2 \]

- From (4) and (5):

\[ \frac{1}{1 + r} = \beta \frac{e_1}{e_2} \]

- Since \( \beta = 1/(1 + \rho) \) and \( e_2/e_1 = (1 + g) \), we get:

\[ 1 + r = (1 + \rho)(1 + g), \text{ or } \]

\[ r \approx \rho + g \]
Add Production to the Model

- \( e_1 > 0, \ e_2 = 0 \).
- storage technology \( y = f(k) \)
or, \( y = g(k) + (1 - \delta)k \)
Agent’s maximization problem

- storage tech. $\rightarrow$ one more choice variable: $k$.

$$\max_{c_1,c_2,k} u(c_1) + \beta u(c_2)$$

$$s.t. \quad c_1 + \frac{c_2}{1+r} = e_1 + \frac{1}{1+r}[f(k) - (1 + r)k]$$

- How to derive this intertemporal b.c.:
  - period 1: $e_1 = c_1 + k + b$
  - period 2: $c_2 = f(k) + (1 + r)b \Rightarrow b = \frac{c_2 - f(k)}{1+r}$ and then substituting $b$ into period 1 constraint

- In equilibrium, $b = 0$. 

Simple Two-Period Model
First order conditions

\[ \frac{\partial L}{\partial c_1} = 0 \iff u'(c_1) = \lambda \]  
\[ (6) \]

\[ \frac{\partial L}{\partial c_2} = 0 \iff \beta u'(c_2) = \frac{\lambda}{1 + r} \]  
\[ (7) \]

\[ \frac{\partial L}{\partial k} = 0 \iff \frac{\lambda}{1 + r} [f'(k) - (1 + r)] = 0 \]  
\[ (8) \]

\[ \frac{\partial L}{\partial \lambda} = 0 \iff b.c. \]  
\[ (9) \]
Euler equation

4 equations to solve for $c_1, c_2, k$ and $\lambda$.

$\Rightarrow (8)$

$$f'(k^*) = 1 + r$$

$\Rightarrow$ Substituting $k^*$ into the b.c.

$$c_1 + \frac{c_2}{1 + r} = e_1 + \frac{1}{1 + r} [f(k^*) - (1 + r)k^*].$$

$\Rightarrow (6)$ and $(7)$ $\Rightarrow$ Euler equation

$$u'(c_1) = (1 + r)\beta u'(c_2).$$

$\Rightarrow (11)$ and $(12)$ $\Rightarrow$ optimal values of $c_1, c_2$ in terms of $r$. 

Simple Two-Period Model
Equilibrium

Definition

A competitive equilibrium for this economy is a price \(r\) and quantities \((c_1, c_2, k)\) such that (i) agents maximize utility given the price (ii) Markets clear.

market clearing conditions

\[(MK1) \quad e_1 = c_1 + k\] (Goods market in pd 1)

\[(MK2) \quad c_2 = f(k)\] (Goods market in pd 2)
Example 2: Add production

\[ u(c) = \ln c \text{ and } f(k) = k^\sigma, \ 0 < \sigma < 1. \]

- \( k^* \) solves \( f'(k) = \sigma k^{\sigma-1} = 1 + r : \)

\[ k^* = \left( \frac{\sigma}{1 + r} \right)^{\frac{1}{1-\sigma}} \]

\[ f(k^*) = \left( \frac{\sigma}{1 + r} \right)^{\frac{\sigma}{1-\sigma}} \]

- Substituting \( k^*, c^* \) into the \((MK1)\):

\[ e_1 = \frac{1}{1 + \beta} \left\{ e_1 + \frac{1}{1 + r} [f(k^*) - (1 + r)k^*] \right\} + k^* \]
Example 2 (cont’d)

- Solving for the equilibrium interest rate:

\[(1 + r) = \sigma \left( \frac{1 + \sigma \beta}{\sigma \beta e_1} \right)^{(1-\sigma)}\]

1. As \(\beta\) rises, \(1 + r\) falls.
2. As \(e_1\) is higher, \(1 + r\) falls.

- Substitute the eqm interest rate \(r\) into the b.c. and Euler eq to get the eqm quantity of \(c_1, c_2, k\) and \(f(k)\).

\[k^* = \frac{\sigma \beta}{1 + \sigma \beta} e_1\]

- \(\frac{dk^*}{d\sigma} > 0\) and \(\frac{dk^*}{d\beta} > 0\).
Add a Government Sector

Assumptions

- Taxes are lump-sum: \((\tau_1, \tau_2)\).
- Government spending, \((g_1, g_2)\), is exogenous and has no effect on the utility of agents.

Agent’s maximization problem

\[
\max_{c_1,c_2,k} u(c_1) + \beta u(c_2) \\
\text{s.t.} \quad c_1 + \tau_1 + \frac{c_2 + \tau_2}{1 + r} = e_1 + \frac{1}{1 + r} [f(k) - (1 + r)k]
\]
Government’s budget constraint

\[
\begin{align*}
\text{period 1} & \quad g_1 - \tau_1 = b \\
\text{period 2} & \quad g_2 + (1 + r)b = \tau_2
\end{align*}
\]

intertemporal budget constraint

\[
g_1 + \frac{g_2}{1 + r} = \tau_1 + \frac{\tau_2}{1 + r}.
\]

or,

\[
\tau_2(r) = (1 + r)(g_1 - \tau_1) + g_2.
\]
Agent’s maximization problem

\[ L = u(c_1) + \beta u(c_2) \]
\[ + \lambda \left\{ e_1 + \frac{1}{1+r} [f(k) - (1+r)k] - c_1 - \tau_1 - \frac{c_2 + \tau_2}{1+r} \right\} \]

F.O.C.

\[ \frac{\partial L}{\partial c_1} = 0 \iff u'(c_1) = \lambda \]
\[ \frac{\partial L}{\partial c_2} = 0 \iff \beta u'(c_2) = \frac{\lambda}{1+r} \]
\[ \frac{\partial L}{\partial k} = 0 \iff f'(k) = 1 + r \]
\[ \frac{\partial L}{\partial \lambda} = 0 \iff c_1 + \frac{c_2}{1+r} = b.c. \]
Market clearing conditions

1. \( e_1 = c_1 + k + g_1 \) (goods market at pd 1)
2. \( c_2 + g_2 = f(k) \) (goods market at pd 2)
3. \( b = g_1 - \tau_1 \) (bond market)
Insight

- substitute govt’s b.c. into the household’s b.c.

\[ c_1 + \frac{c_2}{1 + r} = e_1 + \frac{1}{1 + r} [f(k) - (1 + r)k] - \left[ g_1 + \frac{g_2}{1 + r} \right]. \]

- Agents do not care about the timing of the taxes! Only the present value of the government spending matters.

- Testable Implications:
  For fixed levels of spending (\( g_1 + \frac{g_2}{1+r} \) does not change): Deficits (or, deficit-financed tax cut) have no effect on:
  (i) consumption;
  (ii) interest rates.