Supplementary Appendix to
"Liquidity, Asset Prices, and Credit Constraints"

This supplementary appendix contains all proofs, omitted derivations of equations, and supplementary material.

A Proofs and derivation of equations

Deriving the marginal values of holding money and the real asset in the first subperiod. Differentiating (7) with respect to m and a, respectively, we obtain

$$\begin{aligned} V_m(m,a) &= (1-n)[u'(q_b)\frac{\partial q_b}{\partial m} + W_m(1+\frac{\partial \ell}{\partial m} - p\frac{\partial q_b}{\partial m}) + W_a\frac{\partial a}{\partial m} + W_\ell \frac{\partial \ell}{\partial m}] \\ &\quad + n[-c'(q_s)\frac{\partial q_s}{\partial m} + W_m(1-\frac{\partial d}{\partial m} + p\frac{\partial q_s}{\partial m}) + W_a\frac{\partial a}{\partial m} + W_d\frac{\partial d}{\partial m}], \\ V_a(m,a) &= (1-n)[u'(q_b)\frac{\partial q_b}{\partial a} + W_m(\frac{\partial m}{\partial a} + \frac{\partial \ell}{\partial a} - p\frac{\partial q_b}{\partial a}) + W_a + W_\ell\frac{\partial \ell}{\partial a}] \\ &\quad + n[-c'(q_s)\frac{\partial q_s}{\partial a} + W_m(\frac{\partial m}{\partial a} - \frac{\partial d}{\partial a} + p\frac{\partial q_s}{\partial a}) + W_a + W_d\frac{\partial d}{\partial a}]. \end{aligned}$$

Recall from (3) – (6) that $W_m = \phi$, $W_a = \psi + \rho$, $W_\ell = -\phi(1+i)$, and $W_d = \phi(1+i_d)$. Moreover, $\frac{\partial d}{\partial a} = 0$, and $\frac{\partial d}{\partial m} = 1$ since a seller deposits all his cash when $i > 0$. Also, $\frac{\partial a}{\partial m} = 0$ and $\frac{\partial m}{\partial a} = 0$ because an agent's portfolio (m,a) is determined in the previous period. The binding budget constraint, $m + \ell - pq_b = 0$, implies $1 + \frac{\partial \ell}{\partial m} - p\frac{\partial q_b}{\partial m} = 0$ and $\frac{\partial m}{\partial a} + \frac{\partial \ell}{\partial a} - p\frac{\partial q_b}{\partial a} = 0$, from which one can derive the conditions $\frac{\partial q_b}{\partial m} = (1+\frac{\partial \ell}{\partial m})/p$ and $\frac{\partial q_b}{\partial a} = (\frac{\partial \ell}{\partial a})/p$. Because the quantities produced by a seller are independent of his portfolio, $\frac{\partial q_s}{\partial m} = 0$ and $\frac{\partial q_s}{\partial a} = 0$. Hence,

$$\begin{aligned} V_m(m,a) &= (1-n)[u'(q_b)\frac{\partial q_b}{\partial m} - \phi(1+i)\frac{\partial \ell}{\partial m}] + n\phi(1+i_d), \\ V_a(m,a) &= (\psi+\rho) + (1-n)[\frac{u'(q_b)}{p} - \phi(1+i)]\frac{\partial \ell}{\partial a}. \end{aligned}$$

Substituting $\frac{\partial \ell}{\partial m} = p\frac{\partial q_b}{\partial m} - 1$, into the term, $u'(q_b)\frac{\partial q_b}{\partial m} - \phi(1+i)\frac{\partial \ell}{\partial m}$, we have $[u'(q_b) - \phi(1+i)p]\frac{\partial q_b}{\partial m} + \phi(1+i) = \frac{u'(q_b)}{p}$, because if the buyer is not credit constrained, $\frac{u'(q_b)}{c'(q_s)} = 1+i$, and if he is constrained, $\frac{\partial \bar{\ell}}{\partial m} = 0$ and $\frac{\partial q_b}{\partial m} = \frac{1}{p}$. After some manipulation we obtain equations (13) and (14). ■

Proof of Proposition 1. Substituting $\frac{u'(q_b)}{c'(q_s)} = (1+i)$ from (11) into (16), the second term of the right side in (16) vanishes, and we obtain the asset price in an the unconstrained equilibrium, (17). ■

Deriving $\phi\bar{\ell}$ in (21) under a collateral mechanism. For a buyer entering the second subperiod who repays his loan and holds no money, the expected discounted utility in a stationary equilibrium is

$$W(m,a) = U(x^*) - h_b + \beta V(m_{+1}, a_{+1}),$$

where h_b is a buyer's production in the second subperiod if he repays the loan. Since banks' punishment is confined to the current period, a defaulter will start a new period as nondefaulters, and his continuation payoffs is denoted by $V(\widehat{m}_{+1}, \widehat{a}_{+1})$, where the hat indicates a deviator's optimal choice. A deviating buyer's expected discounted utility is

$$\widehat{W}(m,a) = U(\widehat{x}) - \widehat{h}_b + \beta V(\widehat{m}_{+1}, \widehat{a}_{+1}).$$

In the second subperiod the deviating buyer's problem is

$$\begin{aligned} \widehat{W}(m,a) &= \max_{\widehat{x}, \widehat{h}_b, \widehat{m}_{+1}, \widehat{a}_{+1}} U(\widehat{x}) - \widehat{h}_b + \beta V(\widehat{m}_{+1}, \widehat{a}_{+1}) \\ \text{s.t. } x + \phi \widehat{m}_{+1} + \psi \widehat{a}_{+1} &= \widehat{h}_b + \phi(m+T). \end{aligned}$$

Note that the deviator's constraint has taken into account the loss in collateral, and the benefit of not repaying the debt. The first-order condition are $U'(\widehat{x}) = 1$, $-\phi + \beta \frac{\partial V(\widehat{m}_{+1}, \widehat{a}_{+1})}{\partial \widehat{m}_{+1}} = 0$ and $-\psi + \beta \frac{\partial V(\widehat{m}_{+1}, \widehat{a}_{+1})}{\partial \widehat{a}_{+1}} = 0$, which imply $\widehat{x} = x^*$ and $(\widehat{m}_{+1}, \widehat{a}_{+1}) = (m_{+1}, a_{+1})$. Therefore, a deviator would choose the same portfolio as non-deviators, and he has the expected discounted utility

$$\widehat{W}(m,a) = U(\widehat{x}) - \widehat{h}_b + \beta V(m_{+1}, a_{+1}).$$

The real borrowing constraint $\phi\bar{\ell}$ is the value such that $W(m,a) = \widehat{W}(m,a)$, or, $\widehat{h}_b = h_b$. For a non-deviator and a deviator, the labor used in production is, respectively,

$$h_b = x^* + \phi m_{+1} + \psi a_{+1} - \phi(m + \bar{\ell} - pq_b) - \phi\tau m - (\psi+\rho)a + \phi(1+i)\bar{\ell},$$

and

$$\widehat{h}_b = x^* + \phi m_{+1} + \psi a_{+1} - \phi(m + \bar{\ell} - pq_b) - \phi\tau m.$$

Hence, $\phi\bar{\ell}$ satisfies $(\psi+\rho)a = (1+i)\phi\bar{\ell}$, and we obtain (21). ■

Deriving the existence condition for the unconstrained equilibrium under a collateral mechanism. Recall that $\phi\bar{\ell} = \frac{(\psi+\rho)a}{1+i}$, and in equilibrium $a = A$. Now define $\triangle = \frac{nc'(q_s)q_b(1+i)}{\psi+\rho} - A$.

Notice that in an unconstrained equilibrium, $\phi\ell < \phi\bar{\ell}$, which implies $\triangle < 0$. Substituting $\psi = \frac{\beta\rho}{1-\beta}$ and $i = \frac{\gamma-\beta}{\beta}$into $\triangle$, we have

$$\begin{aligned}
\triangle &= \frac{n\phi p q_b \gamma(1-\beta)}{\rho\beta} - A \\
&= \frac{n\phi M_{-1}\gamma(1-\beta)}{\rho(1-n)\beta} - A \\
&= \frac{n(1-\beta)\phi M}{\rho(1-n)\beta} - A < 0,
\end{aligned}$$

where we have used $(1-n)pq_b = M_{-1}$ and $M = \gamma M_{-1}$ to obtain the first and second equalities. Therefore, if the asset supply is sufficient; i.e.,

$$A > \frac{n(1-\beta)\phi M}{\rho(1-n)\beta} = \overline{A},$$

there exists an equilibrium with unconstrained credit. ■

Proof of proposition 2. Substituting (22) into (16) one obtains (24). Rearranging (24), one can solve for the asset price $\psi = \psi_1$, where ψ_1 is defined in (25). The loan-to-value ratio θ is defined as the ratio of the real loan amount to the real value of collateral, $\frac{\phi\ell}{\psi a}$. Using $\phi\ell = \phi\bar{\ell}$ from (21), we obtain $\theta = \theta_1$, where θ_1 is defined in (26). ■

Proof of proposition 3. Here we derive the effects of inflation under the collateral mechanism. In an unconstrained equilibrium, (q_b, i, ψ) satisfy (17), (18) and (19). Define

$$\begin{aligned}
f^u(q_b, i, \psi; z) &= \frac{u'(q_b)}{c'(q_s)} - 1 - \frac{\gamma-\beta}{\beta}, \\
g^u(q_b, i, \psi; z) &= i - \frac{\gamma-\beta}{\beta}, \\
h^u(q_b, i, \psi; z) &= \psi - \frac{\beta\rho}{1-\beta},
\end{aligned}$$

where $q_s = \frac{1-n}{n} q_b$. Let k_x^u denote $\frac{\partial k}{\partial x}$, where $k = f^u, g^u, h^u$ and $x = q_b, i, \psi, z$, where z denotes the parameter such as γ, ρ and A. Then, $f_{q_b}^u = \frac{u''(q_b)}{c'(q_s)} - \frac{1-n}{n}\frac{u'(q_b)c''(q_s)}{[c'(q_s)]^2} < 0$, $f_i^u = f_\psi^u = g_{q_b}^u = g_\psi^u = h_{q_b}^u = h_i^u = h_\gamma^u = 0$, $f_\gamma^u = g_\gamma^u = -\frac{1}{\beta}$, $g_i^u = h_\psi^u = 1$. Note that

$$\begin{bmatrix} f_{q_b}^u & f_i^u & f_\psi^u \\ g_{q_b}^u & g_i^u & g_\psi^u \\ h_{q_b}^u & h_i^u & h_\psi^u \end{bmatrix} \begin{bmatrix} dq_b \\ di \\ d\psi \end{bmatrix} = - \begin{bmatrix} f_z^u dz \\ g_z^u dz \\ h_z^u dz \end{bmatrix}.$$

Let Λ^u, Λ^u_1, Λ^u_2, Λ^u_3 denote the determinants of the following matrices, respectively:

$$\Lambda^u = \begin{vmatrix} f^u_{q_b} & f^u_i & f^u_\psi \\ g^u_{q_b} & g^u_i & g^u_\psi \\ h^u_{q_b} & h^u_i & h^u_\psi \end{vmatrix}, \ \Lambda^u_1 = \begin{vmatrix} -f^u_\gamma & f^u_i & f^u_\psi \\ -g^u_\gamma & g^u_i & g^u_\psi \\ -h^u_\gamma & h^u_i & h^u_\psi \end{vmatrix},$$
$$\Lambda^u_2 = \begin{vmatrix} f^u_{q_b} & -f^u_\gamma & f^u_\psi \\ g^u_{q_b} & -g^u_\gamma & g^u_\psi \\ h^u_{q_b} & -h^u_\gamma & h^u_\psi \end{vmatrix}, \ \Lambda^u_3 = \begin{vmatrix} f^u_{q_b} & f^u_i & -f^u_\gamma \\ g^u_{q_b} & g^u_i & -g^u_\gamma \\ h^u_{q_b} & h^u_i & -h^u_\gamma \end{vmatrix},$$

and $\Lambda^u < 0$, $\Lambda^u_1 > 0$, $\Lambda^u_2 < 0$, $\Lambda^u_3 = 0$. Thus, $\frac{\partial q_b}{\partial \gamma} = \frac{\Lambda^u_1}{\Lambda^u} < 0$, $\frac{\partial i}{\partial \gamma} = \frac{\Lambda^u_2}{\Lambda^u} > 0$, $\frac{\partial \psi}{\partial \gamma} = \frac{\Lambda^u_3}{\Lambda^u} = 0$. Given $p = \frac{c'(q_s)}{\phi}$ and $\phi = \frac{(1-n)c'(q_s)q_b}{M_{-1}}$, we have

$$\begin{aligned} \frac{\partial \phi}{\partial \gamma} &= \frac{1-n}{M_{-1}}[\frac{1-n}{n}c''(q_s)q_b + c'(q_s)]\frac{\partial q_b}{\partial \gamma} < 0, \\ \frac{\partial p}{\partial \gamma} &= -\frac{p^2(1-n)}{M_{-1}}\frac{\partial q_b}{\partial \gamma} > 0, \end{aligned}$$

because $\frac{\partial q_b}{\partial \gamma} < 0$.

In a constrained equilibrium, (q_b, i, ψ) satisfy (23), (21), and (25). Define

$$\begin{aligned} f(q_b, i, \psi; z) &= (1-n)[\frac{u'(q_b)}{c'(q_s)} - 1] + ni - \frac{\gamma - \beta}{\beta}, \\ g(q_b, i, \psi; z) &= (1+i)nc'(q_s)q_b - (\psi + \rho)A, \\ h(q_b, i, \psi; z) &= (1 - \beta B)\psi - \beta B \rho, \end{aligned}$$

where we have substituted $\phi \ell = nc'(q_s)q_b$. Then,

$$f_{q_b} = (1-n)\{\frac{u''(q_b)}{c'(q_s)} - \frac{1-n}{n}\frac{u'(q_b)c''(q_s)}{[c'(q_s)]^2}\} < 0, \tag{32}$$

$g_{q_b} = (1+i)[(1-n)c''(q_s)q_b + nc'(q_s)] > 0$, $f_i = n$, $g_i = nc'(q_s)q_b$, $g_\psi = -A$, $h_{q_b} = -\frac{\beta(\psi+\rho)}{1+i}f_{q_b} > 0$, $h_i = \frac{\beta(\psi+\rho)(1-n)}{(1+i)^2}\frac{u'(q_b)}{c'(q_s)} > 0$, $f_\psi = g_\gamma = h_\gamma = 0$, $h_\psi = 1 - \beta B$, $f_\gamma = -\frac{1}{\beta}$.

Let Λ, Λ_1, Λ_2, Λ_3 denote the determinants of the following matrices, respectively:

$$\Lambda = \begin{vmatrix} f_{q_b} & f_i & f_\psi \\ g_{q_b} & g_i & g_\psi \\ h_{q_b} & h_i & h_\psi \end{vmatrix}, \ \Lambda_1 = \begin{vmatrix} -f_\gamma & f_i & f_\psi \\ -g_\gamma & g_i & g_\psi \\ -h_\gamma & h_i & h_\psi \end{vmatrix},$$
$$\Lambda_2 = \begin{vmatrix} f_{q_b} & -f_\gamma & f_\psi \\ g_{q_b} & -g_\gamma & g_\psi \\ h_{q_b} & -h_\gamma & h_\psi \end{vmatrix}, \ \Lambda_3 = \begin{vmatrix} f_{q_b} & f_i & -f_\gamma \\ g_{q_b} & g_i & -g_\gamma \\ h_{q_b} & h_i & -h_\gamma \end{vmatrix}.$$

We find that $\Lambda < 0$, $\Lambda_1 > 0$, $\Lambda_2 < 0$ and $\Lambda_3 > 0$ if $u'(q_b) + u''(q_b)q_b > 0$. Thus, $\frac{\partial q_b}{\partial \gamma} = \frac{\Lambda_1}{\Lambda} < 0$ and $\frac{\partial i}{\partial \gamma} = \frac{\Lambda_2}{\Lambda} > 0$. Then, $\frac{\partial \phi}{\partial \gamma} < 0$ and $\frac{\partial p}{\partial \gamma} > 0$. Also, $\frac{\partial \psi}{\partial \gamma} = \frac{\Lambda_3}{\Lambda} = \frac{n(1-n)\psi\theta[u'(q_b)+u''(q_b)q_b]}{\Lambda} < 0$ if $u'(q_b) + u''(q_b)q_b > 0$. Denote $r_p = \frac{\rho}{\psi}$. Given $\theta = \frac{1+r_p}{1+i}$, we have

$$\begin{aligned}
\frac{\partial \theta}{\partial \gamma} &= \frac{-r_p}{(1+i)\psi}\frac{\partial \psi}{\partial \gamma} - \frac{\theta}{1+i}\frac{\partial i}{\partial \gamma} \\
&= \frac{1}{(1+i)\Lambda}\{\frac{-r_p}{\psi}n(1-n)\psi\theta[u^{'}(q_b) + u''(q_b)q_b] - A\psi\theta^2 f_{q_b} + \theta r_p[(1-n)\frac{u'(q_b)}{c'(q_s)}\frac{1}{1+i} + n]g_{q_b}\} \\
&= \frac{1}{(1+i)\Lambda}\{-r_p n(1-n)\theta u''(q_b)q_b - A\psi\theta^2 f_{q_b} + r_p\theta n g_{q_b} \\
&\quad +\theta r_p(1-n)\frac{u'(q_b)}{c'(q_b)}[(1-n)c''(q_s)q_b + nc^{'}(q_s)] - r_p n(1-n)\theta u^{'}(q_b)\} \\
&= \frac{1}{(1+i)\Lambda}\{-n(1-n)r_p\theta u''(q_b)q_b - \psi\theta^2 A f_{q_b} + r_p\theta n g_{q_b} + \frac{(1-n)^2 r_p\theta u'(q_b)c''(q_s)q_b}{c'(q_s)}\} < 0,
\end{aligned}$$

because $\Lambda < 0$, $g_{q_b} > 0$, $f_{q_b} < 0$. ■

Proof of proposition 4. Here we derive the effects of changes in the supply of the real asset and dividend flows for the equilibria under the collateral mechanism. Let Λ_4^u, Λ_5^u, Λ_6^u, Λ_4, Λ_5, Λ_6 denote the determinants of the following matrices, respectively:

$$\begin{aligned}
\Lambda_4^u &= \begin{vmatrix} -f_A^u & f_i^u & f_\psi^u \\ -g_A^u & g_i^u & g_\psi^u \\ -h_A^u & h_i^u & h_\psi^u \end{vmatrix}, \ \Lambda_5^u = \begin{vmatrix} f_{q_b}^u & -f_A^u & f_\psi^u \\ g_{q_b}^u & -g_A^u & g_\psi^u \\ h_{q_b}^u & -h_A^u & h_\psi^u \end{vmatrix}, \ \Lambda_6^u = \begin{vmatrix} f_{q_b}^u & f_i^u & -f_A^u \\ g_{q_b}^u & g_i^u & -g_A^u \\ h_{q_b}^u & h_i^u & -h_A^u \end{vmatrix}, \\
\Lambda_4 &= \begin{vmatrix} -f_A & f_i & f_\psi \\ -g_A & g_i & g_\psi \\ -h_A & h_i & h_\psi \end{vmatrix}, \ \Lambda_5 = \begin{vmatrix} f_{q_b} & -f_A & f_\psi \\ g_{q_b} & -g_A & g_\psi \\ h_{q_b} & -h_A & h_\psi \end{vmatrix}, \ \Lambda_6 = \begin{vmatrix} f_{q_b} & f_i & -f_A \\ g_{q_b} & g_i & -g_A \\ h_{q_b} & h_i & -h_A \end{vmatrix}.
\end{aligned}$$

Because $g_A = -(\psi + \rho)$, $f_A^u = g_A^u = h_A^u = f_A = h_A = 0$, we have $\Lambda_4 < 0$, $\Lambda_5 < 0$, $\Lambda_6 > 0$, $\Lambda_4^u = \Lambda_5^u = \Lambda_6^u = 0$. Thus, in an unconstrained equilibrium, $\frac{\partial q_b}{\partial A} = \frac{\Lambda_4^u}{\Lambda^u} = 0$, $\frac{\partial i}{\partial A} = \frac{\Lambda_5^u}{\Lambda^u} = 0$, and $\frac{\partial \psi}{\partial A} = \frac{\Lambda_6^u}{\Lambda^u} = 0$. With the same argument in the proof of proposition 3, $\frac{\partial \phi}{\partial A} = 0$ and $\frac{\partial p}{\partial A} = 0$, because $\frac{\partial q_b}{\partial A} = 0$. In a constrained equilibrium, $\frac{\partial q_b}{\partial A} = \frac{\Lambda_4}{\Lambda} > 0$, $\frac{\partial i}{\partial A} = \frac{\Lambda_5}{\Lambda} > 0$, and $\frac{\partial \psi}{\partial A} = \frac{\Lambda_6}{\Lambda} < 0$. Also, $\frac{\partial \phi}{\partial A} > 0$

and $\frac{\partial p}{\partial A} < 0$, because $\frac{\partial q_b}{\partial A} > 0$. Denote $r_p = \frac{\rho}{\psi}$. Given $\theta = \frac{1+\frac{\rho}{\psi}}{1+i}$, we have

$$\begin{aligned}
\frac{\partial \theta}{\partial A} &= -\frac{1}{1+i}[\frac{r_p}{\psi}\frac{\partial \psi}{\partial A} + \theta\frac{\partial i}{\partial A}] \\
&= -\frac{r_p\theta}{\Lambda}[-n\beta\psi\theta f_{q_b} - \frac{\beta(1-n)\psi\theta f_{q_b}}{1+i}\frac{u'(q_b)}{c'(q_s)} + \frac{\psi\theta f_{q_b}(1-\beta B)}{r_p}] \\
&= \frac{\psi\theta^2 f_{q_b} r_p}{\Lambda}[-n\beta - \frac{\beta(1-n)}{1+i}\frac{u'(q_b)}{c'(q_s)} + \frac{1-\beta B}{r_p}] \\
&= \frac{\psi\theta^2 f_{q_b} r_p}{\Lambda}[-\beta B + \frac{1-\beta B}{r_p}] = 0.
\end{aligned}$$

The last equality is obtained by substituting $\psi = \frac{\beta B\rho}{1-\beta B}$ into $r_p = \frac{\rho}{\psi}$.

Let Λ_7^u, Λ_8^u, Λ_9^u, Λ_7, Λ_8, Λ_9 denote the determinants of the following matrices:

$$\begin{aligned}
\Lambda_7^u &= \begin{vmatrix} -f_\rho^u & f_i^u & f_\psi^u \\ -g_\rho^u & g_i^u & g_\psi^u \\ -h_\rho^u & h_i^u & h_\psi^u \end{vmatrix}, \ \Lambda_8^u = \begin{vmatrix} f_{q_b}^u & -f_\rho^u & f_\psi^u \\ g_{q_b}^u & -g_\rho^u & g_\psi^u \\ h_{q_b}^u & -h_\rho^u & h_\psi^u \end{vmatrix}, \ \Lambda_9^u = \begin{vmatrix} f_{q_b}^u & f_i^u & -f_\rho^u \\ g_{q_b}^u & g_i^u & -g_\rho^u \\ h_{q_b}^u & h_i^u & -h_\rho^u \end{vmatrix}, \\
\Lambda_7 &= \begin{vmatrix} -f_\rho & f_i & f_\psi \\ -g_\rho & g_i & g_\psi \\ -h_\rho & h_i & h_\psi \end{vmatrix}, \ \Lambda_8 = \begin{vmatrix} f_{q_b} & -f_\rho & f_\psi \\ g_{q_b} & -g_\rho & g_\psi \\ h_{q_b} & -h_\rho & h_\psi \end{vmatrix}, \ \Lambda_9 = \begin{vmatrix} f_{q_b} & f_i & -f_\rho \\ g_{q_b} & g_i & -g_\rho \\ h_{q_b} & h_i & -h_\rho \end{vmatrix}.
\end{aligned}$$

One can show that $h_\rho^u = -\frac{\beta}{1-\beta}$, $g_\rho = -A$, $h_\rho = -\beta B$, $f_\rho^u = g_\rho^u = f_\rho = 0$. Hence, $\Lambda_9^u < 0$, $\Lambda_7 < 0$, $\Lambda_8 < 0$, $\Lambda_7^u = \Lambda_8^u = 0$. In an unconstrained equilibrium, $\frac{\partial q_b}{\partial \rho} = \frac{\Lambda_7^u}{\Lambda^u} = 0$, $\frac{\partial i}{\partial \rho} = \frac{\Lambda_8^u}{\Lambda^u} = 0$, and $\frac{\partial \psi}{\partial \rho} = \frac{\Lambda_9^u}{\Lambda^u} > 0$. Therefore, $\frac{\partial \phi}{\partial \rho} = 0$ and $\frac{\partial p}{\partial \rho} = 0$. In a constrained equilibrium, $\frac{\partial q_b}{\partial \rho} = \frac{\Lambda_7}{\Lambda} = \frac{-nA}{\Lambda} > 0$ and $\frac{\partial i}{\partial \rho} = \frac{\Lambda_8}{\Lambda} = \frac{f_{q_b}A}{\Lambda} > 0$, where f_{q_b} is defined in (32). Hence, $\frac{\partial \phi}{\partial \rho} > 0$, $\frac{\partial p}{\partial \rho} < 0$, and $\frac{\partial B}{\partial \rho} = \frac{f_{q_b}}{1+i}\frac{\partial q_b}{\partial \rho} - \frac{1-n}{(1+i)^2}\frac{u'(q_b)}{c'(q_s)}\frac{\partial i}{\partial \rho} < 0$. Moreover, $\frac{\partial \psi}{\partial \rho} = \frac{\beta B}{1-\beta B} + \frac{\beta\rho}{1-\beta B}\frac{\partial B}{\partial \rho} + \frac{\beta^2 B\rho}{(1-\beta B)^2}\frac{\partial B}{\partial \rho} = \frac{\beta B}{1-\beta B}[1 + \frac{\rho}{B(1-\beta B)}\frac{\partial B}{\partial \rho}] > 0$ if $\left|\frac{\partial B/B}{\partial \rho/\rho}\right| < 1-\beta B$. Similar to the derivation of $\frac{\partial \theta}{\partial A}$,

$$\begin{aligned}
\frac{\partial \theta}{\partial \rho} &= \frac{1}{(1+i)\psi} - \frac{\rho}{(1+i)\psi^2}\frac{\partial \psi}{\partial \rho} - \frac{1+\frac{\rho}{\psi}}{(1+i)^2}\frac{\partial i}{\partial \rho} \\
&= \frac{1}{(1+i)\psi}\left\{1 - r_p\frac{\beta B}{1-\beta B}[1 + \frac{\rho}{B(1-\beta B)}\frac{\partial B}{\partial \rho}] - \psi\theta\frac{\partial i}{\partial \rho}\right\} \\
&= \frac{1}{(1+i)\psi}\left\{\frac{\rho}{B(1-\beta B)}\frac{\partial B}{\partial \rho} - \psi\theta\frac{\partial i}{\partial \rho}\right\},
\end{aligned}$$

where we have used $r_p\frac{\beta B}{1-\beta B} = 1$ by substituting $\psi = \frac{\beta B\rho}{1-\beta B}$ into $r_p = \frac{\rho}{\psi}$ to obtain the last equality. Substituting $\frac{\partial B}{\partial \rho} = \frac{f_{q_b}}{1+i}\frac{\partial q_b}{\partial \rho} - \frac{1-n}{(1+i)^2}\frac{u'(q_b)}{c'(q_s)}\frac{\partial i}{\partial \rho}$ and $\frac{\partial i}{\partial \rho} = \frac{f_{q_b}A}{\Lambda}$ into the above expression, and using $\frac{\partial q_b}{\partial \rho} = \frac{-nA}{\Lambda}$, we obtain $\frac{\partial \theta}{\partial \rho} = 0$. ■

Deriving $\widetilde{W}(m,a)$ and $\phi\bar{\ell}$ in (27) under a combined mechanism. Under a combined mechanism, an economy in which banks take defaulters' collateral and exclude them permanently from the banking system with probability $\zeta \in (0,1]$. If a defaulter will be excluded, he would choose a different portfolio from nondeviators, and trade at a different quantity, $\widetilde{q}_b$. Let $\widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1})$ denote his expected discounted utility from entering the next period, where the tilde indicates the optimal choice. His expected discounted utility in the second subperiod is

$$\widetilde{W}(m,a) = U(\widetilde{x}) - \widetilde{h}_b + \beta \widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1}).$$

With probability $1-\zeta$ a defaulter faces only the punishment of losing his collateral, his expected utility in the second subperiod is $\widehat{W}(m,a)$, from (20).

The continuation payoffs are

$$\begin{aligned} V(m_{+1}, a_{+1}) &= (1-\beta)^{-1}[(1-n)u(q_b) - nc(q_s) + U(x^*) - h], \\ \widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1}) &= (1-\beta)^{-1}[(1-n)u(\widetilde{q}_b) - nc(\widetilde{q}_s) + U(\widetilde{x}) - \widetilde{h}]. \end{aligned} \tag{33}$$

Thus, the expected discounted utility of a deviating buyer entering the second subperiod is

$$\overline{W}(m,a) = \zeta \widetilde{W}(m,a) + (1-\zeta)\widehat{W}(m,a).$$

Consider a deviator who will be excluded from the banking sector. The deviating buyer's problem in the second subperiod is

$$\begin{aligned} \widetilde{W}(m,a) &= \max_{\widetilde{x}, \widetilde{h}_b, \widetilde{m}_{+1}, \widetilde{a}_{+1}} U(\widetilde{x}) - \widetilde{h}_b + \beta \widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1}) \\ \text{s.t. } x + \phi \widetilde{m}_{+1} + \psi \widetilde{a}_{+1} &= \widetilde{h}_b + \phi(m+T). \end{aligned}$$

The first-order condition are $U'(\widetilde{x}) = 1$, which implies $\widetilde{x} = x^*$, $-\phi + \beta \frac{\partial \widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1})}{\partial \widetilde{m}_{+1}} = 0$ and $-\psi + \beta \frac{\partial \widetilde{V}(\widetilde{m}_{+1}, \widetilde{a}_{+1})}{\partial \widetilde{a}_{+1}} = 0$. Note that in the next period, if the deviator becomes a seller, the quantity that he sells is independent of his portfolio; i.e., $\widetilde{q}_s$ satisfies $-c'(\widetilde{q}_s) + p\phi = 0$. So, $\widetilde{q}_s = q_s = \frac{1-n}{n} q_b$; the deviator produces the same amount as non-deviating sellers. Because a deviator cannot borrow or make deposits, his expected utility in the future first subperiod is

$$\widetilde{V}(\widetilde{m}, \widetilde{a}) = (1-n)[u(\widetilde{q}_b) + \widetilde{W}(\widetilde{m} - p\widetilde{q}_b, \widetilde{a})] + n[-c(q_s) + \widetilde{W}(\widetilde{m} + pq_s, \widetilde{a})].$$

The marginal value of holding money for a deviator is

$$\begin{aligned}
\widetilde{V}_m(\widetilde{m},\widetilde{a}) &= (1-n)[u'(\widetilde{q}_b)\frac{\partial \widetilde{q}_b}{\partial m} + \widetilde{W}_m(1-p\frac{\partial \widetilde{q}_b}{\partial m}) + \widetilde{W}_a\frac{\partial \widetilde{a}}{\partial m}] + n[-c'(q_s)\frac{\partial q_s}{\partial m} + \widetilde{W}_m(1+p\frac{\partial q_s}{\partial m}) + \widetilde{W}_a\frac{\partial \widetilde{a}}{\partial m}] \\
&= \phi[(1-n)\frac{u'(\widetilde{q}_b)}{c'(q_s)} + n].
\end{aligned}$$

A deviator's choice of money holdings thus satisfies

$$\frac{\gamma-\beta}{\beta} = (1-n)[\frac{u'(\widetilde{q}_b)}{c'(q_s)} - 1],$$

which is equation (29). Comparing (29) with (15), we find that when $\gamma > \beta$ (which implies $i > 0$),

$$\frac{u'(\widetilde{q}_b)}{c'(q_s)} > \frac{u'(q_b)}{c'(q_s)},$$

implying $\widetilde{q}_b < q_b$. Moreover, because a deviator will be denied credit permanently, his marginal value of holding the real asset is

$$\widetilde{V}_a(\widetilde{m},\widetilde{a}) = (\psi+\rho).$$

For a non-deviator the marginal value of holding the real asset is given by (14). Therefore, if non-deviators hold the real asset, the asset price satisfies

$$\psi = \beta V_a(m,a) = \beta\{(\psi+\rho) + (1-n)\phi[\frac{u'(q_b)}{c'(q_s)} - (1+i)]\frac{\partial \ell}{\partial a}\}.$$

Obviously, $V_a(m,a) > \widetilde{V}_a(\widetilde{m},\widetilde{a})$ when $\frac{u'(q_b)}{c'(q_s)} > (1+i)$. That is, in a constrained equilibrium, $\psi > \beta\widetilde{V}_a(\widetilde{m},\widetilde{a})$, so a deviator choose not to hold the real asset; i.e., $\widetilde{a} = 0$.

We derive the real borrowing constraint $\phi\bar{\ell}$ in an economy under a combined mechanism. For a buyer who repays his loan in the second subperiod, his expected discounted utility is

$$W(m,a) = U(x^*) - h_b + \beta V(m_{+1}, a_{+1}).$$

Existence of an equilibrium with credit requires that borrowers voluntarily repay their loans; i.e., $W(m,a) \geq \overline{W}(m,a)$. From $W(m,a) = \overline{W}(m,a)$, we solve for the real borrowing constraint, $\phi\bar{\ell}$, which leads to

$$U(x^*) - [(1-\zeta)U(\widehat{x}) + \zeta U(\widetilde{x})] + [(1-\zeta)\widehat{h}_b + \zeta\widetilde{h}_b] - h_b + \zeta\beta[V(m_{+1},a_{+1}) - \widetilde{V}(\widetilde{m}_{+1},\widetilde{a}_{+1})] = 0. \quad (34)$$

From $\widetilde{x} = x^* = \widehat{x}$, $\widetilde{q}_s = q_s$, (33), and (34),

$$h_b - [(1-\zeta)\widehat{h}_b + \zeta\widetilde{h}_b] = \frac{\beta\zeta}{1-\beta}\{(1-n)[u(q_b) - u(\widetilde{q}_b)] + \widetilde{h} - h\}. \tag{35}$$

We now derive $h_b - [(1-\zeta)\widehat{h}_b + \zeta\widetilde{h}_b]$ and $\widetilde{h} - h$.

(i) Deriving $h_b - [(1-\zeta)\widehat{h}_b + \zeta\widetilde{h}_b]$, the difference in the production between a non-deviator and a deviator in the subperiod when default occurs:

If the buyer repays his loan, the labor used in production is

$$\begin{aligned} h_b &= x^* + \phi m_{+1} + \psi a_{+1} - \phi(m + \overline{\ell} - pq_b) - \phi\tau m - (\psi + \rho)a + \phi(1+i)\overline{\ell} \\ &= x^* + \phi i\overline{\ell} + \phi pq_b - \rho a, \end{aligned} \tag{36}$$

where we have used $m_{+1} = m + \tau m$ and $a_{+1} = a = A$. If the buyer defaults and will be excluded, he works

$$\begin{aligned} \widetilde{h}_b &= x^* + \phi\widetilde{m}_{+1} + \psi\widetilde{a}_{+1} - \phi(m + \overline{\ell} - pq_b) - \phi\tau m \\ &= x^* + \phi(\widetilde{m}_{+1} - m_{+1}) - \phi\overline{\ell} + \phi pq_b \\ &= x^* + \phi\gamma(\widetilde{m} - m) - \phi\overline{\ell} + \phi pq_b, \end{aligned} \tag{37}$$

where we have used $\widetilde{a}_{+1} = 0$ and the equilibrium condition that a defaulter's money holdings must grow at the rate γ, $\widetilde{m}_{+1} = (1+\tau)\widetilde{m} = \gamma\widetilde{m}$. If a defaulter will not be excluded, he works

$$\begin{aligned} \widehat{h}_b &= x^* + \phi m_{+1} + \psi a_{+1} - \phi(m + \overline{\ell} - pq_b) - \phi\tau m \\ &= x^* - \phi\overline{\ell} + \phi pq_b + \psi a, \end{aligned} \tag{38}$$

where we have used $m_{+1} = m + \tau m$.

From (36), (37), and (38),

$$h_b - [(1-\zeta)\widehat{h}_b + \zeta\widetilde{h}_b] = \phi(1+i)\overline{\ell} - \rho a - (1-\zeta)\psi a - \zeta\phi\gamma(\widetilde{m} - m). \tag{39}$$

(ii) Deriving $h - \widetilde{h}$, the difference in the production between a non-deviator and a deviator in the next period following default:

If a seller never deviated in the past, in the second subperiod, he works

$$
\begin{aligned}
h_s &= x^* + \phi m_{+1} + \psi a_{+1} - \phi(pq_s + \tau m) - (\psi + \rho)a - \phi(1 + i_d)d \\
&= x^* + \phi(m_{+1} - m - \tau m) + \psi(a_{+1} - a) - \phi pq_s - \rho a - \phi im \\
&= x^* - \frac{1-n}{n}\phi i\bar{\ell} - \frac{1-n}{n}\phi pq_b - \rho a,
\end{aligned} \tag{40}
$$

where we have used $i_d = i$, $d = m$ (since $i > 0$), $q_s = \frac{1-n}{n}q_b$, $m = \frac{1-n}{n}\bar{\ell}$, and $m_{+1} = (1+\tau)m = \gamma m$.

From (36) and (40), a non-deviator's expected hours worked are

$$
h = (1-n)h_b + nh_s = x^* - \rho a. \tag{41}
$$

If an agent has defaulted in the previous period, he does not hold any real assets, and he cannot borrow nor make deposits in this period. If he is a buyer, he uses $\widetilde{m}$ money to buy $\widetilde{q}_b$ good in the first subperiod, and in the second subperiod, he chooses money holdings brought to the next period, $\widetilde{m}_{+1}$, receives transfers $\tau\widetilde{m}$, and works

$$
\begin{aligned}
\widetilde{\widetilde{h}}_b &= x^* + \phi\widetilde{m}_{+1} - \phi(\widetilde{m} - p\widetilde{q}_b) - \phi\tau\widetilde{m} \\
&= x^* + \phi p\widetilde{q}_b.
\end{aligned}
$$

where we have used $\widetilde{m}_{+1} = (1+\tau)\widetilde{m}$. If he is a seller, the hours worked is

$$
\begin{aligned}
\widetilde{\widetilde{h}}_s &= x^* + \phi\widetilde{m}_{+1} - \phi(\widetilde{m} + p\widetilde{q}_s) - \phi\tau\widetilde{m} \\
&= x^* - \phi pq_s \\
&= x^* - \frac{1-n}{n}\phi pq_b,
\end{aligned}
$$

where we have used $\widetilde{q}_s = q_s = \frac{1-n}{n}q_b$. Thus, a deviator's expected hours worked are

$$
\widetilde{h} = (1-n)\widetilde{\widetilde{h}}_b + n\widetilde{\widetilde{h}}_s = x^* + (1-n)\phi p(\widetilde{q}_b - q_b). \tag{42}
$$

From (41) and (42),

$$
\widetilde{h} - h = (1-n)\phi p(\widetilde{q}_b - q_b) + \rho a. \tag{43}
$$

Substituting (39) and (43) into (35) and rearranging yields

$$\phi\bar{\ell} = \frac{(1-\beta+\zeta\beta)\rho a + (1-\beta)(1-\zeta)\psi a}{(1-\beta)(1+i)} + \frac{\zeta\beta}{(1-\beta)(1+i)}\{(1-n)\Psi(q_b,\widetilde{q}_b) + \frac{\gamma(1-\beta)}{\beta}c'(q_s)[\widetilde{q}_b-(1-n)q_b]\},$$

where

$$\Psi(q_b,\widetilde{q}_b) = u(q_b) - u(\widetilde{q}_b) - c'(q_s)(q_b-\widetilde{q}_b) \geq 0.$$

In a constrained equilibrium, $\phi\ell = \phi\bar{\ell}$. Substituting (28) into (16), we obtain (30). ■

Proof of Proposition 5. Substituting (28) into (16) one obtains (30) and rearranging (30), we obtain $\psi = \psi_2$. The loan-to-value ratio θ is defined as the ratio of the real loan amount to the real value of collateral, $\frac{\phi\ell}{\psi a}$. Using $\phi\ell = \phi\bar{\ell}$ from (27), we obtain $\theta = \theta_2$, where θ_2 is defined in (31). ■

Proof of Proposition 6. This proof contains two parts. First, we show that if banks buy assets from, instead of lending to, people who need liquidity, it results in identical asset prices and allocation as in the economy under the collateral mechanism. Second, we consider an economy without banks, where buyers and sellers trade the real asset in a competitive asset market in the first subperiod. We will show that the asset prices and allocations are identical to the economy under the collateral mechanism.

(1) Borrowing money from banks under the collateral mechanism and selling assets to banks result in the identical asset price and allocation.

Suppose that in the first subperiod a competitive asset market opens after the consumption shocks are realized but before the trade of goods. We assume, as in the basic model, that sellers do not have the technology to verify the real asset so they do not participate in the asset market, whereas banks have the verification technology. Banks take deposits from the sellers and buy the real asset from the buyers.

We first look at the first-subperiod asset market. Let p_A denote the nominal price (in monetary units) of the real asset in the first subperiod. We will show below that the zero-profit condition for banks implies that $p_A < \frac{\psi+\rho}{\phi}$. Hence, agents who do not need liquidity do not sell the real asset, because they can receive the dividends ρ and the resale price ψ in a frictionless market in the second subperiod. Let a^s denote the amount of the real asset that a buyer wishes to sell in the first-subperiod asset market.

The expected lifetime utility of an agent with portfolio (m, a) entering the first subperiod is

$$V(m,a) = (1-n)[u(q_b) + W(m + p_A a^s - pq_b, a - a^s)] + n[-c(q_s) + W(m - d + pq_s, a, d)], \quad (44)$$

which is similar to (7), except that buyers sell the real asset at the price p_A to acquire funds. A seller's maximization problem is identical to that in the basic model, whereas a buyer's problem becomes

$$\begin{aligned} \max_{q_b, a^s} \quad & u(q_b) + W(m + p_A a^s - pq_b, a - a^s) \\ \text{s.t.} \quad & pq_b \leq m + p_A a^s \\ & a^s \leq a. \end{aligned} \quad (45)$$

Let λ_m and λ_a be the multipliers on the buyer's budget constraint and asset constraint that he cannot sell more assets than what he holds, respectively. The first order conditions are

$$\begin{aligned} u'(q_b) &= c'(q_s)(1 + \frac{\lambda_m}{\phi}), \\ \psi + \rho - p_A\phi &= \lambda_m p_A - \lambda_a. \end{aligned}$$

We have the following cases. Case (i): $\lambda_m = 0$ and $\lambda_a = 0$. In this case, $u'(q_b) = c'(q_s)$; trade is efficient, and $p_A = \frac{\psi+\rho}{\phi}$. Case (ii) $\lambda_m > 0$ and $\lambda_a = 0$. The buyer spends all funds available, $q_b = \frac{m+p_A a^s}{p}$, but the asset constraint does not bind, $a^s < a$. Combining the two first order conditions, we obtain

$$\frac{p_A \phi u'(q_b)}{c'(q_s)} = \psi + \rho. \quad (46)$$

Equation (46) implies that the buyer sells the real asset up to the point at which the marginal benefit of selling an additional unit of assets, $\frac{p_A\phi u'(q_b)}{c'(q_s)}$, equals the marginal cost (the asset's value in the second subperiod), $\psi + \rho$. Case (iii) $\lambda_m > 0$ and $\lambda_a > 0$. Both constraints bind, $q_b = \frac{m+p_A a}{p}$ and $a^s = a$, so we have

$$\frac{p_A \phi u'(q_b)}{c'(q_s)} > \psi + \rho. \quad (47)$$

Buyers wish to acquire more funds to finance consumption, but they are constrained by their asset holdings. Finally, the case with $\lambda_m = 0$ and $\lambda_a > 0$ is not an equilibrium. If the budget constraint does not bind, buyers need not sell all assets, unless the sale price is higher than the asset's value in the second subperiod; i.e., $p_A > \frac{\psi+\rho}{\phi}$. Banks, however, will not buy any assets if $p_A > \frac{\psi+\rho}{\phi}$ (see below).

Banks take deposits d per seller, and buy a^d units of the real asset per buyer, so they face the following resource constraint:

$$(1-n)p_A a^d \leq nd.$$

Given that $i_d > 0$, banks will use all deposits to buy assets, and so the equality holds. Notice that banks' cost of funds is $\phi(1+i_d)nd$, and the revenue from selling assets in the second subperiod is $(\psi+\rho)(1-n)a^d$. The zero-profit condition for competitive banks thus implies $(\psi+\rho)(1-n)a^d - \phi(1+i_d)nd = 0$, from which and the resource constraint we derive

$$p_A = \frac{\psi+\rho}{\phi(1+i_d)}. \tag{48}$$

Note that $p_A < \frac{\psi+\rho}{\phi}$ if $i_d > 0$. The asset market clearing condition is $a^s = a^d = \frac{nd}{(1-n)p_A}$.

An agent's optimal portfolio satisfies

$$\phi_{-1} \geq \beta\phi[(1-n)\frac{u'(q_b)}{c'(q_s)} + n(1+i_d)],$$
$$\psi_{-1} \geq \beta\{\psi+\rho+(1-n)[\frac{p_A\phi u'(q_b)}{c'(q_s)} - \psi - \rho]\frac{\partial a^s}{\partial a}\}.$$

In a stationary equilibrium, the following two conditions must be satisfied:

$$\frac{\gamma-\beta}{\beta} = (1-n)[\frac{u'(q_b)}{c'(q_s)} - 1] + ni_d, \tag{49}$$
$$\frac{1-\beta}{\beta}\psi = \rho + (1-n)[\frac{p_A\phi u'(q_b)}{c'(q_s)} - \psi - \rho]\frac{\partial a^s}{\partial a}. \tag{50}$$

If the asset constraint does not bind, the marginal benefit of selling an additional unit of asset equals the cost, $\frac{p_A\phi u'(q_b)}{c'(q_s)} = \psi+\rho$. The second term of the right side in (50) vanishes, and the asset price ψ is determined by the dividend flows. If the asset constraint binds, $\frac{p_A\phi u'(q_b)}{c'(q_s)} > \psi+\rho$, then the asset price is determined not only by the fundamentals but also by the importance of the asset in financing people's consumption needs.

We now show that the prices and allocations are identical to those in the economy under the collateral mechanism. If the asset constraint does not bind, $a^s < a = A$. From (46), (48) and (49), $i_d = \frac{\gamma-\beta}{\beta}$, and q_b satisfies (19). The asset price is the discounted sum of dividends, $\psi = \frac{\beta\rho}{1-\beta}$.

Thus, the asset price, deposit rate, and allocations are identical to those in the equilibrium with unconstrained credit.

When the asset constraint binds, $a^s = a$. In equilibrium, ψ, p_A, i_d and q_b satisfy (48), (49), (50), and the asset market clearing condition $a^s = \frac{nd}{(1-n)p_A} = A$. Substituting $d = m = M_{-1}$ into the asset market clearing condition, we get

$$p_A = \frac{nM_{-1}}{(1-n)A}. \tag{51}$$

From (48) and (51),

$$1 + i_d = \frac{(\psi + \rho)(1-n)A}{\phi n M_{-1}}, \tag{52}$$

which is identical to (21) by substituting $\phi\ell = \frac{n}{1-n}\phi M_{-1}$ and $a = A$. Moreover, comparing (49) and (15), we find that, because the interest rates in both economies are identical, so is q_b. Substituting $a^s = A$, $\frac{\partial a^s}{\partial a} = 1$ and (48) into (50), we obtain the asset pricing equation as described in (25). The asset price, deposit rate, and allocations are identical to those in the equilibrium with constrained credit.

(2) Borrowing money from banks under the collateral mechanism and selling assets in the financial market result in the identical asset price and allocation.

Assume that in the first subperiod a competitive asset market opens after the consumption shocks are realized but before the trade of goods. People who have consumption needs can liquidate the real asset, and those with idle cash may purchase the asset. We assume that there is no recognizability problem regarding the real asset.

Let a^s and a^d denote the amount of the real asset that a buyer wishes to sell and a seller wishes to buy, respectively. An agent's expected lifetime utility in the first subperiod is

$$V(m,a) = (1-n)[u(q_b) + W(m + p_A a^s - pq_b, a - a^s)] + n[-c(q_s) + W(m - p_A a^d + pq_s, a + a^d)], \tag{53}$$

which is similar to (44), except that sellers may purchase the real asset instead of depositing their cash. A buyer's maximization problem is as described in (45), whereas a seller's problem is

$$\begin{aligned} \max_{q_s, a^d} \quad & -c(q_s) + W(m - p_A a^d + pq_s, a + a^d) \\ \text{s.t.} \quad & p_A a^d \leq m. \end{aligned}$$

Let λ_v denote the multiplier on the investment constraint. The first order conditions are

$$\begin{aligned} -c'(q_s) + pW_m &= 0, \\ -p_A W_m + W_a - p_A \lambda_v &= 0. \end{aligned}$$

If the investment constraint does not bind, $a^d < \frac{m}{p_A}$, then $\lambda_v = 0$ and $p_A = \frac{\psi+\rho}{\phi}$. If $\lambda_v > 0$, then $a^d = \frac{m}{p_A}$ and $p_A < \frac{\psi+\rho}{\phi}$.

In equilibrium, ψ, p_A and q_b satisfy (50),

$$\frac{\gamma - \beta}{\beta} = (1-n)[\frac{u^{'}(q_b)}{c^{'}(q_s)} - 1] + n(\frac{\psi + \rho}{\phi p_A} - 1), \tag{54}$$

and the asset market clearing condition $(1-n)a^s = na^d$.

We have the following cases. Case (i) $\lambda_v = 0$, $\lambda_m = 0$ and $\lambda_a = 0$. All of sellers' investment constraint, buyers' budget constraint and asset constraint are not binding. Then, $p_A = \frac{\psi+\rho}{\phi}$ and $u'(q_b) = c'(q_s)$. Case (ii) $\lambda_v > 0$, $\lambda_m > 0$ and $\lambda_a = 0$. From (46) we know that $p_A = \frac{(\psi+\rho)c'(q_s)}{\phi u'(q_b)}$. Substituting $p_A = \frac{(\psi+\rho)c'(q_s)}{\phi u'(q_b)}$ into (50) and (54), we find that the asset price and q_b are identical to those in the equilibrium with unconstrained credit.

Case (iii) $\lambda_v > 0$, $\lambda_m > 0$ and $\lambda_a > 0$. If buyers' asset constraint binds, they liquidate all assets, $a^s = a$. The asset market clearing condition is $a^s = \frac{nM_{-1}}{(1-n)p_A} = A$, from which we have $p_A = \frac{nM_{-1}}{(1-n)A}$. Substitute $\frac{\partial a^s}{\partial a} = 1$ and $p_A = \frac{nM_{-1}}{(1-n)A}$ into (50) to get the asset price $\psi = \psi_A$, where

$$\psi_A = \frac{\beta n[\rho + \frac{\phi M_{-1} u^{'}(q_b)}{A c^{'}(q_s)}]}{1 - \beta n}. \tag{55}$$

In this economy, there is no deposit interest rate. To make the comparison with the basic model, we substitute $i_d = \frac{(\psi+\rho)(1-n)A}{\phi n M_{-1}} - 1$ from (52) into equations (15) and (25), which determine q_b and the asset price in the economy under the collateral mechanism. After rearranging, we obtain (54) and (55). Thus, the asset price and allocations are identical to those in the constrained equilibrium. ■

B Solving the bank's problem

Since banks are perfectly competitive with free entry, they take as given the loan rate and the deposit rate. There is no strategic interaction among banks or between banks and agents, and no bargaining over the terms of the loan contract. The representative bank solves the following problem per borrower:

$$\begin{aligned} \max_{\ell} \quad & (i - i_d)\ell \\ \text{s.t.} \quad & \ell \leq \bar{\ell}, \quad u(q_b) + W(m, a, \ell, d) \geq \Gamma, \end{aligned}$$

where Γ is the reservation value of the borrower, which is the surplus from obtaining loans at another bank. If banks have full enforcement on repayment, the borrowing constraint is $\bar{\ell} = \infty$. When enforcement is limited, banks choose the credit limit $\bar{\ell}$ to ensure voluntary repayment. The first order condition to the bank's problem is

$$i - i_d - \lambda_L + \lambda_\Gamma [u'(q_b)\frac{dq_b}{d\ell} + W_\ell] = 0,$$

where λ_L and λ_Γ are the Lagrangian multipliers on the lending constraint and borrower's participation constraint, respectively. For $i - i_d > 0$, banks would like to make the largest loan possible to borrowers and, therefore, would choose a loan amount such that $\lambda_\Gamma > 0$.

From (8) and the buyer's budget constraint, $\frac{dq_b}{d\ell} = \frac{\phi}{c'(q_s)}$. We rewrite the first order condition of bank's maximization problem as

$$\frac{u'(q_b)}{c'(q_s)} = 1 + i + \frac{\lambda_L}{\phi \lambda_\Gamma}.$$

If banks can force repayment without any cost, the lending constraint does not bind, and $\lambda_L = 0$. The loan supplied by banks satisfies $\frac{u'(q_b)}{c'(q_s)} = 1 + i$. If $\lambda_L > 0$, the lending constraint binds and $\frac{u'(q_b)}{c'(q_s)} > 1 + i$. With limited enforcement, banks may have to conduct credit rationing.

C The real asset is used as a means of payment and collateral

To concentrate on the role of the real asset as collateral, our paper features fiat money as the unique medium of exchange. In this appendix, we relax this assumption to explore the role of the real asset as a means of payment and as collateral. We will illustrate that the price of an asset reflects its dual role in overcoming the frictions caused by insufficiency of payment instruments and credit market imperfections. We will also show that agents using the real asset as a payment instrument achieve higher consumption and welfare than those using it as collateral.

We assume in the first subperiod there are two locations where agents can trade the consumption good competitively. In location 1 sellers accept the real asset for payment (this can be justified by assuming a costless verification technology available to sellers in location 1, which enables them to fully ascertain the quality of the real asset). Location 2 is reminiscent of the basic model, in that money is the unique means of payment. At the beginning of a period an agent receives a location shock and a preference shock, both of which arrives independently. An agent goes to location 1 with probability α, and to location 2 with the complementary probability $1-\alpha$, where $0<\alpha<1$. For simplicity we assume that buyers in location 1 cannot take loans, but sellers can make deposits if they wish to.[27]

Spatial and informational frictions imply that arbitrage is limited across markets, so that the good may trade at different prices in different locations. Let p_k denote the nominal price of the good in location k, and $q_{b,k}$ and $q_{s,k}$ denote the quantities consumed by a buyer and produced by a seller, respectively, in location $k=1,2$. Let $p_a=\frac{\psi+\rho}{\phi}$ denote the value (including the resale price and dividend) of the real asset in monetary units. An agent with portfolio (m,a) entering the first subperiod has the expected lifetime utility

$$\begin{aligned} V(m,a) &= \alpha\{(1-n)[u(q_{b,1})+W(m+p_a a-p_1 q_{b,1})]+n[-c(q_{s,1})+W(m-d+p_1 q_{s,1})]\} \\ &\quad +(1-\alpha)\{(1-n)[u(q_{b,2})+W(m+\ell-p_2 q_{b,2})]+n[-c(q_{s,2})+W(m-d+p_2 q_{s,2})]\}, \end{aligned}$$

where we have dropped the last three elements in the value function W to reduce notations. The

[27] The assumption that buyers in location 1 cannot make loans is not restrictive because in equilibrium agents prefer using the real asset as a means of payment rather than as collateral. Also, if we assume sellers in location 1 cannot make deposits, the main qualitative results still hold.

interpretation of $V(m,a)$ is similar to (7) except that, with probability α, a buyer can finance his consumption directly with fiat money and the real asset.

The maximization problems and the optimal conditions of sellers and buyers in location 2 are identical to those in the basic model. We discuss briefly the optimal conditions in location 1. The first order condition to seller's problem is

$$p_1 = \frac{c'(q_{s,1})}{\phi}. \tag{56}$$

Since the buyer can use fiat money and the real asset to make purchases, his problem is

$$\begin{array}{ll} \max\limits_{q_{b,1}} & u(q_{b,1}) + W(m + p_a a - p_1 q_{b,1}) \\ \text{s.t.} & p q_{b,1} \leq m + p_a a. \end{array}$$

Let λ_1 denote the multiplier on the buyer's budget constraint. The first order condition is

$$u'(q_{b,1}) = c'(q_{s,1})(1 + \frac{\lambda_1}{\phi})$$

If the budget constraint does not bind, $u'(q_{b,1}) = c'(q_{s,1})$, implying the quantity traded is efficient. If $\lambda_1 > 0$, the buyer spends all money and real assets, and

$$q_{b,1} = \frac{m + p_a a}{p_1}. \tag{57}$$

Finally, the goods market clearing condition at location k is

$$n q_{s,k} = (1-n) q_{b,k}.$$

We focus on the equilibrium in which agents hold money and the real asset. An agent's optimal holdings of money and assets satisfy

$$\begin{aligned} \phi_{-1} &= \beta\phi\{(1-n)[\alpha\frac{u'(q_{b,1})}{c'(q_{s,1})} + (1-\alpha)\frac{u'(q_{b,2})}{c'(q_{s,2})}] + n(1+i)\}, \\ \psi &= \beta\{\alpha(\psi+\rho)[(1-n)\frac{u'(q_{b,1})}{c'(q_{s,1})} + n] + (1-\alpha)\{\psi + \rho + (1-n)\phi[\frac{u'(q_{b,2})}{c'(q_{s,2})} - (1+i)]\frac{\partial \ell}{\partial a}\}\}. \end{aligned}$$

The following two conditions must be satisfied in equilibrium:

$$\frac{\gamma - \beta}{\beta} = (1-n)\{\alpha[\frac{u'(q_{b,1})}{c'(q_{s,1})} - 1] + (1-\alpha)[\frac{u'(q_{b,2})}{c'(q_{s,2})} - 1]\} + ni, \tag{58}$$

$$\frac{1 - \beta B_m}{\beta}\psi = B_m \rho + (1-\alpha)(1-n)\phi[\frac{u'(q_{b,2})}{c'(q_{s,2})} - (1+i)]\frac{\partial \ell}{\partial a}, \tag{59}$$

where

$$B_m = 1 + \alpha(1-n)[\frac{u'(q_{b,1})}{c'(q_{s,1})} - 1].$$

If the buyer's budget constraint binds; i.e., the amounts of fiat money and real assets are not sufficient to buy the efficient quantity, then $B_m > 1$.

As in the basic model, under full enforcement the interest rate that clears the market is

$$i = \frac{u'(q_{b,2})}{c'(\frac{1-n}{n}q_{b,2})} - 1. \tag{60}$$

Substituting (60) into (58) and (59), we obtain

$$\frac{\gamma - \beta}{\beta} = \alpha(1-n)[\frac{u'(q_{b,1})}{c'(\frac{1-n}{n}q_{b,1})} - 1] + (1 - \alpha + \alpha n)i, \tag{61}$$

and the asset price $\psi = \psi_u^m$, where

$$\psi_u^m = \frac{\beta B_m \rho}{1 - \beta B_m}. \tag{62}$$

The asset pricing equation (62) is the discounted sum of dividends, with βB_m as the "effective" discount factor. The term B_m depends on whether the amounts of fiat money and real assets are sufficient to achieve the efficiency, and it also reflects the liquidity return from facilitating trades. When the buyer's budget constraint binds, $B_m > 1$, and $\psi_u^m > \psi^u$. The asset price under full enforcement is influenced by factors besides fundamentals, and the liquidity premium arises from the service provided by the asset to facilitate trades.

We discuss equilibria under the collateral mechanism; other cases under the combined mechanism may be analyzed in a similar way. Since buyers in location 1 do not take loans, in an equilibrium with $i > 0$ bank lending satisfies

$$\phi\ell = \frac{n}{(1-\alpha)(1-n)}\phi M_{-1}.$$

In an equilibrium with unconstrained credit, the asset price is ψ_u^m, defined in (62). The loan rate and allocations are identical to those in the economy with full enforcement. Next, we study the constrained equilibrium.

Definition 4 *A monetary equilibrium with constrained credit is* $(q_{b,1}, q_{b,2}, i, \psi)$ *satisfying*

$$\begin{aligned}
q_{b,1} &= \frac{m + p_a a}{p_1} \\
\frac{\gamma - \beta}{\beta} &= (1-n)\{\alpha[\frac{u'(q_{b,1})}{c'(q_{s,1})} - 1] + (1-\alpha)[\frac{u'(q_{b,2})}{c'(q_{s,2})} - 1]\} + ni \\
\frac{1 - \beta B_m}{\beta}\psi &= B_m \rho + (1-\alpha)(1-n)[\frac{u'(q_{b,2})}{c'(q_{s,2})} - (1+i)](\frac{\psi + \rho}{1+i}),
\end{aligned}$$

such that $0 < \phi\ell = nc'(q_{s,2})q_{b,2} = \phi\bar{\ell}$, *where* $\phi\bar{\ell}$ *satisfies (21), and* $q_{s,2} = \frac{1-n}{n} q_{b,2}$.

Proposition 7 *When the real asset is used as a means of payment and collateral, in a constrained equilibrium under collateral mechanism, the asset price is* $\psi = \psi_c$, *where*

$$\psi_c = \frac{\beta B_c \rho}{1 - \beta B_c}, \tag{63}$$

and

$$B_c = 1 + (1-n)\left\{\alpha[\frac{u'(q_{b,1})}{c'(q_{s,1})} - 1] + (1-\alpha)[\frac{u'(q_{b,2})}{c'(q_{s,2})}\frac{1}{1+i} - 1]\right\}.$$

In the asset pricing equation (63), the "effective" discount factor βB_c takes into account the insufficiency of payment instruments and the frictions due to credit market imperfection: the first term in the big bracket of B_c reflects whether the amount of the means of payment is sufficient to achieve the efficiency, and the second term captures the severity of credit rationing. If any of the budget constraint or the credit constraint binds, $B_c > 1$ and ψ_c is higher than the fundamental value. The liquidity premium stems from the liquidity services provided by the asset to secure loans and to facilitate trades as a means of payment.

From the numerical examples we find that, in both constrained and unconstrained equilibrium, buyers in location 1 enjoy higher expected utility than those in location 2.[28] This implies that, using the real asset as a payment instrument achieves higher consumption and welfare than using it as collateral. The reason is that a borrower has to pay an interest, whereas there is no such a cost if he uses the asset directly as a means of payment. That is, the interest payments reduce

[28] In numerical examples the utility function is $u(q_{b,k}) = \frac{(q_{b,k})^{0.8}}{0.8}$, and the cost function is $c(q_{s,k}) = q_{s,k}$. The parameter values for the benchmark are $n = .6$, $\beta = .95$, $\gamma = 1.01$, $\alpha = .4$, $C = 2.537$, $\rho = .01$, and $A = 2$. There exists a constrained equilibrium in which $q_{b,1} = .7347 > q_{b,2} = .7048$. When $\rho = .015$, there exists an unconstrained equilibrium in which $q_{b,1} = .863765 > q_{b,2} = .714567$.

the amount that agents can borrow so that they cannot consume as much if they use the asset to borrow as if they use it as a means of payment.