A Unified Framework for Monetary Theory and Policy Analysis

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Introduction

• Reduced-form monetary macro models: not explicit about the role of money in overcoming spatial, temporal or informational frictions.

• Search models have explicit micro-foundations.

• Previous search models: ill-suited for the analysis of monetary policy due to the extreme restrictions on money holding.

• This model: no extreme restrictions on money holding.
Main feature of the model

- Previous models without restrictions on money holding are complicated by the endogenous distribution of money holding, $F(m)$.

- Assumption of quasi-linear preference makes $F(m)$ degenerate: No wealth effects in the demand for money.

- This framework is as easy to use as standard reduced-form models (e.g. study the cost of inflation)
Model: market structure and preferences

Market structure \( \left\{ \begin{array}{l}
\text{Day – DM (search): special goods} \\
\text{Night – CM (Walrasian): general goods}
\end{array} \right. \)

- Preferences: \( U(x, h, X, H) = u(x) - c(h) + U(X) - H. \)
  day night
- \( x, X: \) consumption. \( h, H: \) labor supply.
- \( \exists q^* \in (0, \infty) \text{ s.t. } u'(q^*) = c'(q^*). \)
- \( \exists X^* \in (0, \infty) \text{ s.t. } U'(X^*) = 1 \text{ with } U(X^*) > X^* \)
Model: DM

- **DM**: decentralized and anonymous $\rightarrow$ no credit.
  $\alpha$: prob of meeting.

- **special goods**:
  prob(double coincidence of wants) $= \delta$.
  prob(single coincidence of wants) $= \sigma$.
  prob(neither wants the other produces) $= 1 - 2\sigma - \delta$. 
Model: CM

- CM: All agents produce and consume a general good.
- Special goods and general goods are divisible and non-storable
  → no commodity money.
Model: distribution of money holdings

- money: perfectly divisible and storable in any non-negative quantity. $M$: total money stock

- $F_t(\tilde{m}) (G_t(\tilde{m}))$: measure of agents starting the DM (CM) holding $m \leq \tilde{m}$, $F_0, G_0$ exogenously given.

- $\int m \, dF_t(m) = \int m \, dG_t(m) = M, \forall t$.

- $\phi_t$: value of money in terms of general goods in CM.

- No uncertainty in the basic model except for random matching.

- Aggregate variables such as $F_t, G_t$ and prices are taken as given, an agent’s decisions depend only on his money holdings, $m$. 
Value function: DM

An agent with $m$ entering DM:

$$V_t(m) = \alpha \sigma \int \{ u[q_t(m, \tilde{m})] + W_t[m - d_t(m, \tilde{m})] \} dF_t(\tilde{m})$$

$$+ \alpha \sigma \int \{ -c[q_t(\tilde{m}, m)] + W_t[m + d_t(\tilde{m}, m)] \} dF_t(\tilde{m})$$

$$+ \alpha \delta \int B_t(m, \tilde{m}) \ dF_t(\tilde{m})$$

$$+ (1 - 2\alpha \sigma - \alpha \delta)W_t(m).$$

(1)
Value function: CM

An agent with $m$ entering CM:

$$W_t(m) = \max_{X,H,m'} \{ U(X) - H + \beta V_{t+1}(m') \}$$ \hspace{1cm} (2)

s.t. $X = H + \phi_t m - \phi_t m'$

$$X \geq 0, 0 \leq H \leq \bar{H}, m' \geq 0.$$ 

$m'$: money taken out of the market.

- Assume interior solution for $X, H$, characterize equilibrium and then check $0 < H < \bar{H}$ is satisfied.
Bargaining: agents with $m$ meets someone with $\tilde{m}$

- In a double-coincidence-of-wants meeting:
  symmetric Nash bargaining with the continuation value as the threat point:

  $$B_t(m, \tilde{m}) = u(q^*) - c(q^*) + W_t(m).$$

- In a single-coincidence-of-wants meeting:
  Nash bargaining with the continuation value as the threat point, buyer’s bargaining power $\theta$:

  $$\max_{q, d} \left[ u(q_t) + W_t(m - d_t) - W_t(m) \right]^\theta$$
  $$\left[ -c(q_t) + W_t(\tilde{m} + d_t) - W_t(\tilde{m}) \right]^{1-\theta}$$
  s.t. $d \leq m, q \geq 0$.

- definition of equilibrium (p.468).
How to find an equilibrium?

1. Derive some properties of the solution to the CM problem.

2. Solve the bargaining problem.

3. Simplify $V_t$ and solve for individual’s problem of choosing $m'_t(m)$: $m'_t = M$ for all agents regardless of $m_t$, $\Rightarrow F_{t+1}$ degenerate

4. Combine the solutions to CM and DM problems to reduce the model to a single difference equation.
Linearity of $W(m)$

- Substitute for $H$ from the budget equation to write (2) as

$$W_t(m) = \phi_t m + \max_{X,m'} \{ U(X) - X - \phi_t m' + \beta V_{t+1}(m') \}$$

where

$$\max_{X,m'} \{ U(X) - X - \phi_t m' + \beta V_{t+1}(m') \} \equiv W(0).$$

- Notes:
  - $X(m) = X^*$ where $U'(X^*) = 1$.
  - $m'_t(m)$ does not depend on $m$. (quasi-linear utility rules out the wealth effect)
  - $W_t$ is linear in $m$ with slope $\phi_t$: $W(m) = W(0) + \phi m$. 
Bargaining problem

Given that \( W(m) = W(0) + \phi m \), the bargaining problem,

\[
\max_{q,d} \left[ u(q_t) + W_t(m - d_t) - W_t(m) \right]^\theta \\
\left[ -c(q_t) + W_t(\tilde{m} + d_t) - W_t(\tilde{m}) \right]^{1-\theta}
\]

s.t. \( d \leq m, q \geq 0 \),

becomes:

\[
\max_{q,d} \left[ u(q) - \phi_t d \right]^\theta \left[ -c(q) + \phi_t d \right]^{1-\theta}
\]

s.t. \( d \leq m, q \geq 0 \).
Bargaining solution

- Bargaining solution:

\[ q_t(m, \tilde{m}) = \begin{cases} 
\hat{q}_t(m) & \text{if } m < m_t^* \\
q^* & \text{if } m \geq m_t^* 
\end{cases} \]

\[ d_t(m, \tilde{m}) = \begin{cases} 
m & \text{if } m < m_t^* \\
m^* & \text{if } m \geq m_t^* 
\end{cases} \]

- \( \hat{q}_t(m) \) solves \( \phi_t m = z(q_t) \)

\[ z(q) \equiv \frac{\theta c(q)u'(q) + (1 - \theta)u(q)c'(q)}{\theta u'(q) + (1 - \theta)c'(q)}. \]

\[ m_t^* = z(q^*) / \phi_t \]
Verifying the bargaining solution

• Ignoring the constraint \( d \leq m \), the necessary and sufficient conditions for a solution are

\[
\theta[-c(q_t) + \phi_t d_t]u'(q_t) = (1 - \theta)[u(q_t) - \phi_t d_t]c'(q_t)
\]

\[
\theta[-c(q_t) + \phi_t d_t] = (1 - \theta)[u(q_t) - \phi_t d_t]
\]

• If the constraint is not binding \((m \geq m^*_t)\): \( q_t = q^* \), 
\( d_t = m^*_t = [\theta c(q^*) + (1 - \theta)u(q^*)]/\phi_t \).
(spend \( m^*_t \) dollars to get \( q^* \))

• If the constraint is binding: \( q_t \) is given by \( \hat{q}_t(m) \) with 
\( d_t = m \Rightarrow \phi_t m = z(q_t) \).
(spend all his money to get \( \hat{q}_t(m) \))

• Solutions do not depend on sellers’ money holdings \( \tilde{m} \)!
Property of the bargaining solution

For all $m < m^*_t$, $q'_t(m) = \frac{\phi_t}{z'(q_t)}$, where

$$z' = \frac{u'c'[\theta u' + (1 - \theta)c'] + \theta(1 - \theta)(u - c)(u'c'' - c'u'')}{[\theta u' + (1 - \theta)c']^2} > 0.$$ 

• Note: $\hat{q}_t(m) \to q^*$ as $m \to m^*_t$. Hence, $q_t(m) = \hat{q}_t(m)$ is strictly increasing for $m < m^*_t$, is continuous at $m^*_t$, and is constant at $q_t(m) = q^*$ for all $m > m^*_t$. 
Simplifying equation (1)

\[ V_t(m) = \nu_t(m) + \phi_t m + \max_{m'}\{-\phi_t m' + \beta V_{t+1}(m')\} \]

where

\[ \nu_t(m) \equiv \alpha \sigma \{ u[q_t(m)] - \phi_t d_t(m) \} + \alpha \sigma \int \{ \phi_t d_t(\tilde{m}) - c[q_t(\tilde{m})] \} dF_t(\tilde{m}) + \alpha \delta[u(q^*) - c(q^*)] + U(X^*) - X^* \]

By repeated substitution we have

\[ V_t(m_t) = \nu_t(m_t) + \phi_t m_t \]

\[ + \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}}\{-\phi_j m_{j+1} + \beta[\nu_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1}]\} \] (3)

Reduces the choice of \( \{m_{t+1}\} \) to a sequence of problems defined in terms of primitives, since \( \nu_{t+1} \) is a known function.
Equilibrium property

\[
\nu'_{t+1}(m_{t+1}) = \alpha \sigma \{ u'[q_{t+1}(m_{t+1})]q'_{t+1}(m_{t+1}) - \phi_{t+1}d'_{t+1}(m_{t+1}) \}.
\]

- \(\nu'_{t+1}(m_{t+1}) = 0\) for all \(m_{t+1} \geq m^*_{t+1}\) by the bargaining solution.
  \(\Rightarrow \phi_t < \beta \phi_{t+1}\) implies that the problem of choosing \(m_{t+1}\) in (3) has no solution, since the objective function is strictly increasing for all \(m_{t+1} \geq m^*_{t+1}\).
  \(\Rightarrow\) Any equilibrium must satisfy \(\phi_t \geq \beta \phi_{t+1}\).

- Therefore, the minimum inflation rate consistent with equilibrium is \(\frac{\phi_t}{\phi_{t+1}} = \beta\), which is the Friedman rule.
Equilibrium path: $\phi_t$

- $\theta \approx 1$ or $u'$ is log concave ($u'u'' \leq (u'')^2$)
  \[ \Rightarrow \nu''_{t+1} < 0 \]
  \[ \Rightarrow \text{a unique choice of } m_{t+1} \text{ in any equilibrium;} \ i.e. \ F_{t+1} \text{ degenerate at } m_{t+1} = M. \]

- Thus, $d_{t+1} = M$, the buyer exchanges all his money, and $q_{t+1} = \hat{q}_{t+1}(M)$.

- In any monetary equilibrium, FOC evaluated at $m_{t+1} = M$ is $\phi_t = \beta[v'_{t+1}(M) + \phi_{t+1}]$, or
  \[ \phi_t = \beta\{\alpha\sigma u'[q_{t+1}(M)]q'_{t+1}(M) + (1 - \alpha\sigma)\phi_{t+1}\} \quad (4) \]
Equilibrium path: $q_t$

- Inserting $\phi_t = z(q_t)/M$ and $q'_t(M) = \phi_t/z'(q_t)$ from the bargaining sol, (4) $\Rightarrow$

$$z(q_t) = \beta z(q_{t+1}) \left[ \alpha \sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha \sigma \right]. \quad (5)$$

a difference equation in $q_t$.

- A monetary equilibrium is characterized by any path $\{q_t\}$ satisfying (5) that stays in $(0, q^*)$, since $q_t < q^*$ follows from $m_t < m^*$. 
Steady State

(5) ⇒

\[
\frac{u'(q)}{z'(q)} = 1 + \frac{1 - \beta}{\alpha \sigma \beta}.
\] (6)

• \(\theta = 1\): \(z(q) = c(q)\) \(\exists! q > 0\) solves (6) if, e.g. \(u'(0) = \infty\).

• \(\theta < 1\): \(\frac{u'(q)}{z'(q)}\) is monotone if, e.g. \(\theta \approx 1\) or \(c\) is linear and \(u'\) log concave.

If there is a unique solution, \(q\), then \(\frac{\partial q}{\partial \theta} > 0, \frac{\partial q}{\partial \sigma} > 0, \frac{\partial q}{\partial \alpha} > 0\) and \(\frac{\partial q}{\partial \beta} > 0\).

• \(\theta = 1\): \(q \to q^*\) as \(\beta \to 1\)

• \(\theta < 1\): \(q < q^*\) as \(\beta \to 1\)

• Steady state is efficient iff \(q = q^*\), which requires \(\beta = 1\) and \(\theta = 1\).
Changes in the money Supply

- New money is injected in CM: \( M_{t+1} = (1 + \tau)M_t \).

- Consider S-S where \( q \) and real balances \( \phi M = z(q) \) are constant; i.e., \( \phi_t / \phi_{t+1} = 1 + \tau \).

- S-S condition:
  \[
  \frac{u'(q)}{z'(q)} = 1 + \frac{1 + \tau - \beta}{\alpha \sigma \beta}.
  \] (7)

- \( 1 + i = (1 + r)(1 + \pi) \); \( \pi = \tau \): equilibrium inflation rate \( r = \frac{1 - \beta}{\beta} \) equilibrium real interest rate.

(7) \( \Rightarrow \)

\[
\frac{u'(q)}{z'(q)} = 1 + \frac{i}{\alpha \sigma}
\]

- Assume a unique monetary S.S: \( \frac{\partial q}{\partial \tau} < 0; \frac{\partial q}{\partial i} < 0 \).
Results

- \( \theta = 1 : z(q) = c(q) \), get \( q^* \) iff \( \tau = \tau^F \) (\( i = 0 \))

- \( \theta < 1 : q < q^* \) at \( \tau^F \) since a necessary condition for monetary equilibrium is \( \tau \geq \tau^F \) (\( i \geq 0 \)). The Friedman rule is optimal here but does not achieve the efficient outcomes \( q^* \).

- Why?
Two types of inefficiencies

- due to $\beta < 1 : q < q^*$
- due to $\theta < 1$: holdup problem.

Hosios (1990) condition for efficiency:
The bargaining solution should split the surplus so that each party is compensated for his contribution to the surplus in a match.

- The surplus in a single-match is all due to the buyer, since the outcome depends on $m$ but not on $\tilde{m}$. Hence, efficiency requires $\theta = 1$ here.

- The wedge due to $\theta < 1$ is important for issues such as the welfare cost of inflation.
Welfare cost of moderate inflation
Welfare cost of inflation

- Calibrate the model to standard observations and use it to measure the cost of inflation.
- Going from 10 percent to 0 percent inflation is worth between 3 and 5 percent of consumption – much higher than previous estimates.
- The empirical relevance of the holdup problem is important to assessing the welfare cost of inflation.